

1. Find $\|f\|_{\text{sup}}$ if $f(x) = x^3$ on $(-2, 1)$.

$$\|f\|_{\text{sup}} = \sup\{|f(x)| \mid x \in (-2, 1)\} = \sup\{|x^3| \mid -2 < x < 1\} = 8$$

2. Find all the cluster points of the set \mathbb{Q} of all rational numbers.

a is a cluster point for \mathbb{Q} if $\forall \delta > 0 \exists q \in \mathbb{Q}$ so that $0 < |q - a| < \delta$. Since \mathbb{Q} is dense in \mathbb{R} , every $a \in \mathbb{R}$ is a cluster point for \mathbb{Q} . So the set of all cluster points of \mathbb{Q} is \mathbb{R} .

3. Let P be the set of all functions f on $[1, 3]$ such that $\int_1^3 f(x) dx \geq 0$. Is P a vector space?

No. This set is not closed under scalar multiplication. For example, $f(x) = 1$ is in P since $\int_1^3 f(x) dx = 2 > 0$. But $(-1) \cdot f(x) = -1$ is not in P since $\int_1^3 (-1) \cdot f(x) dx = -2 < 0$.

4. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \in \mathbb{Q}, \\ x^2 - 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all points p at which f is continuous.

Let $p \in \mathbb{R}$, let $x_n \rightarrow p$ be a sequence in \mathbb{Q} converging to p , and let $y_n \rightarrow p$ be a sequence in $\mathbb{R} \setminus \mathbb{Q}$ converging to p . Then $f(x_n) \rightarrow 1 - p^2$ and $f(y_n) \rightarrow p^2 - 1$ as $n \rightarrow \infty$. So f is continuous at p if and only if $1 - p^2 = p^2 - 1$, i.e., $p = 1$ or $p = -1$.

5. Let $f_n(x) = 1 - x^n$ on $[0, 1]$. Find the pointwise limit of f_n , and determine whether or not f_n converges uniformly on $[0, 1]$.

If $0 \leq x < 1$, then $f_n(x) \rightarrow 1$ as $n \rightarrow \infty$. If $x = 1$, then $f_n(1) = 0 \rightarrow 0$ as $n \rightarrow \infty$. So

$$f_n \rightarrow f \text{ pointwise, where } f(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & x = 1. \end{cases}$$

Since f is not continuous on $[0, 1]$, f_n does not converge uniformly on $[0, 1]$.

6. Let $f \in \mathcal{C}[0, 1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Is there a real number $c \in [0, 1]$ such that $f(c) = \sqrt{c}$?

Note that $\sqrt{0} = 0$ and $\sqrt{1} = 1$. So if $f(0) = 0$ or $f(1) = 1$, there is a $c \in [0, 1]$ with $f(c) = \sqrt{c}$.

If $f(0) > 0$ and $f(1) < 1$, let $g(x) = f(x) - \sqrt{x}$. Then $g(0) > 0$ and $g(1) < 0$. Since g is continuous on $[0, 1]$ (can you prove this?), by the Intermediate Value Theorem, there is a $c \in [0, 1]$ for which $g(c) = 0$, i.e., $f(c) = \sqrt{c}$.

7. If f is a bounded function on $[0, 1]$ with $|f| \in \mathcal{R}[0, 1]$, must it be the case that $f \in \mathcal{R}[0, 1]$?

No. For example, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ -1 & x \in [0, 1] \setminus \mathbb{Q} \cap [0, 1] \end{cases}$ is not in $\mathcal{R}[0, 1]$ (prove this claim),

but $|f(x)| = 1$ is in $\mathcal{R}[0, 1]$. Another such example is given in problem #4.

- A. Suppose that f is continuous at p and that $f(p) < c$. Prove: there exists $\delta > 0$ such that $x \in D_f \cap (p - \delta, p + \delta)$ implies $f(x) < c$. (Recall that D_f denotes the domain of f .)

Since f is continuous at p , $\forall \epsilon > 0$, $\exists \delta > 0$ so that $x \in D_f \cap (p - \delta, p + \delta)$ implies $f(x) \in (f(p) - \epsilon, f(p) + \epsilon)$. Let $\epsilon = c - f(p)$. (This is positive since $f(p) < c$.) Then, $\exists \delta > 0$ so that $x \in D_f \cap (p - \delta, p + \delta)$ implies $f(x) \in (f(p) - (c - f(p)), f(p) + (c - f(p)))$. In particular $f(x) < c$.

- B. Let $f(x) = \frac{1}{x}$. Prove that f is uniformly continuous on $(1, \infty)$, but is not uniformly continuous on $(0, \infty)$.

f is uniformly continuous on $(1, \infty)$ if $\forall \epsilon > 0$, $\exists \delta > 0$ so that if $x_1, x_2 \in (1, \infty)$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. Note that if $x \in (1, \infty)$, then $\frac{1}{x} < 1$. Given $\epsilon > 0$, let $\delta = \epsilon$. Then if $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{|x_1 x_2|} < |x_1 - x_2| < \delta = \epsilon$.

f is not uniformly continuous on $(0, \infty)$: Take $\epsilon = 1$, and for any $\delta > 0$ find $n \in \mathbb{N}$ so that $\frac{1}{n} < \delta$. Let $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+1}$. Then $|x_1 - x_2| < \frac{1}{n} < \delta$, but $|f(x_1) - f(x_2)| = 1 \geq \epsilon$.

- C. Let $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial of even degree. Prove that p has a minimum value on \mathbb{R} .

Write $p(x) = x^{2n} \cdot g(x)$, where $g(x) = 1 + \frac{a_{2n-1}}{x} + \cdots + \frac{a_2}{x^{2n-2}} + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}}$. Note that $g(x) \rightarrow 1$ and $x^{2n} \rightarrow \infty$ as $|x| \rightarrow \infty$. Consequently, we can find a real number $M > 0$ so that if $|x| > M$, then $g(x) > \frac{1}{2}$ and $x^{2n} > 2|a_0|$. For $|x| > M$, we then have $p(x) > |a_0| \geq a_0 = p(0)$.

Since p is continuous on the closed interval $[-M, M]$, the Extreme Value Theorem implies that p has a minimum on this interval, say at $x_0 \in [-M, M]$. So for $x \in [-M, M]$, we have $p(x_0) \leq p(x)$, in particular, $p(x_0) \leq p(0) = a_0$. Since $p(x) > a_0 = p(0)$ for $|x| > M$, $p(x_0)$ is the minimum value of p on all of \mathbb{R} .

- D. If f and g are in $\mathcal{R}[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, prove that $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.

Let $h(x) = g(x) - f(x)$. Then $h \in \mathcal{R}[a, b]$ and $h(x) \geq 0$ for $x \in [a, b]$. Let P be a partition of $[a, b]$, and $\{\bar{x}_i\}$ a collection of evaluation points. Then $P(h, \{\bar{x}_i\}) = \sum_{i=1}^n h(\bar{x}_i)\Delta x_i \geq 0$ since $h(\bar{x}_i) \geq 0$ and $\Delta x_i > 0$ for each i . Thus, 0 is a lower bound for any Riemann sum $P(h, \{\bar{x}_i\})$, and $0 \leq \int_a^b h(x) dx = \int_a^b g(x) dx - \int_a^b f(x) dx$. So $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.

STATISTICS:

Score	Number
90	3
80	3
70	3
60	2
50	4
less	4
total	19

average: 67
median: 64