# MATH 2090 — Answers to the Review Problems for Exam 3 — Fall 2001

**1**. Consider the vectors $\vec{v}_1 = (1,3,4)$, $\vec{v}_2 = (0,1,1)$, $\vec{v}_3 = (-1,0,-1)$, $\vec{v}_4 = (5,3,-2)$ in $\mathbb{R}^3$.

(a) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\mathbb{R}^3$?

$\boxed{\text{This set of vectors } \textit{does} \text{ span } \mathbb{R}^3}$

(b) Is this set of vectors is linearly independent?

$\boxed{\text{No. For instance, } \vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}}$

(c) Does this set of vectors form a basis for $\mathbb{R}^3$? Explain.

$\boxed{\text{No, it is not a linearly independent set.}}$

**2**. Recall that the trace of a square matrix is the sum of the diagonal entries. Let $S$ be the set of all $2 \times 2$ matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ real, and } a + d = 0 \right\}$.

(a) Show that $S$ is a subspace of $\mathbb{M}_2(\mathbb{R})$, the vector space of all $2 \times 2$ matrices with real entries.

> Show that $S$ is closed under vector addition and scalar multiplication:
> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be in $S$. Then $a + d = 0$, $x + w = 0$, and $A + B = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$.
> Now $A + B$ is in $S$ if $(a + x) + (d + w) = 0$. But $(a + x) + (d + w) = a + d + x + w = 0 + 0 = 0$.
> So $A + B$ is in $S$, and $S$ is closed under vector addition.
> If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $S$ and $k$ is a scalar, then $a + d = 0$ and $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$.
> Now $kA$ is in $S$ if $ka + kd = 0$. But $ka + kd = k(a + d) = k \cdot 0 = 0$.
> So $A + B$ is in $S$, and $S$ is closed under scalar multiplication.

(b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $S$.

> Write $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
> Check that $c_1A_1 + c_2A_2 + c_3A_3 = \mathbf{0}$ has only the trivial solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ to see that $\{A_1, A_2, A_3\}$ is a linearly independent set.
> Check that $c_1A_1 + c_2A_2 + c_3A_3 = A$ is consistent whenever $a + d = 0$ to see that $\{A_1, A_2, A_3\}$ spans $S$.
> Note that these are systems of four linear equations in three unknowns.

(c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).
In other words, express the matrix $A$ as a linear combination of the matrices given in part (b).

$\boxed{A = (2)A_1 + (-1)A_2 + (1)A_3.}$

**3**.(a) Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where $k$ is a scalar.
Find all values of $k$ for which $T$ is a linear transformation. Find the matrix of $T$ for these values of $k$.

$\boxed{T \text{ is a linear transformation if } k = 0. \text{ If } k = 0, \text{ the matrix of } T \text{ is } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.}$

(b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1,0,0) = 1$, $T(1,1,0) = 3$, $T(1,1,1) = 5$, find $T(2,4,6)$.

$\boxed{\text{Since } (2,4,6) = -2(1,0,0) - 2(1,1,0) + 6(1,1,1),\ T(2,4,6) = -2T(1,0,0) - 2T(1,1,0) + 6T(1,1,1) = 22.}$

**4**. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Ker}(T)$, the kernel of the linear transformation $T$. What is the dimension of $\mathrm{Ker}(T)$?

$\boxed{\{(3,-4,6)\} \text{ is a basis for } \mathrm{Ker}(T). \quad \dim \mathrm{Ker}(T) = 1.}$

(b) Find a basis for $\mathrm{Rng}(T)$, the range of the linear transformation $T$. What is the dimension of $\mathrm{Rng}(T)$?

$\boxed{\{(2,2,-4), (0,3,0)\} \text{ is a basis for } \mathrm{Rng}(T). \quad \dim \mathrm{Rng}(T) = 2.}$

(c) Do the rows of the matrix $A$ form a basis for $\mathbb{R}^3$? Explain.

$\boxed{\text{No. e.g., they are linearly dependent.}}$