Exam 2 will take place on Thursday, November 3. It will cover Chapter 15 and sections 16.1 and 16.2. (Cylindrical and spherical coordinates, used in section 15.4, are introduced in section 12.7.) Some remarks concerning this material are included below. Books, notes, calculators, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class on Tuesday, November 1. You may also make use of my office hours, and the free tutoring available in 141 Middleton Library (hours: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available.

Some review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. All the relevant WeBWorK assignments will remain open through the exam. You can use them to review and/or improve your homework grade. Some potentially relevant problems from the Chapter Review Exercises are:

Ch. 15 (p. 937) #5–17 odd, 19, 20, 22, 26, 27, 29, 31, 33–39, 43 Ch. 16 (p. 1001) #1–13 odd, 17–21, 25–27

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Chapter 15. The focus of this chapter is on extending the basic notions of integral calculus to functions of two or more variables.

I will try to avoid lengthy techniques of integration problems on the exam (you were presumably tested on these in Calculus II). But standard techniques such as $u$-substitution and integration by parts are fair game.

§15.1 Double integrals over rectangles are introduced in this section, and Fubini's Theorem for calculating these via iterated integrals is discussed. I will not ask you to work directly with the definition of a double integral (involving Riemann sums) on the exam. But you should be aware of this definition, properties of double integrals, and the geometric interpretations, related notions, and applications discussed in this chapter.

§15.2 Double integrals over simple regions may also be calculated using iterated integrals, see Theorem 2 on page 888. These are our main computational tools for dealing with double integrals, and you should be comfortable with them. Be able to (quickly) decide what is the best way to express a given region, to set up iterated integrals over vertically and horizontally simple regions, to change the order of integration if necessary (and to recognize when this is necessary), and of course to calculate iterated integrals.

§15.3 Triple integrals over boxes and more generally simple solids may also be calculated via iterated integrals, see for instance Theorem 2 on page 901. Again, be able to decide what is the best way to express a given solid, to set up and evaluate iterated integrals over simple solids. Applications of triple integrals include volume, average value (of a function of three variables), mass, center of mass. . .

§15.4 Integration in polar, cylindrical, and spherical coordinates. If $D$ is a polar region, $D = \{(r,\theta) \mid \theta_1 \le \theta \le \theta_2,\ \alpha(\theta) \le r \le \beta(\theta)\}$, and $f$ is continuous on $D$, you should be able to calculate $\iint_D f(x,y)\,dA$ by switching to polar coordinates, see page 912. When carrying this out, do not forget that the "polar element of area" is $r\,dr\,d\theta$. If $W$ is a solid of the form $\theta_1 \le \theta \le \theta_2,\ \alpha(\theta) \le r \le \beta(\theta),\ z_1(r,\theta) \le z \le z_2(r,\theta)$, and $f$ is continuous on $W$, the triple integral $\iiint_W f(x,y,z)\,dV$ may be calculated by switching to cylindrical coordinates, see page 914. In particular, the "element of volume" is $r\,dz\,dr\,d\theta$. If $W$ is of the form $\theta_1 \le \theta \le \theta_2,\ \phi_1 \le \phi \le \phi_2,\ \rho_1(\phi,\theta) \le \rho \le \rho_2(\phi,\theta)$, and $f$ is continuous on $W$, the triple integral $\iiint_W f(x,y,z)\,dV$ may be calculated by switching to spherical coordinates, see page 917. In particular, the "element of volume" is $\rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta$. Be prepared to set up and evaluate multiple integrals by changing to one of these coordinate systems. For this, you will need to be able to make translations between rectangular and polar, cylindrical, and spherical coordinates. You should also develop enough experience with these regions and solids and multiple integrals to recognize when switching to polar, cylindrical, or spherical coordinates is appropriate.

§15.5 Change of variables. If $\Phi(u,v) = \big(x(u,v), y(u,v)\big)$ is a transformation that maps a region $D_0$ in the $uv$-plane to a region $D$ in the $xy$-plane and $f$ is continuous on $R$, then under certain conditions (see page 929), $\displaystyle\iint_D f(x,y)\,dA = \iint_{D_0} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$, where $\displaystyle\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$ is the Jacobian of $\Phi$. Given a transformation, you should be able to use this to evaluate double integrals.

Chapter 16. The focus of this chapter is on the calculus of vector fields, including line integrals.

