1. [11 points] For the function $f(x, y) = \sqrt{3x - y}$, sketch the domain in the $xy$-plane, and sketch the level curves $f(x, y) = c$ for $c = 0, 1, 2$..

   The domain $\{(x, y) \mid y \leq 3x\}$ is the set of all points in the $xy$-plane to the right of the line $y = 3x$ (including the line). The level curve $\sqrt{3x - y} = c$ $(k \geq 0)$ is the line $y = 3x - c^2$ in the $xy$-plane with slope 3 and $y$-intercept $-c^2 \leq 0$.

2. [11 points] Consider the function $f(x, y) = \dfrac{5y^2}{x^2 + 4y^2}$.

   (a) Find the limit, or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{5y^2}{x^2 + 4y^2}$

   It does not exist. For instance, along the line $y = 0$, $f(x, 0) = 0$ tends to 0 as $(x, y) \to (0, 0)$; while along the line $x = 0$, $f(0, y) = \frac{5}{4}$ tends to $\frac{5}{4}$ as $(x, y) \to (0, 0)$. Since $0 \neq \frac{5}{4}$, the limit does not exist.

   (b) Is the function $f(x, y)$ continuous at the point $(0, 2)$? Explain.

   Yes. $\displaystyle\lim_{(x,y)\to(0,2)} f(x, y) = \frac{5}{4} = f(0, 2)$.

3. [15 points] The function $f(x, y) = 2x^2 + y^2 - x^2y$ has first partial derivatives

$$f_x(x, y) = 4x - 2xy \quad \text{and} \quad f_y(x, y) = 2y - x^2.$$

   (a) Find all critical points of this function.

   $f_x = 4x - 2xy = 2x(2 - y)$, so $f_x = 0$ if $x = 0$ or $y = 2$. If $x = 0$, $f_y = 0$ says $y = 0$. If $y = 2$, $f_y = 0$ says $x = \pm 2$. So the critical points are $(0, 0)$, $(2, 2)$, and $(-2, 2)$.

   (b) Find an equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 2, 4)$.

   $z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = 4 + 0(x - 1) + 3(y - 2)$

   (c) Use linear approximation to estimate the value of $f(1.1, 2.1)$.

   $f(1.1, 2.1) \approx f(1, 2) + f_x(1, 2)(1.1 - 1) + f_y(1, 2)(2.1 - 2) = 4 + 0(1.1 - 1) + 3(2.1 - 2) = 4.3$

4. [12 points] If $z = y^2 \sin(2x)$; $x = st$, and $y = \sqrt{t}$, use the Chain Rule to find the value of the partial derivative $\dfrac{\partial z}{\partial t}$ when $s = \pi$ and $t = 1$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2y^2 \cos(2x) \cdot s + 2y \sin(2x) \cdot \frac{1}{2}t^{-1/2}$$

   When $s = \pi$ and $t = 1$, $x = \pi$ and $y = 1$, so $\dfrac{\partial z}{\partial t} = 2\pi \cos(2\pi) + \sin(2\pi) = 2\pi$.

5. [15 points] Consider the function $F(x, y, z) = xe^{yz}$, and the point $P = (2, 4, 0)$.

   $\nabla F = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$, $\quad \nabla F(2, 4, 0) = \langle 1, 0, 8 \rangle$

   (a) Find the directional derivative of $F$ at $P$ in the direction of the unit vector $\mathbf{u} = \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

   $D_{\mathbf{u}}F(2, 4, 0) = \nabla F(2, 4, 0) \bullet \mathbf{u} = 6$

   (b) Find the maximum rate of change of $F$ at $P$, and the direction in which it occurs.

   The maximum rate of change is $||\nabla F(2, 4, 0)|| = \sqrt{65}$;
   the direction is that of $\nabla F(2, 4, 0) = \langle 1, 0, 8 \rangle$.

(c) The point $P$ lies on the level surface $F(x, y, z) = 2$. Find an equation of the tangent plane to this surface at the point $P$.

The tangent plane has equation $\nabla F(2, 4, 0) \bullet \langle x - 2, y - 4, z - 0\rangle = 0$,
i.e., $1(x - 2) + 0(y - 4) + 8(z - 0) = 0$.

6. [14 points] Consider the function $f(x, y) = x^2 + 7y^2 - 2xy + 2y^3 + 5$.

(a) Find all second order partial derivatives of this function.

$f_x = 2x - 2y \quad f_y = 14y - 2x + 6y^2 \quad f_{xx} = 2 \quad f_{yy} = 14 + 12y \quad f_{xy} = f_{yx} = -2$

(b) The critical points of this function are $(0, 0)$ and $(-2, -2)$. Find the local maximum and minimum values and all saddle points of this function.

Use the second derivatives test, $D = f_{xx}f_{yy} - f_{xy}^2 = 24 + 24y$.
$D(0, 0) = 24 > 0$ and $f_{xx}(0, 0) = 2 > 0$ implies that $f(0, 0) = 5$ is a local minimum.
$D(-2, -2) = -24 < 0$ implies that $(-2, -2, f(-2, -2)) = (-2, -2, 13)$ is a saddle point.

7. [11 points] Find the maximum value of the function $f(x, y) = 2x + y - 3xy$ on the unit square $\mathcal{D} = [0, 1] \times [0, 1] = \{(x, y) : 0 \le x, y \le 1\}$.

$f_x = 2 - 3y$ and $f_y = 1 - 3x$, so $(\frac{1}{3}, \frac{2}{3})$ is the only critical point of $f$.
Note that $(\frac{1}{3}, \frac{2}{3})$ is in $\mathcal{D}$, and that $f(\frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$.
The boundary of $\mathcal{D}$ has four pieces:
$x = 0,\ 0 \le y \le 1$, here $f(0, y) = y$ has a maximum of 1 at $(0, 1)$;
$x = 1,\ 0 \le y \le 1$, here $f(1, y) = 2 - 2y$ has a maximum of 2 at $(1, 0)$;
$0 \le x \le 1,\ y = 0$, here $f(x, 0) = 2x$ has a maximum of 2 at $(1, 0)$;
$0 \le x \le 1,\ y = 1$, here $f(x, 3) = 1 - x$ has a maximum of 1 at $(0, 1)$.
So the maximum value of $f$ on $\mathcal{D}$ is $f(1, 0) = 2$.

8. [11 points] Use the method of Lagrange multipliers to find the point in the plane $x + 2y - 3z = 14$ that is closest to the origin. This can be done by minimizing the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x + 2y - 3z - 14 = 0$.

Solve $\nabla f = \lambda \nabla g$: $\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 2, -3\rangle$,
$x = \lambda/2,\ y = \lambda,\ z = -3\lambda/2$. The constraint $x + 2y - 3z = 14$ then says $\lambda = 2$.
So $x = 1,\ y = 2,\ z = -3$, and the point $(1, 2, -3)$ is closest to the origin.

MATH 2057-4 EXAM 1 STATISTICS FALL 2011

| Score | Number |
|---|---|
| 90 – 100 | 5 |
| 80 – 89 | 4 |
| 70 – 79 | 9 |
| 60 – 69 | 10 |
| 50 – 59 | 5 |
| 0 – 49 | 5 |
| total | 38 |

average: 68
median: 68