Exam 2 will take place on Friday, October 28. It will cover the material from Chapters 4 and 5 we've discussed in class (primarily §4.1–4.3 and §5.1–5.4). Books, notes, calculators, etc. may **not** be used on the exam. We can spend some time in class this coming week reviewing as necessary. I will also have my office hours at the usual times (M F 9:00 - 10:00, W 11:30 - 12:30).

The focus of these sections, and this exam, is on vector spaces and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. While I don't anticipate asking you to prove that a given set, together with operations of addition and scalar multiplication, is a vector space (i.e., verify the ten vector space axioms), you should have a good working understanding of what a vector space is, and be prepared to work in this context. In an exam setting, you can assume that sets such as $\mathbb{R}^n$, P_n (polynomials of degree at most n with real coefficients), $M_{m,n}(\mathbb{R})$ ($m \times n$ matrices with real entries), all equipped with the "standard" operations, are vector spaces.

Be comfortable, familiar, and ready to work with the various notions we've developed in the framework of vector spaces. Definitions you should know (well) include the following:

(1) A **subspace** of a vector space is......
(2) If $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ is a set of vectors in a vector space V, then **span**(S) is......
(3) The set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ in the vector space V **spans** V if......
(4) The set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ in the vector space V is **linearly independent** if......
and is **linearly dependent** if......
(5) A **basis** for a vector space V is......
(6) The **dimension** of a finite dimensional vector space V is......
(7) A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if......
A **linear operator** is......
etc.

Types of problems you should be prepared for include the following:

(1) Prove that a given subset of a vector space is a subspace, or show that it is not.
(2) Determine if a set of vectors in a vector space V spans V.
(3) Determine if a set of vectors in a vector space V is linearly dependent or independent.
(4) Determine if a set of vectors in a vector space V is a basis for V.
(5) Find the coordinates of a given vector $\vec{v} \in V$ relative to a basis.
(6) Given a finite dimensional vector space V, exhibit a basis for V and find the dimension of V.
(7) Prove that a given map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, or show that it is not.
(8) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, find the standard matrix of T; determine if T is one-to-one and/or onto. If T is a linear operator, determine if T has an inverse (and find the matrix of the inverse if so).
etc.

I will not ask you about things like inner products, norms, and eigenvalues on this exam.

A few review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, the WeBWorK assignments, and the Supplementary Exercises at the ends of Chapters 4 and 5. The *Discussion & Discovery* problems at the ends of the sections also provide useful materials for reviewing conceptual aspects of this subject matter.

1. Consider the polynomials $p_1 = 1 + 3x + 4x^2$, $p_2 = x + x^2$, $p_3 = -1 - x^2$, $p_4 = 5 + 3x - 2x^2$ in P_2.

(a) Does the set $\{p_1, p_2, p_3, p_4\}$ span P_2?

(b) Determine if this set is linearly independent.

(c) Does this set form a basis for P_2? Explain.

2. Recall that the trace of a square matrix is the sum of the diagonal entries. Let S be the set of all 2×2 matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ a, b, c, d \text{ real, and } a + d = 0 \right\}$.

(a) Show that S is a subspace of $M_{2\times 2}(\mathbb{R})$, the vector space of all 2×2 matrices with real entries.

(b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

(c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).
In other words, express A as a linear combination of the matrices given in part (b).

3. (a) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where k is a scalar.
Find all values of k for which T is a linear transformation.
Find the standard matrix of T for these values of k.

(b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1, 0, 0) = 1$, $T(1, 1, 0) = 3$, $T(1, 1, 1) = 5$, find $T(2, 4, 6)$ and find the standard matrix of T. Let S be the subset of $\mathbb{R}^3$ defined by $S = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = 0\}$. Show that S is a subspace of $\mathbb{R}^3$, find a basis for S, and determine the dimension of S.

4. Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 + 4x_2 - x_3, 2x_1 + 7x_2 + x_3, x_1 + 3x_2).$$

(a) Determine whether this linear operator is one-to-one; if so, find the standard matrix for the inverse operator T^{-1}.

(b) Find all vectors $\vec{x} = (x_1, x_2, x_3)$ for which $T(\vec{x}) = (2, 4, 6)$.

(c) Does the set of vectors $\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)\}$ form a basis for $\mathbb{R}^3$? Explain.

1. Consider the polynomials $p_1 = 1 + 3x + 4x^2$, $p_2 = x + x^2$, $p_3 = -1 - x^2$, $p_4 = 5 + 3x - 2x^2$ in P_2.

(a) Does the set $\{p_1, p_2, p_3, p_4\}$ span P_2?

YES, this set does span P_2. For any "vector" $q = a+bx+cx^2$, one can find scalars k_1, k_2, k_3, k_4 so that $k_1 \cdot p_1 + k_2 \cdot p_2 + k_3 \cdot p_3 + k_4 \cdot p_4 = q$. Equating coefficients of powers of x, one obtains a system of 3 equations in the four unknowns k_1, k_2, k_3, k_4 which is consistent for any a, b, c.

