Exam 1 will take place on Thursday, September 22. It will cover material from chapter 14. You will have the entire class period to do the exam. Some remarks concerning the material on the exam are included below.

Books, notes, calculators, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class next Tuesday. You may also make use of my office hours, and the free tutoring available in 141 Middleton Library (hours: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available.

Some review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. All the relevant WeBWorK assignments have been reopened, and will remain open until the exam. You can use them to review and/or improve your homework grade. Some potentially relevant problems from the Chapter 14 Review Exercises are: #1–3, 7, 9, 11, 13, 16–20, 23, 25, 26, 29, 33, 35, 36, 39–41, 46, 47, 49, 52, 53

Chapter 14. Basic notions of differential calculus are extended to functions of two or more variables.

§14.1 Functions of several variables, their domains, ranges... are introduced here. These are the basic objects of study, so be familiar with these ideas, and notions such as traces, level curves, and level surfaces. In particular, if a function of several variables is given by a formula, be able to determine the domain, sketch it (if appropriate), and describe the traces, level curves, or surfaces (by a sketch or in words, again if approriate).

§14.2 Be able to calculate limits of functions of several variables (provided they exist) using the natural generalizations of the basic limit laws and techniques (see Chapter 2), or show that these limits do not exist using methods such at those illustrated in class and in the §14.2 Examples. Also understand what it means for a function of several variables to be continuous. I won't ask you to work with the δ-ϵ definition of a limit.

§14.3 Be able to (quickly and easily) calculate the first, second... partial derivatives of functions of several variables. Be aware of the geometric significance of the first partial derivatives of $f(x, y)$ at a point, and also of "Clairaut's Theorem" which guarentees the equality of the mixed second partials under certain conditions.

§14.4 If $f(x, y)$ has continuous first partials at (a, b), one equation for the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. The right-hand side of this equation is the linearization of f at (a, b). Be able to work with these notions, to use the linearization to approximate functional values, and to use $\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$ to approximate the increment (the change in functional value). Note that linearization extends naturally to functions of three or more variables.

§14.5 If f is a function of two or three variables, and P is a point in the plane or space, the directional derivative of f at P in the direction of the unit vector $\mathbf{u}$ is $D_{\mathbf{u}}f(P) = \nabla f(P) \bullet \mathbf{u}$, where $\nabla f(P)$ is the gradient of f at P. If $f = f(x, y)$, the gradient is $\nabla f = \langle f_x, f_y\rangle$, and if $f = f(x, y, z)$, the gradient is $\nabla f = \langle f_x, f_y, f_z\rangle$. Be aware of the geometric significance of the gradient, its relation to directional derivatives, level curves and surfaces, and be able to use it to find equations for tangent planes, etc.

§14.6 Be comfortable working with the various forms of the chain rule we've encountered (including the chain rule for paths in §14.5). Tree diagrams are useful here. One good way to test your understanding of the chain rule is to use it to calculate second partial derivatives (see, for instance, problem #5 below). I will not ask you about the chain rule approach to implicit differentiation on the exam.

§14.7 Given a function of two variables $f(x, y)$, be able to find the critical points, and classify them as local maxima, local minina, or saddle points using the second derivatives test. Also be prepared to find the extreme values of a continuous function $f = f(x, y)$ defined on a closed, bounded set D in the plane by analyzing the critical points of f in D and the behavior of f on the boundary of D, and to solve applied optimization problems as in Example 6 (p. 853) and a number of the exercises.

§14.8 Lagrange multipliers. To find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, find all values of x, y, z, and λ such that $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$ and $g(x, y, z) = 0$. Then evaluate f at the points (x, y, z) you found, and identify the maximum and minimum values, if any. If f has an extremum subject to the constraint $g = 0$ at $P = (x_0, y_0, z_0)$ (and $\nabla g_P \neq 0$), there is a scalar λ_0 so that $\nabla f_P = \lambda_0\nabla g_P$. There is an analogous 2-variable version of this method, see e.g., Example 1. I will not ask you to consider problems involving two constraints (see Example 4) on the exam.

Remarks.

I will expect you to know the identity $\sin^2\theta + \cos^2\theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponential and logarithmic functions.

Review Problems.

1. Consider the function $f(x,y) = \sqrt{4 - x^2 - y^2}$.
   (a) Find the domain and range of $f(x,y)$.
   (b) Describe the level curves of $f(x,y)$.

2. Consider the function $f(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$.
   (a) Evaluate the following limits or show that they do not exist: $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ and $\lim\limits_{(x,y)\to(3,4)} f(x,y)$.
   (b) Is f continuous at $(0,0)$? Is f continuous at $(3,4)$? Explain.

3. Let $z = g(x,y) = \sin(\pi x y^2)$.
   (a) Find an equation of the plane tangent to the surface $z = \sin(\pi x y^2)$ at the point $(1,1,0)$.
   (b) Compute the pure second partial derivative, $z_{yy} = \dfrac{\partial^2 z}{\partial y^2}$, of z with respect to y.
   (c) Use linear approximation to estimate the value of g at the point $(0.95, 1.02)$.

4. Consider the function $f(x,y) = x^2y - 2y^2 - x^2$.
   (a) Find the critical points of this function, and classify them using the second derivatives test.
   (b) Find the absolute maximum and minimum values of this function on the closed rectangular region with vertices $(0,0)$, $(4,0)$, $(0,2)$, and $(4,2)$.

