1. Find a number $\delta > 0$ small enough so that $|a-b| < \delta$ and $|b-c| < \delta$ implies $|a-c| < \frac{1}{5}$.

   Since $|a-c| \leq |a-b| + |b-c|$, if $|a-b| < \delta = \frac{1}{10}$ and $|b-c| < \delta = \frac{1}{10}$, then $|a-c| < \delta = \frac{1}{5}$.

2. Give examples of divergent sequences x_n and y_n such that $x_n y_n$ converges.

   For instance, if $x_n = (-1)^n$ and $y_n = (-1)^n$, then x_n and y_n are divergent sequences, but $x_n y_n = (-1)^{2n} = 1$ is a constant convergent sequence, so is convergent.

3. Give an example of a subset $E \subset \mathbb{R}$ for which E is neither open nor closed.

   One example is the interval $E = [a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$. E is not open since for every $\epsilon > 0$, the open interval $(a-\epsilon, a+\epsilon)$ is *not* contained in E. Similarly, E is not closed since its complement $\mathbb{R} \setminus E = (-\infty, a) \cup [b, \infty)$ is not open.

4. Consider the sequence $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

   This sequence is of the form $0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, -\frac{6}{7} \dots$
   $\limsup x_n = \inf \sup T_n = \inf\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \dots\} = 1$
   $\liminf x_n = \sup \inf T_n = \sup\{-1, -1, -1, \dots\} = -1$

5. The sequence x_n begins as follows: $0, 1, \frac{3}{2}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{13}{4}, \frac{14}{4}, \frac{15}{4}, 4, \dots$ and continues according to the same pattern. Find $\lim\limits_{n\to\infty} |x_{n+1} - x_n|$ and determine if x_n is a Cauchy sequence.

   The sequence $x_{n+1} - x_n$ looks like $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots$, and $\lim\limits_{n\to\infty} |x_{n+1} - x_n| = 0$.

   The original sequence x_n is *not* Cauchy. For instance, it is not bounded.

6. Find an open cover of the interval $(0,2)$ that has no finite subcover.

   If $O_n = (\frac{1}{n}, 2)$, check that $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ is an open cover of $(0,2)$. For a finite subset $\{O_{n_1}, \dots, O_{n_k}\}$ of $\mathcal{O}$, if $N > \max\{n_1, \dots, n_k\}$, then $\frac{1}{N} \in (0,2)$, but $\frac{1}{N} \notin O_{n_1} \cup \cdots \cup O_{n_k}$.

7. Let I be the set of irrational numbers. Is I a countable set?

   The set of irrational numbers is *not* countable. Since the set $\mathbb{Q}$ of rational numbers is countable, if I was a countable set, it would follow that $\mathbb{R} = \mathbb{Q} \cup I$ is also countable. But we know that $\mathbb{R}$ is not countable.

A. Prove: If $s_n \leq t_n \leq u_n$ for all n, and if both $s_n \to L$ and $u_n \to L$, then $t_n \to L$ as $n \to \infty$. (Here $L \in \mathbb{R}$ is a real number. This is sometimes called the *squeeze theorem* for sequences.)

   Given $\epsilon > 0$, we must find $N \in \mathbb{N}$ so that $n \geq N \implies |t_n - L| < \epsilon$. Since $s_n \to L$ and $u_n \to L$, we can find $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ so that $n \geq N_1 \implies |s_n - L| < \frac{\epsilon}{3}$ and $n \geq N_1 \implies |u_n - L| < \frac{\epsilon}{3}$. Let $N = \max\{N_1, N_2\}$. For $n \geq N$, we have

$$|t_n - L| \leq |t_n - s_n| + |s_n - L| \leq |u_n - s_n| + |s_n - L| \leq |u_n - L| + |L - s_n| + |s_n - L| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

B. Suppose that the sequence x_n has no convergent subsequences. Let $M > 0$. Prove that there are at most finitely many values of n such that $x_n \in [-M, M]$. Explain why this implies that $|x_n| \to \infty$ as $n \to \infty$.

For a contradiction, assume that there are infinitely many values of n such that $x_n \in [-M, M]$. Then the sequence x_n has a bounded subsequence. By the Bolzano-Weierstrass theorem, this subsequence has a convergent subsequence, so the original sequence x_n has a convergent subsequence. This contradicts the hypothesis.
For each $M > 0$, there are infinitely many values of n so that $|x_n| > M$. Thus, $\exists N \in \mathbb{N}$ so that $n \geq N \implies |x_n| > M$.

C. Prove that every open interval $(a, b) \subset \mathbb{R}$, with $b - a > 0$, must contain a rational number.

Let $x \in (a, b)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence x_n in $\mathbb{Q}$ converging to x. Find $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \subset (a, b)$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ so that $n \geq N \implies |x_n - x| < \epsilon$. So for $n \geq N$, we have $x_n \in (x - \epsilon, x + \epsilon)$, so $x_n \in (a, b)$.

D. Prove that the set $\{\frac{m}{2^n} \mid m \in \mathbb{Z},\, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

Let $S = \{\frac{m}{2^n} \mid m \in \mathbb{Z},\, n \in \mathbb{N}\}$. Let $x \in \mathbb{R}$. There is an integer m so that $m \leq x \leq m + 1$. Let $a_1 = m$ and $b_1 = m + 1$, and note that $a_1, b_1 \in S$. Let $c = (a_1 + b_1)/2$. If $c > x$, let $a_2 = a_1$ and $b_2 = c$. Otherwise, let $a_2 = c$ and $b_2 = b_1$. In either case, we get a closed interval $[a_2, b_2]$ with $a_2 \leq x \leq b_2$ and $a_2, b_2 \in S$. Continue interval halving to obtain a sequence of nested intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_k, b_k] \supset \ldots$$

which satisfy $b_k - a_k \to 0$ as $k \to \infty$, $x \in [a_k, b_k]$ for each k, and $a_k, b_k \in S$ for each k (this may be proved, for instance, by induction on k). By the Nested Intervals theorem, there is a unique point $c \in \bigcap_{k=1}^{\infty} [a_k, b_k]$, and $a_k \to c$, $b_k \to c$ as $k \to \infty$. Since $x \in [a_k, b_k]$ for each k, we have $c = x$, and, for instance, a_k is a sequence in S that converges to x. So S is dense in $\mathbb{R}$.

STATISTICS:

Score	Number
90	8
80	3
70	2
60	3
50	3
less	3
total	22

average: 73
median: 78