MATH 2090 — Answers to the Review Problems for Exam 4 — Fall 2001

1. For each of the matrices A below, find a real-valued fundamental set of solutions (explain why your solutions form a fundamental set), a fundamental matrix, and the general solution of the system $\vec{x}\,' = A\,\vec{x}$. Determine if the matrix A is defective or diagonalizable in each case. If A is diagonalizable, find a diagonal matrix D and an invertible matrix S so that $S^{-1}AS = D$. Also, solve the initial value problem in part (a).

(a) $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & -5 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 7 & -4 \\ 4 & -1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ (e) $A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}$

(a) fund. set: $\left\{ \vec{x}_1 = e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{x}_2 = e^{-3t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$, $W[\vec{x}_1, \vec{x}_2] = 4e^{2t} \neq 0$, fund. matrix: $\begin{bmatrix} e^{5t} & -e^{-3t} \\ 2e^{5t} & 2e^{-3t} \end{bmatrix}$

A is diagonalizable. $S = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ $\vec{x} = 2e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{-3t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ solves the IVP

(b) fund. set: $\left\{ \vec{x}_1 = \begin{bmatrix} e^{3t}\cos 2t - 2e^{3t}\sin 2t \\ e^{3t}\cos 2t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} e^{3t}\sin 2t + 2e^{3t}\cos 2t \\ e^{3t}\sin 2t \end{bmatrix} \right\}$, $W[\vec{x}_1, \vec{x}_2] = -2e^{6t} \neq 0$,

fund. matrix: $\begin{bmatrix} e^{3t}\cos 2t - 2e^{3t}\sin 2t & e^{3t}\sin 2t + 2e^{3t}\cos 2t \\ e^{3t}\cos 2 & e^{3t}\sin 2t \end{bmatrix}$

A is diagonalizable. $S = \begin{bmatrix} 1+2\mathrm{i} & 1-2\mathrm{i} \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3+2\mathrm{i} & 0 \\ 0 & 3-2\mathrm{i} \end{bmatrix}$

(c) fund. set: $\left\{ \vec{x}_1 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}_2 = te^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right\}$, $W[\vec{x}_1, \vec{x}_2] = -e^{6t}/4 \neq 0$,

fund. matrix: $\begin{bmatrix} e^{3t} & te^{3t} + e^{3t}/4 \\ e^{3t} & te^{3t} \end{bmatrix}$ A is defective. The eigenvalue $\lambda = 3$ has algebraic multiplicity 2 and geometric multiplicity 1.

(d) fund. set: $\left\{ \vec{x}_1 = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix}, \vec{x}_2 = e^{2t}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \vec{x}_3 = e^{4t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = -4e^{6t} \neq 0$,

fund. matrix: $\begin{bmatrix} -3 & e^{2t} & e^{4t} \\ 9 & 3e^{2t} & e^{4t} \\ 5 & e^{2t} & e^{4t} \end{bmatrix}$ A is diagonalizable. $S = \begin{bmatrix} -3 & 1 & 1 \\ 9 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(e) fund. set: $\left\{ \vec{x}_1 = e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = e^{3t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = e^{3t}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, $W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = 2e^{7t} \neq 0$,

fund. matrix: $\begin{bmatrix} e^{t} & -3e^{3t} & 2e^{3t} \\ e^{t} & e^{3t} & 0 \\ e^{t} & 0 & e^{3t} \end{bmatrix}$ A is diagonalizable. $S = \begin{bmatrix} 1 & -3 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. The differential equation $x^3y''' + x^2y'' - 2xy' + 2y = 0$, $x > 0$ has three solutions of the form $y(x) = x^r$. Find these solutions, show that they are linearly independent on the interval $(0, \infty)$, and find the general solution of this differential equation.

The functions $y_1 = x$, $y_2 = x^2$, and $y_3 = x^{-1}$ are solutions of this DE.
The Wronskian $W[x, x^2, x^{-1}] = 6/x$ is non-zero on $I = (0, \infty)$, so $\{y_1, y_2, y_3\}$ is linearly independent on I.
The general solution of the DE is $y = c_1x + c_2x^2 + c_3x^{-1}$

3. For each of the non-homogeneous linear differential equations $P(D)y = F(x)$ below, find the complementary solution, find the annihilator of the function $F(x)$, and determine the form of a particular solution. In parts (a) and (b), also find the general solution of the differential equation, and solve the initial value problem in part (a).

