

The final exam will take place on Thursday, December 8, 7:30–9:30 am in our usual classroom (138 Lockett). The final will be comprehensive. I anticipate being available for questions in my office pretty much every morning of finals week. As you know by now, I am also willing to (attempt to) answer questions by email at essentially any time. You may also make use of the free tutoring available in room 141 Middleton Library (the hours during the semester were: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available, or what the hours during finals week will be. As usual, books, notes, calculators, etc. may not be used on the exam. Some remarks and problems concerning the material covered since the second in-class exam are included below.

In addition to the material covered on the in-class exams, the final will also cover material from sections 16.3 through 16.5 and Chapter 17. Some potentially relevant problems from the Chapter Review Exercises are:

Ch. 16 (p. 1001) #9, 13, 15, 23, 25, 27, 37, 41, 43 Ch. 17 (p. 1043) #1 - 5 odd, 9, 10, 18, 19, 22, 23

This is **not** a comprehensive list. See the WeBWorK assignments and the section Exercises for more problems.

Chapter 16. The focus of this chapter is on the calculus of vector fields, including line and surface integrals. (See the Exam 2 review for additional remarks.)

§16.3 This section addresses a number of conceptual aspects of line integrals. These include the fundamental theorem for line integrals involving gradient vector fields: Briefly, if $\mathbf{F} = \nabla\phi$ is a conservative vector field that is continuous on a neighborhood of a curve C with initial point P and terminal point Q , then $\int_C \mathbf{F} \cdot d\mathbf{s} = \phi(Q) - \phi(P)$. The notion of independence of path is introduced in this section, and conditions (on the vector field \mathbf{F} and the region D on which it is defined) which insure that that \mathbf{F} is independent of path in D (resp., conservative on D) are discussed. I will expect you to understand these notions and conditions; to be able to draw conclusions from them; to be able to find a potential function for a conservative vector field; to understand when the fundamental theorem may be used to evaluate line integrals, etc.

§16.4 and §16.5 These sections introduce surface integrals of functions and vector fields over parameterized surfaces, see page 982 and page 992. You should be prepared to find parameterizations of surfaces that are graphs of functions, that are naturally expressed in cylindrical or spherical coordinates, etc., to determine the normal vector from the parameterization, and to compute surface integrals using these.

Chapter 17. The focus of this chapter is on the three main theorems of vector calculus, Green's Theorem, Stokes' Theorem, and the Divergence Theorem, which may be viewed as generalizations of the Fundamental Theorem of Calculus.

§17.1 Green's Theorem is discussed in this section. I will expect you to thoroughly know the statement(s) of this theorem (the precise conditions when it applies, and the conclusion), and be able to use this theorem in various ways. The punchlines of this theorem are: $\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ and

$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl}(\mathbf{F}) \cdot \mathbf{k} dA$ (you should state the necessary conditions on P , Q , ∂D , D , and \mathbf{F} , as well as identify $\text{curl}(\mathbf{F}) \cdot \mathbf{k} = \text{curl}_z(\mathbf{F})$). For instance, the first of these versions may be used to trade a line integral for double integral (and visa versa), to compute area using line integrals, etc.

§17.2 Stokes' Theorem is discussed in this section. See page 1019 for the definition of the curl of a vector field, and see page 1020 for the precise statement of the theorem. The punchline is $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

§17.3 The Divergence Theorem is discussed here. See page 1031 for the definition of the divergence of a vector field, and see page 1032 for the precise statement of the theorem. The punchline is $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div}(\mathbf{F}) dV$.

Remarks. Regarding sections 17.2 and 17.3, you should be able to compute the curl and divergence of a vector field. I will not hold you responsible for the precise statements of Stokes' Theorem (§17.2) or the Divergence Theorem (§17.3) on the final. These are, however, important results, and I am pleased that we were able to briefly discuss them. Also, problems from these sections and the corresponding bonus WeBWorK assignment are very useful for review, as they address line integrals, surface integrals, and triple integrals (sometimes two of these types of integrals at a time, e.g., "Verify that Stokes' Theorem is true for..." and "Verify that the Divergence Theorem is true for..."). Some of the Chapter 17 Review Exercises above are of this nature.

