1. Recall that $\mathbb{R}^3$ denotes the vector space of all ordered 3-tuples of real numbers.
Consider the vectors $\vec{v}_1 = (1,2,1)$, $\vec{v}_2 = (-1,2,3)$, $\vec{v}_3 = (1,0,-1)$ in $\mathbb{R}^3$.

(a) Can the vector $\vec{v} = (3,4,1)$ be expressed as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$?
YES. The system of equations $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{v}$ is consistent. This system reads

$$k_1 \begin{bmatrix}1\\2\\1\end{bmatrix} + k_2 \begin{bmatrix}-1\\2\\3\end{bmatrix} + k_3 \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\2 & 2 & 0\\1 & 3 & -1\end{bmatrix} = \begin{bmatrix}3\\4\\1\end{bmatrix}.$$

Row reduce the corresponding augmented matrix to see that the system has infinitely many solutions:

$$\begin{bmatrix}1 & -1 & 1 & | & 3\\2 & 2 & 0 & | & 4\\1 & 3 & -1 & | & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1/2 & | & 5/2\\0 & 1 & -1/2 & | & -1/2\\0 & 0 & 0 & | & 0\end{bmatrix}$$

(b) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for $\mathbb{R}^3$? Explain.
NO. The matrix $A = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$ is singular (since it is not row equivalent to the identity matrix, as illustrated above). So the columns of A, i.e., the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, are linearly dependent and do not span $\mathbb{R}^3$. So they do not form a basis.

2.(a) Is the mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (2x_1 - 3x_3, 9x_2 - 4)$ a linear transformation? Explain.
NO. Check that $T(\vec{x} + \vec{y}) \neq T(\vec{x}) + T(\vec{y})$, or that $T(k \cdot \vec{x}) \neq k \cdot T(\vec{x})$.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with $T(1,0) = (1,3,-2)$ and $T(0,1) = (1,0,5)$, find $T(2,3)$.
Since $(2,3) = 2(1,0) + 3(0,1)$ and T is a linear transformation, we have
$T(2,3) = T(2(1,0) + 3(0,1)) = 2T(1,0) + 3T(0,1) = 2(1,3,-2) + 3(1,0,5) = (5,6,11)$.

(c) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is any linear transformation, what is $T(\vec{0})$? Explain.
$T(\vec{0}) = \vec{0}$. For instance, $T(k \cdot \vec{x}) = k \cdot T(\vec{x})$ for any scalar k and any vector $\vec{x}$. If $k = 0$, this reads $T(\vec{0}) = T(0 \cdot \vec{x}) = 0 \cdot T(\vec{x}) = \vec{0}$, since $0 \cdot \vec{v} = \vec{0}$ for any vector $\vec{v}$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ 2x_1 + 2x_2,\ x_1 + 3x_2 - x_3).$$

(a) Find the standard matrix of T and use it to compute $T(\vec{x})$ for $\vec{x} = (-2,4,9)$.

$$A = [T] = \begin{bmatrix}1 & -1 & 1\\2 & 2 & 0\\1 & 3 & -1\end{bmatrix} \qquad T(\vec{x}) = A\vec{x} = \begin{bmatrix}1 & -1 & 1\\2 & 2 & 0\\1 & 3 & -1\end{bmatrix} \begin{bmatrix}-2\\4\\9\end{bmatrix} = \begin{bmatrix}3\\4\\1\end{bmatrix}$$

(b) Determine if the linear operator T is one-to-one.
T is one-to-one if and only if the matrix of T is invertible. Since $A = [T]$ is singular (see problem **1**), T is not one-to-one.

(c) Is the range of T equal to $\mathbb{R}^3$? Explain.
NO. The range of T is equal to $\mathbb{R}^3$ if and only if the matrix of T is invertible, and $A = [T]$ is still singular.

4. Recall that $M_{2\times 2}(\mathbb{R})$ denotes the vector space of all 2×2 matrices with real entries.

Consider the matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$.

(a) Show that the set $\{A_1, A_2, A_3\}$ is linearly independent.
We must show that $k_1A_1 + k_2A_2 + k_3A_3 = 0$ only when $k_1 = k_2 = k_3 = 0$. This matrix equation reads

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} k_1 & k_2+k_3 \\ k_2+k_3 & k_1+k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $k_1 = 0$, $k_2 + k_3 = 0$, and $k_1 + k_3 = 0$. Solving this homogeneous system, we get $k_1 = k_2 = k_3 = 0$.

(b) Explain why the set $\{A_1, A_2, A_3\}$ does not span $M_{2\times 2}(\mathbb{R})$.
From the above calculation, any linear combination of the matrices A_1, A_2, and A_3 is symmetric. Since there are 2×2 matrices that are not symmetric, the set $\{A_1, A_2, A_3\}$ does not span $M_{2\times 2}(\mathbb{R})$.
Alternatively, if the set $\{A_1, A_2, A_3\}$ did span $M_{2\times 2}(\mathbb{R})$, then it would be a basis for $M_{2\times 2}(\mathbb{R})$ (since it is linearly independent by part (a)). But the dimension of $M_{2\times 2}(\mathbb{R})$ is 4, and the set $\{A_1, A_2, A_3\}$ only has 3 elements.

5. Let W be the set of vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ satisfying the condition $x_1 - x_2 - 2x_3 = 0$.
In other words,

$$W = \Big\{(x_1, x_2, x_3) \text{ in } \mathbb{R}^3 \;:\; x_1 - x_2 - 2x_3 = 0\Big\} = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3 \;:\; x_1 - x_2 - 2x_3 = 0\right\}.$$

(a) Show that W is a subspace of $\mathbb{R}^3$.
Note that W is a non-empty set, since, for instance $\vec{0} = (0,0,0)$ is in W. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in W, and let k be a scalar. Then $u_1 - u_2 - 2u_3 = 0$ and $v_1 - v_2 - 2v_3 = 0$. The components of $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ satisfy $(u_1 + v_1) - (u_2 + v_2) - 2(u_3 + v_3) = u_1 - u_2 - 2u_3 + v_1 - v_2 - 2v_3 = 0 + 0 = 0$. So $\vec{u} + \vec{v}$ is in W. The components of the vector $k \cdot \vec{u} = (ku_1, ku_2, ku_3)$ satisfy $ku_1 - ku_2 - 2ku_3 = k(u_1 - u_2 - 2u_3) = k(0) = 0$. So $k \cdot \vec{u}$ is in W. Therefore, W is a non-empty subset of $\mathbb{R}^3$ that is closed under vector addition and scalar multiplication, so W is a subspace of $\mathbb{R}^3$.

(b) Find a basis for W, and determine the dimension of W.
A vector $\vec{x} = (x_1, x_2, x_3)$ in W satisfies $x_1 - x_2 - 2x_3 = 0$. Solving this homogeneous "system", we get $\vec{x} = (s+2t, s, t) = s(1,1,0) + t(2,0,1)$. Consequently, the set $\{(1,1,0), (2,0,1)\}$ spans W, and is clearly linearly independent, so is a basis for W. Thus, $\dim W = 2$.

FALL 2005 MATH 2085-1 EXAM 2 STATISTICS

Score	Number
90	4
80	2
70	3
60	6
50	3
less	5
total	23

average: 64
median: 64