MATH 2090 — Review Problems for Exam 4 — Fall 2001

1. For each of the matrices A below, find a real-valued fundamental set of solutions (explain why your solutions form a fundamental set), a fundamental matrix, and the general solution of the system $\vec{x}\,' = A\,\vec{x}$. Determine if A is defective or diagonalizable in each case. If A is diagonalizable, find a diagonal matrix D and an invertible matrix S so that $S^{-1}AS = D$. Also, solve the initial value problem indicated in part (a).

(a) $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & -5 \\ 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 7 & -4 \\ 4 & -1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix}$

2. The differential equation $x^3y''' + x^2y'' - 2xy' + 2y = 0,\ x > 0$ has three solutions of the form $y(x) = x^r$. Find these solutions, show that they are linearly independent on the interval $(0, \infty)$, and find the general solution of this differential equation.

3. For each of the non-homogeneous linear differential equations $P(D)y = F(x)$ below, find the complementary solution, find the annihilator of the function $F(x)$, and determine the form of a particular solution. In parts (a) and (b), also find the general solution of the differential equation, and solve the initial value problem in part (a).

(a) $(D-1)(D-2)(D-3)y = 6e^{4x},\ y(0) = 4,\ y'(0) = 10,\ y''(0) = 30$

(b) $y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x}$

(c) $(D^2 - 2D + 2)(D-3)^2(D+2)y = x^2 - e^x \cos x + 3e^{3x}$

4. Suppose λ is an eigenvalue of the matrix A, and that $\vec{v}$ is a corresponding eigenvector.

(a) Show that λ is an eigenvalue of the matrix A^T.

(b) Show that λ^2 is an eigenvalue of the matrix A^2, and that $\vec{v}$ is a corresponding eigenvector.

(c) If A is non-singular, show that $1/\lambda$ is an eigenvalue of A^{-1}, and that $\vec{v}$ is a corresponding eigenvector.

5. Let $X(t) = [\,\vec{x}_1(t)\ \vec{x}_2(t)\ \cdots\ \vec{x}_n(t)\,]$ be a fundamental matrix for the system $\vec{x}\,' = A(t)\,\vec{x}$ on the interval I.

(a) Show that $X'(t) = A(t)\,X(t)$.

(b) Show that the general solution of the system can be written as $\vec{x}(t) = X(t)\,\vec{c}$, where $\vec{c}$ is a constant vector.

(c) If t_0 is in I, show that the solution of the IVP $\vec{x}\,' = A(t)\,\vec{x},\ \vec{x}(t_0) = \vec{\alpha}$ can be written as $\vec{x}(t) = X(t)X^{-1}(t_0)\vec{\alpha}$.