Exam 3 will take place on Friday, December 2. It will cover material from Chapters 5, 6, and 7, primarily §5.5–5.6, §6.1–6.3, and §7.1–7.2. Books, notes, calculators, etc. may **not** be used on the exam. We can spend some time in class this coming week reviewing as necessary. I will also have my office hours at the usual times (M F 9:00 - 10:00, W 11:30 - 12:30).

Sections 5.5 and 5.6 address the "fundamental subspaces" of a matrix, the row space, column space, nullspace, and the nullspace of the transpose, as well as the rank and nullity of a matrix. We developed algorithms for finding the dimensions of these subspaces (using the rank and nullity), and for finding bases for these subspaces. Be prepared to implement these.

Sections 6.1 through 6.3 concern inner products, orthogonality, and related ideas. While I don't anticipate asking you to verify the inner product axioms, you should be familiar and comfortable with them, and related properties of inner products (e.g., what is the Cauchy-Schwarz inequality? the triangle inequality?). Be able to calculate with a given inner product: the inner product of two vectors, norms, the (cosine of the) angle between vectors, etc. Be prepared to work with orthogonality. If W is a subspace of a real inner product space V, what is the orthogonal complement, $W^{\perp}$, of W? Can you prove that $W^{\perp}$ is a subspace of V? find a basis for $W^{\perp}$? etc. In the context of the standard (dot) inner product on Euclidean space, how does this notion relate to the fundamental subspaces of a matrix?

Given a linear independent set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ in a real inner product space V, you should be able to use the Gram-Schmidt process to produce an orthogonal set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and an orthonormal set of vectors $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ with

$$\text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}.$$

If W is a subspace of a real inner product space V, and $\vec{u}$ is a vector in V, you should be able to find the orthogonal projection, $\text{proj}_W \vec{u}$, of $\vec{u}$ on W (the vector in W that is closest to $\vec{u}$). Having an orthogonal or orthonormal basis for W is very useful for problems such as this. Explain why. (Note that we discussed elements of §6.4 in this context.)

Sections 7.1 and 7.2 deal with the eigenvalue problem. Given a square matrix A, you should be able to find the characteristic polynomial, the eigenvalues, and associated eigenvectors. You should also be able to determine if A is diagonalizable, and, if so, construct an invertible matrix P and a diagonal matrix Λ so that $P^{-1}AP = \Lambda$.

As usual, and to reiterate, be comfortable, familiar, and ready to work with the notions we've developed in the various contexts above. Also be prepared to interpret and discuss the implications of a given calculation. For instance, what conclusions can you draw if the rank of a 4×4 matrix A is 4? less than 4? What if the columns of A form an orthogonal set? What if 0 is an eigenvalue of A? etc.

A few review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, the WeBWorK assignments, and the Supplementary Exercises at the ends of Chapters 5, 6, and 7. The *Discussion & Discovery* problems at the ends of the sections also provide useful materials for reviewing conceptual aspects of this subject matter.

1. Let $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$.

(a) Find bases for the row space, column space, nullspace, and "left nullspace" (the nullspace of the transpose) for the matrix A. If you want extra practice with the Gram-Schmidt process, find orthonormal bases for these subspaces.

(b) What is the relationship between the row space and nullspace of A? Explain.

2. Let P_2 be the vector space of all polynomials of degree at most two with real coefficients.

(a) For $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$, define $\langle \mathbf{p}, \mathbf{q} \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Show that this is an inner product on P_2.

(b) For $\mathbf{p} = 1 + x + x^2$ and $\mathbf{q} = 1 + 2x + x^2$, use the inner product from part (a) to compute the norms $||\mathbf{p}||$ and $||\mathbf{q}||$, and to find the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$. Is this angle acute or obtuse?

(c) For $\mathbf{p} = 1 + x + x^2$ and $\mathbf{q} = 1 + 2x + x^2$ and the inner product from part (a), verify the Cauchy-Schwarz inequality and the triangle inequality.

3. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{u}_1 = (1, 1, 1, -1)$, $\vec{u}_2 = (2, -1, -1, 1)$, $\vec{u}_3 = (-1, 2, 2, 1)$, and let $\vec{u} = (1, 2, 6, 4)$.