As usual, I will expect you to know the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

| | | | | | |
|---------------|---|--------------|--------------|--------------|---------|
| θ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
| $\sin \theta$ | 0 | 1/2 | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1 |
| $\cos \theta$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | 1/2 | 0 |

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponentials and logarithms.

The in-class exams (including answers) and corresponding reviews are posted. All WeBWorK assignments will remain open through the final.

Review Problems.

- Let \mathcal{S} be the portion of the cone $z = \sqrt{x^2 + y^2}$ which lies below the plane $z = 3$.
 - If $f(x, y, z) = x^2 z^2$, evaluate the surface integral $\iint_{\mathcal{S}} f(x, y, z) dS$.
 - If $\mathbf{F} = \langle x, y, z^4 \rangle$ and S is oriented down, evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$.
- Consider the vector field $\mathbf{F} = \langle yz(2x + y), xz(x + 2y), xy(x + y) \rangle$ defined on all of \mathbb{R}^3 .
 - Calculate the curl and divergence of \mathbf{F} .
 - What does " \mathbf{F} is conservative" mean? Is the vector field \mathbf{F} given above conservative (on \mathbb{R}^3)?
 - What does " $\int_C \mathbf{F} \bullet d\mathbf{s}$ is independent of path" mean? Are line integrals involving the vector field \mathbf{F} given above independent of path (on \mathbb{R}^3)?
 - Evaluate $\int_C \mathbf{F} \bullet d\mathbf{s}$ for the curve C given by $\mathbf{c}(t) = \langle 1 + t, 1 + 2t^2, 1 + 3t^2 \rangle$, $0 \leq t \leq 1$.
- State Green's Theorem.
 - Use Green's Theorem to find the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$, and the area inside the hypocycloid parameterized by $x = a \cos^3 t$, $y = a \sin^3 t$, $0 \leq t \leq 2\pi$.
Evaluate the following in two ways: (i) directly; and (ii) using Green's Theorem
 - $\oint_C x^2 y dx + x dy$, where C is the triangle with vertices $(0, 0)$, $(1, 2)$, $(0, 2)$ oriented counterclockwise.
 - $\oint_C x^2 y dx + x dy$, where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.
- Let \mathcal{S} be the portion of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, inside the cylinder $x^2 + y^2 = 1$, oriented up, and let $\mathbf{F} = \langle xz, yz, xy \rangle$. Verify Stokes' Theorem for this oriented surface and this vector field.
- Let \mathcal{W} be the solid bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the plane $z = 4$, and let $\mathbf{F} = \langle x^2, xy, z \rangle$. Verify the Divergence Theorem for this solid and this vector field.

Many additional problems may be found in the Chapter 16 and 17 Review Exercises, and in the WeBWorK assignments. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, go to the course WeBWorK page and click the Guest Login button.

Selected Review Problem Answers.

1. One parameterization of \mathcal{S} is $\Phi(r, \theta) = (r \cos \theta, r \sin \theta, r)$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$. For this parameterization, the normal vector $\mathbf{n} = T_\theta \times T_r = \langle r \cos \theta, r \sin \theta, -r \rangle$ points down, and $|\mathbf{n}| = \sqrt{2}r$.

$$(a) \iint_{\mathcal{S}} x^2 z^2 dS = \int_0^{2\pi} \int_0^3 r^2 \cos^2 \theta \cdot r^2 \cdot \sqrt{2}r dr d\theta = \frac{243\sqrt{2}\pi}{2}$$

$$(b) \text{ Note that } \mathbf{F} \bullet \mathbf{n} = \langle r \cos \theta, r \sin \theta, r^4 \rangle \bullet \langle r \cos \theta, r \sin \theta, -r \rangle = r^2 - r^5.$$

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \int_0^{2\pi} \int_0^3 (r^2 - r^5) dr d\theta = -225\pi$$

2. (a) $\text{curl}(\mathbf{F}) = \mathbf{0} = \langle 0, 0, 0 \rangle$ $\text{div}(\mathbf{F}) = 2yz + 2xz$

$$(b) \text{ This vector field is conservative (on } \mathbb{R}^3 \text{): } \mathbf{F} = \nabla\phi \text{ for } \phi(x, y, z) = x^2yz + xy^2z$$

$$(c) \text{ Line integrals involving this vector field are independent of path (on } \mathbb{R}^3 \text{)}$$

$$(d) \int_C \mathbf{F} \bullet d\mathbf{s} = \phi(\mathbf{c}(1)) - \phi(\mathbf{c}(0)) = 118$$

3. (a) The ellipse $x^2/a^2 + y^2/b^2 = 1$ can be parameterized by $x = a \cos t$, $y = b \sin t$, $0 \leq t \leq 2\pi$.