(b) Determine if this set is linearly independent.

This set is LINEARLY DEPENDENT. For instance, $p_1 - 3p_2 + p_3 = 0$. In the case where $a = b = c = 0$, the aforementioned system is a homogeneous system of 3 equations in 4 unknowns, which necessarily has infinitely many solutions.

(c) Does this set form a basis for P_2? Explain.

NO, it is not a linearly independent set.

2. Recall that the trace of a square matrix is the sum of the diagonal entries. Let S be the set of all 2×2 matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ real, and } a + d = 0 \right\}$.

(a) Show that S is a subspace of $M_{2\times 2}(\mathbb{R})$, the vector space of all 2×2 matrices with real entries.

Denote the trace of a square matrix A by $\mathrm{tr}(A)$. It is enough to check that S is closed under "vector" addition and scalar multiplication. If A and B are square matrices and k is a scalar, then it follows from the definitions of matrix operations that $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(k \cdot A) = k \cdot \mathrm{tr}(A)$. Consequently, if $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(B) = 0$, then $\mathrm{tr}(A + B) = 0 + 0 = 0$ and $\mathrm{tr}(k \cdot A) = k \cdot 0 = 0$. So if A and B are in S, then $A + B$ is in S, and $k \cdot A$ is in S.

(b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

Check that

$$k_1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + k_2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + k_3 \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

has a unique solution $k_1 = a + b + c$, $k_2 = -a - b + 2c$, $k_3 = b - c$ for any real numbers a, b, c. So the given set of trace-zero matrices spans S. In the case where $a = b = c = 0$, the unique solution of the above matrix equation is $k_1 = k_2 = k_3 = 0$. So the given set is linearly independent.

(c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).

In other words, express A as a linear combination of the matrices given in part (b).

From part (b), $k_1 = a + b + c = 4$, $k_2 = -2$, $k_3 = 1$.

3. (a) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where k is a scalar. Find all values of k for which T is a linear transformation.

Find the standard matrix of T for these values of k.

T is a linear transformation if $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c \cdot \vec{x}) = c \cdot T(\vec{x})$. This is the case only when $k = 0$. The standard matrix of T is

$$[\,T\,] = [\,T(\vec{e}_1) \vdots T(\vec{e}_1) \vdots T(\vec{e}_1)\,] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. (b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1,0,0) = 1$, $T(1,1,0) = 3$, $T(1,1,1) = 5$, find $T(2,4,6)$ and find the standard matrix of T. Let S be the subset of $\mathbb{R}^3$ defined by $S = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = 0\}$. Show that S is a subspace of $\mathbb{R}^3$, find a basis for S, and determine the dimension of S.

Since $(2,4,6) = 6 \cdot (1,1,1) - 2 \cdot (1,1,0) - 2 \cdot (1,0,0)$, we have

$$T(2,4,6) = 6 \cdot T(1,1,1) - 2 \cdot T(1,1,0) - 2 \cdot T(1,0,0) = 30 - 6 - 2 = 22.$$

Checking that $T(\vec{e}_1) = 1$, $T(\vec{e}_2) = 2$, and $T(\vec{e}_3) = 2$, the standard matrix of T is $[\,T\,] = [\,1\ 2\ 2\,]$.
As in problem **2**. (a), check that $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + 2x_2 + 2x_3 = 0\}$ is closed under vector addition and scalar multiplication to show that it is a subspace. Solving the homogeneous "system" $x_1 + 2x_2 + 2x_3 = 0$ yields the basis $\{(-2,1,0), (-2,0,1)\}$ for S, and $\dim(S) = 2$.

4. Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 + 4x_2 - x_3, 2x_1 + 7x_2 + x_3, x_1 + 3x_2).$$

(a) Determine whether this linear operator is one-to-one; if so, find the standard matrix for the inverse operator T^{-1}.

The standard matrix $A = [\,T\,]$ of T (given below) is invertible. Consequently, T is one-to-one. The standard matrix of T^{-1} is the inverse of the matrix of T, $[\,T^{-1}\,] = [\,T\,]^{-1} = A^{-1}$.

$$A = [\,T\,] = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix} \qquad A^{-1} = [\,T^{-1}\,] = \frac{1}{2} \begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ -1 & 1 & -1 \end{bmatrix}$$

(b) Find all vectors $\vec{x} = (x_1, x_2, x_3)$ for which $T(\vec{x}) = (2,4,6)$.

$\vec{x} = (24, -6, -2)$

(c) Does the set of vectors $\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)\}$ form a basis for $\mathbb{R}^3$? Explain.

YES, since A is invertible, the columns of A form a basis for $\mathbb{R}^3$.