(a) Use the Gram-Schmidt process to find orthogonal and orthonormal bases for W.

(b) Verify that the vector $\vec{u}$ is not in the subspace W, and find $\text{proj}_W \vec{u}$, the orthogonal projection of $\vec{u}$ on W. Also find the (smallest) distance between $\vec{u}$ and W.

(c) Show that $W^\perp$, the orthogonal complement of W, is a subspace of $\mathbb{R}^4$ and find a basis for $W^\perp$.

4. For each of the following matrices, find the characteristic polynomial, the eigenvalues, and bases for the corresponding eigenspaces. Which of these matrices is diagonalizable?

(a) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 1 & 2 \\ -1 & 7 & 0 \\ 1 & -1 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

5. Let $\vec{x}_1 = (0, 3, -2)$, $\vec{x}_2 = (-3, -3, 0)$, and $\vec{x}_3 = (-2, 0, 3)$ be eigenvectors of the matrix A which correspond to the eigenvalues $\lambda_1 = -2$, $\lambda_2 = 0$, and $\lambda_3 = 4$ respectively.

(a) Express the vector $\vec{x} = (-10, -3, 4)$ as a linear combination of $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, and find $A\vec{x}$.

(b) Find the dimension of the nullspace of A.

(c) Find the eigenvalues and corresponding eigenvectors of the matrix A^3.

(d) Find the matrix A.

1. Let $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$. This matrix is row equivalent to $R = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find bases for the row space, column space, nullspace, and "left nullspace" (the nullspace of the transpose) for the matrix A. If you want extra practice with the Gram-Schmidt process, find orthonormal bases for these subspaces.

basis for the row space: $\{(1,0,1,2,1),(0,1,1,1,2)\}$
the nonzero rows of R form a basis for the row space
orthonormal basis: $\{\frac{1}{\sqrt{7}}(1,0,1,2,1), \frac{1}{2\sqrt{42}}(-5,7,2,-3,9)\}$

basis for the column space: $\{(1,3,2),(4,-2,3)\}$
columns of A corresponding to leading one columns of R form a basis for the column space
orthonormal basis: $\{\frac{1}{\sqrt{14}}(1,3,2), \frac{1}{\sqrt{1365}}(26,-20,17)\}$

basis for the nullspace: $\{(-1,-2,0,0,1),(-2,-1,0,1,0),(-1,-1,1,0,0)\}$
solve $A\vec{x} = \vec{0}$ as usual
orthonormal basis: $\{\frac{1}{\sqrt{6}}(-1,-2,0,0,1), \frac{1}{\sqrt{30}}(-4,1,0,3,-2), \frac{1}{2\sqrt{30}}(-1,-1,10,-3,-3)\}$

basis for the left nullspace: $\{(-13,-5,14)\}$
solve $A^{\top}\vec{y} = \vec{0}$
orthonormal basis: $\{\frac{1}{\sqrt{390}}(-13,-5,14)\}$

(b) What is the relationship between the row space and nullspace of A? Explain.

The nullspace of A is the orthogonal complement of the row space of A (with respect to the standard Euclidean inner product). A vector in the nullspace of A is orthogonal to every row of A, so is orthogonal to any linear combination of rows of A, so is in the orthogonal complement of the row space of A. Conversely, a vector that is orthogonal to the row space of A is orthogonal to every row of A, so is in the nullspace of A.

2. Let P_2 be the vector space of all polynomials of degree at most two with real coefficients.

(a) For $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$, define $\langle \mathbf{p}, \mathbf{q} \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.
Show that this is an inner product on P_2.

$$\begin{aligned}\langle \mathbf{p}, \mathbf{q} \rangle &= p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\ &= q(-1)p(-1) + q(0)p(0) + q(1)p(1) = \langle \mathbf{q}, \mathbf{p} \rangle\end{aligned}$$

$$\begin{aligned}\langle \mathbf{p}+\mathbf{q}, \mathbf{r} \rangle &= (p(-1)+q(-1))r(-1) + (p(0)+q(0))r(0) + (p(1)+q(1))r(1) \\ &= p(-1)r(-1) + p(0)r(0) + p(1)r(1) + q(-1)r(-1) + q(0)r(0) + q(1)r(1) \\ &= \langle \mathbf{p}, \mathbf{r} \rangle + \langle \mathbf{q}, \mathbf{r} \rangle\end{aligned}$$