5. Consider the function $f(x,y,z) = ye^{xz} + z^2$ and the point $P = (0,2,3)$
   (a) Find an equation of the tangent plane to the surface $ye^{xz} + z^2 = 11$ at P.
   (b) Find the directional derivative of f at P in the direction of the vector $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
   (c) What is the value of the largest directional derivative of f at P?
   (d) If the value of $f(x,y,z)$ gives the temperature at the point (x,y,z), in what direction does the temperature increase the most rapidly at the point P?

6. Let $w = f(x,y)$ be a function with continuous partial derivatives, and let $x = 2 + 3v$ and $y = 9u - 4v$.
   (a) Use the chain rule to compute $\dfrac{\partial w}{\partial u}$. Express your answer in terms of the first order partial derivatives of w with respect to x and y.
   (b) Compute $\dfrac{\partial^2 w}{\partial v \partial u}$. Express your answer in terms of the second order partial derivatives of w with respect to x and y.

7. Use Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $z^2 - xy = 1$.

Selected Review Problem Answers.

1. $f(x,y) = \sqrt{x^2+y^2-4}$.
   (a) Domain: $\{(x,y) \mid x^2+y^2 \geq 4\}$. This is all points in the xy-plane of distance at least 2 from the origin. Range: $[0,\infty)$ i.e., $0 \leq f(x,y) < \infty$.
   (b) For $k \geq 0$, the level curve $f(x,y) = k$ is a circle of radius $\sqrt{4+k^2}$ centered at the origin. (If $k < 0$, $f(x,y) = k$ does not make sense.)

2. $f(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$.
   (a) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. You can use, for instance, polar coordinates to see this. Alternatively, as $(x,y) \to (0,0)$ along the line $x = 0$, $f(0,y) = y/\sqrt{y^2} = y/|y|$, and this function has no limit as $y \to 0$.

   (b) f is not continuous at $(0,0)$ (e.g., $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist).

   f is continuous at $(3,4)$, $\lim_{(x,y)\to(3,4)} f(x,y) = \dfrac{4}{5} = f(3,4)$

3. $z = g(x,y) = \sin(\pi xy^2)$.
   (a) $z = -\pi(x-1) - 2\pi(y-1)$
   (b) $z_{yy} = 2\pi x\cos(\pi xy^2) - 4\pi^2x^2y^2\sin(\pi xy^2)$
   (c) $g(0.95, 1.02) \approx -\pi(0.95-1) - 2\pi(1.02-1) = \pi/100$.

4. $f(x,y) = x^2y - 2y^2 - x^2$
   (a) f has a local maximum $f(0,0) = 0$ at $(0,0)$, and has saddle points at $(2,1,f(2,1)) = (2,1,-2)$ and $(-2,1,f(-2,1)) = (-2,1,-2)$.
   (b) $f(4,2) = 8$ is the maximum value, and $f(4,0) = -16$ is the minimum value.

5. $f(x,y,z) = ye^{xz} + z^2$, $P = (0,2,3)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
   (a) $6(x-0) + 1(y-2) + 6(z-3) = 0$
   (b) $\nabla f(0,2,3) \bullet \mathbf{u} = \langle 6,1,6\rangle \bullet \langle \frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\rangle = 45/7$
   (c) $|\nabla f(0,2,3)| = \sqrt{73}$
   (d) In the direction of $\nabla f(0,2,3) = \langle 6,1,6\rangle$

6. $w = f(x,y)$, $x = 2+3v$, $y = 9u - 4v$.
   (a) $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} = 9\dfrac{\partial w}{\partial y}$
   (b) $\dfrac{\partial^2 w}{\partial v \partial u} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial w}{\partial u}\right) = \dfrac{\partial}{\partial v}\left(9\dfrac{\partial w}{\partial y}\right) = 9\left[\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)\dfrac{\partial x}{\partial v} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial y}\right)\dfrac{\partial y}{\partial v}\right] = 27\dfrac{\partial^2 w}{\partial x \partial y} - 36\dfrac{\partial^2 w}{\partial y^2}$

7. Use Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $z^2 - xy = 1$.
   The system of equations one must solve, $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, $g(x,y,z) = 0$, is:
   $$2x = -\lambda y, \quad 2y = -\lambda x, \quad 2z = 2\lambda z, \quad z^2 - xy - 1 = 0.$$
   From the third of these equations, either $z = 0$ or $\lambda = 1$.
   If $z = 0$, then $y = -1/x$ (from the fourth equation). Use this in the first two equations to see that $x^4 = 1$, so $x = 1$ or $x = -1$. If $x = 1$, then $y = -1$. If $x = -1$, then $y = 1$. So we must consider the points $(1,-1,0)$ and $(-1,1,0)$.
   If $\lambda = 1$, use the first two equations to show that $x = 0$ and $y = 0$. Then the fourth equation reads $z^2 = 1$. So $z = 1$ or $z = -1$, and we must consider the points $(0,0,1)$ and $(0,0,-1)$.
   $f(1,-1,0) = f(-1,1,0) = 2$ is the maximum, and $f(0,0,1) = f(0,0,-1) = 1$ is the minimum

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Many additional problems may be found in the Chapter 14 Review Exercises, and in the WeBWorK assignments. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, go to the course WeBWorK page and click the Guest Login button.