(a) $(D-1)(D-2)(D-3)y = 6e^{4x}$, $y(0) = 4$, $y'(0) = 10$, $y''(0) = 30$ (b) $y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x}$

(c) $(D^2 - 2D + 2)(D-3)^2(D+2)y = x^2 - e^x \cos x + 3e^{3x}$

(a) complementary solution: $y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ $D-4$ annihilates of $6e^{4x}$ trial solution: $y_p = Ae^{4x}$
general solution: $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + e^{4x}$ solution of the IVP: $y = e^x + e^{2x} + e^{3x} + e^{4x}$

(b) complementary solution: $y_c = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$ $(D+1)(D-2)$ annihilates $2e^{-x} + 3e^{2x}$
trial solution: $y_p = Ax^3 e^{-x} + Be^{2x}$ general solution: $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + \frac{1}{3}x^3 e^{-x} + \frac{1}{9}e^{2x}$

(c) complementary solution: $y_c = c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{3x} + c_4 x e^{3x} + c_5 e^{-2x}$
$D^3(D^2 - 2D + 2)(D-3)$ annihilates $x^2 - e^x \cos x + 3e^{3x}$
trial solution: $y_p = A + Bx + Cx^2 + Dxe^x \cos x + Exe^x \sin x + Fx^2 e^{3x}$

4. Suppose λ is an eigenvalue of the matrix A, and that $\vec{v}$ is a corresponding eigenvector.

(a) Show that λ is an eigenvalue of the matrix A^T.

(b) Show that λ^2 is an eigenvalue of the matrix A^2, and that $\vec{v}$ is a corresponding eigenvector.

(c) If A is non-singular, show that $1/\lambda$ is an eigenvalue of A^{-1}, and that $\vec{v}$ is a corresponding eigenvector.

(a) Recall that the determinant of a matrix is equal to the determinant of its transpose.
Since $\det[A - \lambda I] = \det[(A - \lambda I)^T] = \det[A^T - (\lambda I)^T] = \det[A^T - \lambda I]$,
the matrices A and A^T have the same eigenvalues.

(b) If $A\vec{v} = \lambda\vec{v}$, then $A^2\vec{v} = A \cdot A\vec{v} = A\lambda\vec{v} = \lambda A\vec{v} = \lambda\lambda\vec{v} = \lambda^2\vec{v}$.
Thus λ^2 is an eigenvalue of A^2 and $\vec{v}$ is a corresponding eigenvector.

(c) Note that if A is non-singular, then the eigenvalues of A are non-zero (why?).
If $A\vec{v} = \lambda\vec{v}$ and A is non-singular, then $\vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$. So $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$.
Thus $1/\lambda$ is an eigenvalue of A^{-1} and $\vec{v}$ is a corresponding eigenvector.

5. Let $X(t) = [\,\vec{x}_1(t)\ \vec{x}_2(t)\ \cdots\ \vec{x}_n(t)\,]$ be a fundamental matrix for the system $\vec{x}\,' = A(t)\,\vec{x}$ on the interval I.

(a) Show that $X'(t) = A(t)\,X(t)$.

(b) Show that the general solution of the system can be written as $\vec{x}(t) = X(t)\,\vec{c}$, where $\vec{c}$ is a constant vector.

(c) If t_0 is in I, show that the solution of the IVP $\vec{x}\,' = A(t)\,\vec{x}$, $\vec{x}(t_0) = \vec{\alpha}$ can be written as $\vec{x}(t) = X(t)X^{-1}(t_0)\vec{\alpha}$.

(a) Each column of $X(t)$ is a solution, so $\vec{x}_1{}' = A\,\vec{x}_1$, $\vec{x}_2{}' = A\,\vec{x}_2$, $\ldots$, $\vec{x}_n{}' = A\,\vec{x}_n$.
Thus $X' = [\,\vec{x}_1{}'\ \vec{x}_2{}'\ \cdots\ \vec{x}_n{}'\,] = [\,A\,\vec{x}_1\ A\,\vec{x}_2\ \cdots\ A\,\vec{x}_n\,] = A\,[\,\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n\,] = A\,X$

(b) The columns of X form a fundamental set of solutions. So the general solution is
$\vec{x}(t) = c_1\,\vec{x}_1(t) + c_2\,\vec{x}_2(t) + \cdots + c_n\,\vec{x}_n(t) = X(t)\,\vec{c}$, where $\vec{c} = [\,c_1\ c_2\ \ldots\ c_n\,]^T$

(c) From part (b), the general solution of the system of DE is $\vec{x}(t) = X(t)\,\vec{c}$.
To solve the IVP, we must find $\vec{c}$ so that $\vec{x}(t_0) = X(t_0)\,\vec{c} = \vec{\alpha}$.
Since the columns of $X(t)$ form a fundamental set of solutions and t_0 is in I, the Wronskian
$W\,[\,\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\,]\,(t_0)$ is different from zero. In other words, $\det X(t_0) \neq 0$. So $X(t_0)$ is non-singular,
and the solution of the system of equations $X(t_0)\,\vec{c} = \vec{\alpha}$ is $\vec{c} = X^{-1}(t_0)\,\vec{\alpha}$.
Thus the solution of the IVP is $\vec{x}(t) = X(t)\,\vec{c} = X(t)X^{-1}(t_0)\,\vec{\alpha}$.