§16.1 This section introduces vector fields, functions that assign to each point in some domain a vector. The gradient of a function of two or three variables is a familiar example. As you should know, a vector field $\mathbf{F}$ is a gradient vector field if $\mathbf{F} = \nabla\phi$ for some function $\phi$. In this situation, $\phi$ is a "potential function" for $\mathbf{F}$.

§16.2 Line integrals in the plane and in space are introduced in this section. Given a parameterized curve and a function or vector field, you should be able to evaluate a relevant line integral by expressing it in terms of the parameter. (You should be able to parameterize certain curves, such as line segments, portions of circles, etc., yourself.) Applications of line integrals include mass, center of mass, work...

Remarks.

I will expect you to know the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

| $\theta$ | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $1/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | $1$ |
| $\cos\theta$ | $1$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $0$ |

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponentials and logarithms.

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Review Problems.

1. Let $D$ be the region in the first quadrant bounded by $x = y^2$, $y = 0$, and $x = 1$. Evaluate $\displaystyle\iint_D y\sin(\pi x^2)\,dA$.

2. Use a double integral to find the area of the region inside one loop of the three-leaved rose $r = \cos(3\theta)$.

3. A thin metal plate occupies the region $D$ in the $xy$-plane bounded by the $x = 1$, $y = 1$, and $xy = 2$. The density at the point $(x, y)$ in this region is given by $f(x, y) = 4xy$. Find the center of mass of the plate.

4. Consider the solid bounded by the surface $z = 1 - y^2$ and the planes $x = 0$, $z = 0$, $y = 1$, and $y = x$.
   (a) Make a beautiful sketch of this solid.
   (b) Use a double integral to find the volume of this solid.

5. Use cylindrical or spherical coordinates, whichever is most appropriate.
   (a) Let $E$ be the solid in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 2$. Evaluate $\iiint_E xyz\,dV$.
   (b) Find the volume common to the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + (z - 2)^2 = 4$

6. Let $S$ be the square in the $uv$-plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Let $\Phi$ be the transformation given by $x = 3u - v$ and $y = u + v$. Let $R$ be the image in the $xy$-plane of $S$ under this transformation. Sketch the region $R$, compute the Jacobian of $\Phi$, and use this transformation to evaluate the integral $\iint_R (x^2 - y^2)\,dA$.

7. Sketch the vector field $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j} = \langle y, x^2\rangle$. Is $\mathbf{F}$ a gradient vector field? Explain.

8. Let $C$ be the curve parameterized by $\mathbf{c}(t) = 6t\,\mathbf{i} + 3\sqrt{2}t^2\,\mathbf{j} + 2t^3\,\mathbf{k}$, $0 \le t \le 1$.
   (a) Find the mass of a thin wire in the shape of $C$ if $\rho(x, y, z) = xz$ is the density at the point $(x, y, z)$.
   (b) Find the work done by the vector (force) field $\mathbf{F} = z\,\mathbf{i} + (y/x)\,\mathbf{j} - \mathbf{k}$ in moving an object along $C$.

Many additional problems may be found in the Chapter 15 and 16 Review Exercises, and in the WeBWorK assignments. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, go to the course WeBWorK page and click the Guest Login button.

Selected Review Problem Answers.

1. Let $D$ be the region in the first quadrant bounded by $x = y^2$, $y = 0$, and $x = 1$. Evaluate $\iint_D y\sin(\pi x^2)\,dA$.

   $D = \{(x,y) \mid y^2 \le x \le 1,\ 0 \le y \le 1\} = \{(x,y) \mid 0 \le y \le \sqrt{x},\ 0 \le x \le 1\}$

   $$\iint_D y\sin(\pi x^2)\,dA = \int_0^1 \int_{y^2}^1 y\sin(\pi x^2)\,dx\,dy = \int_0^1 \int_0^{\sqrt{x}} y\sin(\pi x^2)\,dy\,dx = \frac{1}{2\pi}$$

   Evaluate the second of the two iterated integrals above (the one on the right).

2. Use a double integral to find the area of the region inside one loop of the three-leaved rose $r = \cos(3\theta)$. Let $D$ be the region inside the loop of $r = \cos(3\theta)$ that lies in the first and fourth quadrants (see next page). This region is given by $D = \{(r,\theta) \mid 0 \le r \le \cos(3\theta), -\pi/6 \le \theta \le \pi/6\}$. The area of $D$ is $A = \iint_D 1\,dA$.