If the region inside the ellipse is D , then

$$\text{area}(D) = \iint_D 1 dA = \frac{1}{2} \oint_{\partial D} x dy - y dx = \frac{1}{2} \int_0^{2\pi} ab dt = ab\pi$$

If D' is the region inside the hypocycloid, then

$$\text{area}(D') = \iint_{D'} 1 dA = \oint_{\partial D'} x dy = \int_0^{2\pi} a \cos^3 t \cdot 3a \sin^2 t \cos t dt = \frac{3a^2\pi}{8}$$

$$(b) \oint_C x^2y dx + x dy = \iint_D (1 - x^2) dA = \frac{5}{6} \quad \text{here } D = \{(x, y) \mid 2x \leq y \leq 2, 0 \leq x \leq 1\}$$

$$(c) \oint_C x^2y dx + x dy = \iint_D (1 - x^2) dA = 0 \quad \text{here } D = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

4. For variety, we will use spherical coordinates to parameterize the surface \mathcal{S} in this problem.

$$\Phi(\phi, \theta) = (2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi), \quad 0 \leq \phi \leq \pi/6, \quad 0 \leq \theta \leq 2\pi.$$

For this parameterization, the normal $\mathbf{n} = T_\phi \times T_\theta = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi \rangle$ points up.

Check that $\text{curl}(\mathbf{F}) = \langle x - y, x - y, 0 \rangle = \langle 2 \sin \phi (\cos \theta - \sin \theta), 2 \sin \phi (\cos \theta - \sin \theta), 0 \rangle$

$$\text{curl}(\mathbf{F}) \bullet \mathbf{n} = \langle 2 \sin \phi (\cos \theta - \sin \theta), 2 \sin \phi (\cos \theta - \sin \theta), 0 \rangle \bullet \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi \rangle \\ = 8 \sin^3 \phi (\cos^2 \theta - \sin^2 \theta)$$

$$\iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \bullet d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/6} 8 \sin^3 \phi (\cos^2 \theta - \sin^2 \theta) d\phi d\theta = 0$$

The boundary $\partial\mathcal{S}$ of \mathcal{S} is the circle $x^2 + y^2 = 1$, $z = \sqrt{3}$, oriented counterclockwise when viewed from above. This (oriented) circle may be parameterized by $\mathbf{c}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$, $0 \leq t \leq 2\pi$. Note that $\mathbf{c}'(t) = \langle -\sin t, \cos t, 0 \rangle$, and that $\mathbf{F}(\mathbf{c}(t)) = \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \sin t \cos t \rangle$.

$$\oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \sin t \cos t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle dt = 0$$

5. The boundary of \mathcal{W} is the union of two surfaces, $\partial\mathcal{W} = \mathcal{S}_1 \cup \mathcal{S}_2$, where \mathcal{S}_1 is the portion of the paraboloid $z = x^2 + y^2$ where $z \leq 4$ and \mathcal{S}_2 is the disk $x^2 + y^2 \leq 4$, $z = 4$. The outward pointing normal is $\mathbf{n}_1 = \langle 2x, 2y, -1 \rangle$ on \mathcal{S}_1 , and is $\mathbf{n}_2 = \langle 0, 0, 1 \rangle$ on \mathcal{S}_2 . Let $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$. Then

$$\begin{aligned} \iint_{\partial\mathcal{W}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \mathbf{n}_1 dA + \iint_D \mathbf{F} \cdot \mathbf{n}_2 dA \\ &= \iint_D (2x^3 + 2xy^2 - x^2 - y^2) dA + \iint_D 4 dA = -8\pi + 16\pi = 8\pi \end{aligned}$$

Check that $\operatorname{div}(\mathbf{F}) = 3x + 1$. Then

$$\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F}) dV = \iint_D \int_{x^2+y^2}^4 (3x+1) dz dA = \iint_D (3x+1)(4-x^2-y^2) dA = 8\pi$$