1. [18 points] The matrix $A = \begin{bmatrix} 5 & 3 & 2 & 8 \\ 3 & 2 & 1 & 5 \\ 2 & 1 & 1 & 3 \end{bmatrix}$ is row equivalent to $R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank and nullity of A, and find the dimensions of the row space of A, the column space of A, the nullspace of A, and the nullspace of $A^\top$.

$\text{rank}(A) = 2$ $\dim(\text{column space}(A)) = 2$ $\dim(\text{row space}(A)) = 2$
$\text{nullity}(A) = 2$ $\dim(\text{nullspace}(A)) = 2$ $\dim(\text{nullspace}(A^\top)) = 1$

(b) Find bases for the row space of A and the column space of A.

row space: $\{(1, 0, 1, 1), (0, 1, -1, 1)\}$ column space: $\{(5, 3, 2), (3, 2, 1)\}$

2. [14 points] Let P_2 be the vector space of all polynomials of degree at most 2 with real coefficients. If $\mathbf{p} = p(x) = a_0 + a_1x + a_2x^2$ and $\mathbf{q} = q(x) = b_0 + b_1x + b_2x^2$ are two polynomials in P_2, then $\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + 4a_2b_2$ defines an inner product on P_2.

Let $\mathbf{p} = 2 - x + x^2$ and $\mathbf{q} = 5 + 3x - 2x^2$. Use this inner product to calculate the norms $||\mathbf{p}||$ and $||\mathbf{q}||$, and to find the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$. Are $\mathbf{p}$ and $\mathbf{q}$ orthogonal?

$||\mathbf{p}|| = 3$, $||\mathbf{q}|| = 5\sqrt{2}$, $\cos\theta = \dfrac{\langle \mathbf{p}, \mathbf{q} \rangle}{||\mathbf{p}||\,||\mathbf{q}||} = \dfrac{-1}{15\sqrt{2}}$, $\mathbf{p}$ and $\mathbf{q}$ are not orthogonal since $\langle \mathbf{p}, \mathbf{q} \rangle \neq 0$

3. [14 points] Consider the vectors $\vec{v}_1 = (1, 0, 1, 1)$ and $\vec{v}_2 = (0, 1, -1, 1)$ in $\mathbb{R}^4$ (with the standard Euclidean inner product). Let $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

(a) Find the dimension of the orthogonal complement $W^\perp$ of W.

The subspace W may be realized as the row space of the matrix A (or R) from problem **1**, $W = \text{row space}(A)$. So the orthogonal complement of W is the nullspace of A, which has dimension two, $\dim(W^\perp) = \dim(\text{nullspace}(A)) = 2$.

(b) Let $\vec{u} = (1, 5, 2, 3)$. Find $\text{proj}_W \vec{u}$, the orthogonal projection of the vector $\vec{u}$ on W.

Note that the vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. Consequently,

$$\text{proj}_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 = (2, 2, 0, 4).$$

4. [16 points] Let $\vec{u}_1 = (1, 2, 2)$, $\vec{u}_2 = (-1, 4, 1)$, $\vec{u}_3 = (4, -1, -1)$. Apply the Gram-Schmidt process to $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ to find an orthogonal basis for $\mathbb{R}^3$ (with the Euclidean inner product).

$$\vec{v}_1 = \vec{u}_1 = (1, 2, 2)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 = (-2, 2, -1)$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 = (2, 1, -2)$$

5. [18 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{bmatrix}$.

(a) Find the eigenvalues of the matrix A.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & -2 \\ 0 & \lambda - 2 & -1 \\ 0 & -5 & \lambda + 2 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 2 & -1 \\ -5 & \lambda + 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda + 3)$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = -3$

(b) Can (only) the answer to part (a) be used to determine if the matrix A is invertible? Explain.
YES. If all the eigenvalues of A are nonzero, then A is invertible.

(c) Can the answer to part (a) be used to determine if the matrix A is diagonalizable? Explain.
YES. If all the eigenvalues of A are distinct, then A is diagonalizable.

6. [20 points] The matrix $A = \begin{bmatrix} 2 & 3 & 3 \\ -3 & -4 & -3 \\ 3 & 3 & 2 \end{bmatrix}$ is diagonalizable.

The characteristic polynomial of A is $\det(\lambda I - A) = \lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2)$.

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.
The eigenvalues of A are $\lambda_1 = \lambda_2 = -1$, and $\lambda_3 = 2$. Corresponding eigenvectors:

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(a) Find an invertible matrix P and a diagonal matrix Λ so that $P^{-1}AP = \Lambda$.

$$P = [\,\vec{x}_1\ \vec{x}_2\ \vec{x}_3\,] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) What are the eigenvalues of the matrix A^3? Explain.
If $A\vec{x} = \lambda\vec{x}$, then $A^3\vec{x} = \lambda^3\vec{x}$. So, since the eigenvalues of A are -1 and 2, the eigenvalues of A^3 are $(-1)^3 = -1$ and $2^3 = 8$.

FALL 2005 MATH 2085-1 EXAM 3 STATISTICS

Score	Number
100	1
90	4
80	5
70	7
less	3
total	20

average: 79
median: 80