$$\begin{aligned}\langle k\mathbf{p}, \mathbf{q} \rangle &= kp(-1)q(-1) + kp(0)q(0) + kp(1)q(1) \\ &= k(p(-1)q(-1) + p(0)q(0) + p(1)q(1)) = k\langle \mathbf{q}, \mathbf{p} \rangle\end{aligned}$$

$$\langle \mathbf{p}, \mathbf{p} \rangle = (p(-1))^2 + (p(0))^2 + (p(1))^2 \geq 0$$

If $\mathbf{p} = p(x) = a + bx + cx^2$ and $\langle \mathbf{p}, \mathbf{p} \rangle = 0$, then $p(-1) = a - b + c = 0$, $p(0) = a = 0$, and $p(1) = a + b + c = 0$. Solve this homogeneous system to see that $a = b = c = 0$, i.e., $\mathbf{p} = p(x) = 0 + 0x + 0x^2$.

(b) For $\mathbf{p} = 1 + x + x^2$ and $\mathbf{q} = 1 + 2x + x^2$, use the inner product from part (a) to compute the norms $||\mathbf{p}||$ and $||\mathbf{q}||$, and to find the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$. Is this angle acute or obtuse?

$$||\mathbf{p}|| = \sqrt{11} \qquad ||\mathbf{q}|| = \sqrt{17} \qquad \cos\theta = \frac{\langle \mathbf{p}, \mathbf{q} \rangle}{||\mathbf{p}||\,||\mathbf{q}||} = \frac{13}{\sqrt{11}\sqrt{17}} > 0 \qquad \theta \text{ is acute}$$

(c) For $\mathbf{p} = 1 + x + x^2$ and $\mathbf{q} = 1 + 2x + x^2$ and the inner product from part (a), verify the Cauchy-Schwarz inequality and the triangle inequality.

Cauchy-Schwarz: $|\langle \mathbf{p}, \mathbf{q} \rangle| \le ||\mathbf{p}||\,||\mathbf{q}||$ says $13 \le \sqrt{11}\sqrt{17}$

triangle: $||\mathbf{p} + \mathbf{q}|| \le ||\mathbf{p}|| + ||\mathbf{q}||$ says $\sqrt{54} \le \sqrt{11} + \sqrt{17}$

3. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{u}_1 = (1, 1, 1, -1)$, $\vec{u}_2 = (2, -1, -1, 1)$, $\vec{u}_3 = (-1, 2, 2, 1)$, and let $\vec{u} = (1, 2, 6, 4)$.

(a) Use the Gram-Schmidt process to find orthogonal and orthonormal bases for W.

orthogonal basis: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\vec{v}_1 = \vec{u}_1 = (1, 1, 1, -1)$, $\quad \vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 = (\frac{9}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4})$

$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 = (0, 1, 1, 2)$

orthonormal basis: $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ $\qquad \vec{q}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$,

$\vec{q}_2 = \frac{1}{||\vec{v}_2||}\vec{v}_2 = (\frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$, $\qquad \vec{q}_3 = \frac{1}{||\vec{v}_3||}\vec{v}_3 = (0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$

(b) Verify that the vector $\vec{u}$ is not in the subspace W, and find $\text{proj}_W \vec{u}$, the orthogonal projection of $\vec{u}$ on W. Also find the (smallest) distance between $\vec{u}$ and W.

Check that the system of equations $k_1\vec{u}_1 + k_2\vec{u}_2 + k_3\vec{u}_3 = \vec{u}$ is inconsistent to verify that $\vec{u}$ is not in the subspace $W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

One way to find the orthogonal projection of $\vec{u}$ on W is by using the orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ for W:

$\text{proj}_W \vec{u} = \langle \vec{u}, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}, \vec{q}_2 \rangle \vec{q}_2 + \langle \vec{u}, \vec{q}_3 \rangle \vec{q}_3 = (1, 4, 4, 4)$.

You could also use the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for W, or the original basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for W. For instance, for the last one, build $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$. Then $\text{proj}_W \vec{u} = A(A^\top A)^{-1}A^\top \vec{u}$.

