MATH 2090 **Answers to the Review Problems for Exam 3** **Fall 2001**

1. Consider the vectors $\vec{v}_1 = (1, 3, 4)$, $\vec{v}_2 = (0, 1, 1)$, $\vec{v}_3 = (-1, 0, -1)$, $\vec{v}_4 = (5, 3, -2)$ in $\mathbb{R}^3$.

(a) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\mathbb{R}^3$?

This set of vectors *does* span $\mathbb{R}^3$

(b) Is this set of vectors is linearly independent?

No. For instance, $\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$

(c) Does this set of vectors form a basis for $\mathbb{R}^3$? Explain.

No, it is not a linearly independent set.

2. Recall that the trace of a square matrix is the sum of the diagonal entries. Let S be the set of all 2×2 matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ real, and } a + d = 0 \right\}$.

(a) Show that S is a subspace of $\mathbb{M}_2(\mathbb{R})$, the vector space of all 2×2 matrices with real entries.

Show that S is closed under vector addition and scalar multiplication:
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be in S. Then $a + d = 0$, $x + w = 0$, and $A + B = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$.
Now $A + B$ is in S if $(a + x) + (d + w) = 0$. But $(a + x) + (d + w) = a + d + x + w = 0 + 0 = 0$.
So $A + B$ is in S, and S is closed under vector addition.
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in S and k is a scalar, then $a + d = 0$ and $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$.
Now kA is in S if $ka + kd = 0$. But $ka + kd = k(a + d) = k \cdot 0 = 0$.
So $A + B$ is in S, and S is closed under scalar multiplication.

(b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

Write $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Check that $c_1A_1 + c_2A_2 + c_3A_3 = \mathbf{0}$ has only the trivial solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ to see that $\{A_1, A_2, A_3\}$ is a linearly independent set.
Check that $c_1A_1 + c_2A_2 + c_3A_3 = A$ is consistent whenever $a + d = 0$ to see that $\{A_1, A_2, A_3\}$ spans S.
Note that these are systems of four linear equations in three unknowns.

(c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).
In other words, express the matrix A as a linear combination of the matrices given in part (b).

$A = (2)A_1 + (-1)A_2 + (1)A_3$.

3.(a) Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where k is a scalar.
Find all values of k for which T is a linear transformation. Find the matrix of T for these values of k.

T is a linear transformation if $k = 0$. If $k = 0$, the matrix of T is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1, 0, 0) = 1$, $T(1, 1, 0) = 3$, $T(1, 1, 1) = 5$, find $T(2, 4, 6)$.

Since $(2, 4, 6) = -2(1, 0, 0) - 2(1, 1, 0) + 6(1, 1, 1)$, $T(2, 4, 6) = -2T(1, 0, 0) - 2T(1, 1, 0) + 6T(1, 1, 1) = 22$.

4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$.

(a) Find a basis for $\text{Ker}(T)$, the kernel of the linear transformation T. What is the dimension of $\text{Ker}(T)$?

$\{(3, -4, 6)\}$ is a basis for $\text{Ker}(T)$. $\dim \text{Ker}(T) = 1$.

(b) Find a basis for $\text{Rng}(T)$, the range of the linear transformation T. What is the dimension of $\text{Rng}(T)$?

$\{(2, 2, -4), (0, 3, 0)\}$ is a basis for $\text{Rng}(T)$. $\dim \text{Rng}(T) = 2$.

(c) Do the rows of the matrix A form a basis for $\mathbb{R}^3$? Explain.

No. e.g., they are linearly dependent.