   Using polar coordinates, we have $A = \int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r\,dr\,d\theta = \frac{\pi}{12}$

3. A thin metal plate occupies the region $D$ in the $xy$-plane bounded by the $x = 1$, $y = 1$, and $xy = 2$. The density at the point $(x, y)$ in this region is given by $f(x, y) = 4xy$. Find the center of mass of the plate.

   The mass is $m = \iint_D f(x,y)\,dA = \int_1^2 \int_1^{2/x} 4xy\,dy\,dx = 8\ln(2) - 3$. The center of mass is $(\bar{x}, \bar{y})$, where

   $$\bar{x} = \iint_D x\cdot f(x,y)\,dA = \frac{1}{m}\int_1^2 \int_1^{2/x} 4x^2y\,dy\,dx = \frac{10}{3m} \text{ and } \bar{y} = \iint_D y\cdot f(x,y)\,dA = \frac{1}{m}\int_1^2 \int_1^{2/x} 4xy^2\,dy\,dx = \frac{10}{3m}$$

4. Consider the solid bounded by the surface $z = 1 - y^2$ and the planes $x = 0$, $z = 0$, and $y = x$.
   (a) Make a beautiful sketch of this solid.
   See next page.
   (b) Use a double integral to find the volume of this solid.
   $$V = \int_0^1 \int_x^1 (1 - y^2)\,dy\,dx = \int_0^1 \int_0^y (1 - y^2)\,dx\,dy = \frac{1}{4}$$

5. Use cylindrical or spherical coordinates...

   (a) $$\int_0^{\pi/2} \int_0^2 \int_r^2 r^3\, z\, \sin\theta\, \cos\theta\, dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2/\cos\phi} \rho^5\, \sin^3\phi\, \cos\phi\, \sin\theta\, \cos\theta\, d\rho\,d\phi\,d\theta = \frac{4}{3}$$

   (b) $$\int_0^{2\pi} \int_0^{3\sqrt{7}/4} \int_{2-\sqrt{4-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_{3\sqrt{7}/4}^{2} \int_{2-\sqrt{4-r^2}}^{2+\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \frac{63\pi}{8}$$

   $$\int_0^{2\pi} \int_0^{\cos^{-1}(3/4)} \int_0^3 \rho^2\, \sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi} \int_{\cos^{-1}(3/4)}^{\pi/2} \int_0^{4\cos\phi} \rho^2\, \sin\phi\,d\rho\,d\phi\,d\theta = \frac{63\pi}{8}$$

6. The image of the square $S$ under the transformation $T$ is the rectangle $R$ in the $xy$-plane with vertices $(0,0)$, $(3,1)$, $(2,2)$, and $(-1,1)$.

   The Jacobian of $\Phi$ is 4.

$$\iint\limits_R (x^2 - y^2)dx\,dy = \iint\limits_S \big((3u-v)^2 - (u+v)^2\big)4\,du\,dv = 4\int_0^1\int_0^1 (8u^2 - 8uv)du\,dv = \frac{8}{3}$$

7. $\mathbf{F} = \langle F_1, F_2\rangle = \langle y, x^2\rangle$ is not a gradient vector field since $\dfrac{\partial F_1}{\partial y} = 1 \neq 2x = \dfrac{\partial F_2}{\partial x}$.

8. If $\mathbf{c}(t) = 6t\,\mathbf{i} + 3\sqrt{2}t^2\,\mathbf{j} + 2t^3\,\mathbf{k}$, then $\mathbf{c}'(t) = 6\mathbf{i} + 6\sqrt{2}t\mathbf{j} + 6t^2\mathbf{k}$ and $|\mathbf{c}'(t)| = \sqrt{36 + 72t^2 + 36t^4} = 6(1+t^2)$.

   (a) $$m = \int_C \rho(x,y,z)\,ds = \int_C \rho(\mathbf{c}(t))|\mathbf{c}'(t)|\,dt = \int_0^1 (6t)(2t^3)\big(6(1+t^2)\big)dt = 72\int_0^1 (t^4 + t^6) = \frac{72}{5} + \frac{72}{7}$$

   (b) $$W = \int_C \mathbf{F} \bullet d\mathbf{s} = \int_0^1 \langle 2t^3, (3\sqrt{2}t^2)/(6t), -1\rangle \bullet \langle 6, 6\sqrt{2}t, 6t^2\rangle dt = \int_0^1 12t^3\,dt = 3$$