The smallest distance between $\vec{u}$ and W is $||\vec{u} - \text{proj}_W \vec{u}|| = 2\sqrt{2}$.

(c) Show that $W^\perp$, the orthogonal complement of W, is a subspace of $\mathbb{R}^4$ and find a basis for $W^\perp$.

To show that $W^\perp$ is a subspace of $\mathbb{R}^4$, check that $W^\perp$ is closed under vector addition and scalar multiplication (and is nonempty) as usual.

A basis for $W^\perp$ is $\{(0, -1, 1, 0)\}$. One way to see this is to realize W as the column space of the matrix $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ mentioned above. Then $W^\perp$ is the nullspace of $A^\top$.

4. For each of the following matrices, find the characteristic polynomial, the eigenvalues, and bases for the corresponding eigenspaces. Which of these matrices is diagonalizable?

(a) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ characteristic polynomial: $\lambda^3 - 8\lambda^2 + 19\lambda - 12$ eigenvalues: $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

This matrix is diagonalizable.

(b) $\begin{bmatrix} 7 & 1 & 2 \\ -1 & 7 & 0 \\ 1 & -1 & 6 \end{bmatrix}$ characteristic polynomial: $\lambda^3 - 20\lambda^2 + 132\lambda - 288$ eigenvalues: $\lambda_1 = 8, \lambda_2 = \lambda_3 = 6$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

This matrix is not diagonalizable.

(c) $\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix}$ characteristic polynomial: $\lambda^3 - 2\lambda^2 - 20\lambda - 24$ eigenvalues: $\lambda_1 = 6, \lambda_2 = \lambda_3 = -2$ $\quad \vec{x}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

This matrix is diagonalizable.

(d) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ characteristic polynomial: $(\lambda-2)(\lambda-3)(\lambda-4)^2$ e-values: $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = \lambda_4 = 4$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

This matrix is not diagonalizable.

5. Let $\vec{x}_1 = (0, 3, -2)$, $\vec{x}_2 = (-3, -3, 0)$, and $\vec{x}_3 = (-2, 0, 3)$ be eigenvectors of the matrix A which correspond to the eigenvalues $\lambda_1 = -2$, $\lambda_2 = 0$, and $\lambda_3 = 4$ respectively.

(a) Express the vector $\vec{x} = (-10, -3, 4)$ as a linear combination of $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, and find $A\vec{x}$.

$\vec{x} = \vec{x}_1 + 2\vec{x}_2 + 2\vec{x}_3 \quad A\vec{x} = A\vec{x}_1 + 2A\vec{x}_2 + 2A\vec{x}_3 = -2\vec{x}_1 + 2 \cdot 0\vec{x}_2 + 2 \cdot 4\vec{x}_3 = (-16, -6, 28)$

(b) Find the dimension of the nullspace of A.

The nullspace of A is the eigenspace associated to the eigenvalue $\lambda_2 = 0$ which has dimension 1.

(c) Find the eigenvalues and corresponding eigenvectors of the matrix A^3.

If $A\vec{x} = \lambda\vec{x}$, then $A^3\vec{x} = \lambda^3\vec{x}$. So the eigenvalues of A^3 are $(-2)^3 = -8$, $0^3 = 0$, and $4^3 = 64$. We can take $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ as associated eigenvectors.

(d) Find the matrix A.

The matrix A is diagonalizable (e.g., it is implicit that A is 3×3, and A has 3 distinct eigenvalues). So there is an invertible matrix P and a diagonal matrix Λ so that $P^{-1}AP = \Lambda$, or $A = P\Lambda P^{-1}$. Use the given eigenvalues and eigenvectors to build Λ and P:

$$P = [\vec{x}_1\ \vec{x}_2\ \vec{x}_3] = \begin{bmatrix} 0 & -3 & -2 \\ 3 & -3 & 0 \\ -2 & 0 & 3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

$$A = P\Lambda P^{-1} = \begin{bmatrix} \frac{16}{13} & -\frac{16}{13} & -\frac{24}{13} \\ \frac{18}{13} & -\frac{18}{13} & \frac{12}{13} \\ -\frac{36}{13} & \frac{36}{13} & \frac{28}{13} \end{bmatrix}$$