

1. Write the function $f(z) = z^2 + \frac{1}{z}$ in the form $f(z) = u(x, y) + i v(x, y)$.

$$f(z) = z^2 + \frac{1}{z} = (x + iy)^2 + \frac{1}{x + iy} \frac{x - iy}{x - iy} = x^2 - y^2 + \frac{x}{x^2 + y^2} + i \left(2xy - \frac{y}{x^2 + y^2} \right)$$

2. Find all cube roots of $8i$, i.e., solve $z^3 = 8i$. Express your answer in rectangular coordinates.

Note that $8i = 8 \exp(\pi i/2 + 2k\pi i)$, where $k \in \mathbb{Z}$ is an integer. If $z = re^{i\theta}$, then $z^3 = r^3 e^{3i\theta}$ and $z^3 = 8i$ says $r^3 = 8$ and $3\theta = \pi/2 + 2k\pi$. So $r = 2$ and $\theta = \pi/6 + 2k\pi/3$. Choosing $k = 0, 1, 2$ yields the three distinct cube roots of $8i$, which are

$$z_1 = 2e^{\pi i/6} = \sqrt{3} + i, \quad z_2 = 2e^{5\pi i/6} = -\sqrt{3} + i, \quad z_3 = 2e^{3\pi i/2} = -2i.$$

3. If z lies on the circle of radius 3 centered at the origin, i.e., $|z| = 3$, show that $\left| \frac{z+2}{z^2-2z+1} \right| \leq \frac{5}{4}$.

If $|z| = 3$, then $|z+2| \leq |z| + 2 = 3 + 2 = 5$ and $|z^2 - 2z + 1| = |z-1|^2 \geq (|z| - 1)^2 = 4$.

$$\text{So } \left| \frac{z+2}{z^2-2z+1} \right| \leq \frac{5}{4}.$$

4. For $f(z) = \frac{\bar{z}-1}{z-1}$, evaluate $\lim_{z \rightarrow 1} f(z)$ or show that this limit does not exist.

This limit does not exist. For instance, consider the limit of $f(z)$ as z approaches 1 along the real axis $y = 0$. One has $\lim_{(x,0) \rightarrow (1,0)} \frac{x-iy-1}{x+iy-1} = \lim_{(x,0) \rightarrow (1,0)} \frac{x-1}{x-1} = 1$. But if z approaches 1 along the line $x = 1$,

one has $\lim_{(1,y) \rightarrow (1,0)} \frac{x-iy-1}{x+iy-1} = \lim_{(1,y) \rightarrow (1,0)} \frac{-iy}{iy} = -1$. Since $1 \neq -1$, the limit cannot exist.

5. In each part below, determine where the function is differentiable.

A function $f(z) = f(x+iy) = u(x, y) + i v(x, y)$ is differentiable at points where its component functions have continuous partials and satisfy the Cauchy-Riemann equations, see section 22 in the text.

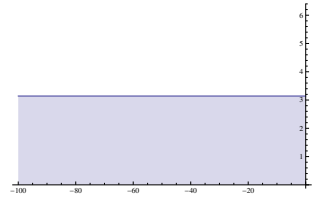
- (a) $f(z) = |z|^2 = |x+iy|^2 = x^2 + y^2$, so $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. $u_x = v_y$ and $u_y = -v_x$ if $2x = 0$ and $2y = 0$. So $x = 0$ and $y = 0$ and $f(z)$ is differentiable only at $z = 0$.
- (b) $g(z) = \exp(\bar{z}) = \exp(x-iy) = e^x(\cos(-y) + i \sin(-y)) = e^x \cos(y) - i e^x \sin(y)$. $u_x = v_y$ and $u_y = -v_x$ if $e^x \cos(y) = -e^x \cos(y)$ and $-e^x \sin(y) = e^x \sin(y)$. The first of these says that $\cos(y) = 0$, and the second says that $\sin(y) = 0$. There are no solutions to this system of equations, so $g(z)$ is nowhere differentiable.
- (c) $h(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2)$. Note that $u(x, y)$ and $v(x, y)$ are polynomial functions of x and y , so have continuous partials. The Cauchy-Riemann equations hold for all x and y , so $h(z)$ is differentiable everywhere, i.e., on all of \mathbb{C} .

6. Let S be the semi-infinite strip $S = \{z = x + iy \mid x < 0, 0 < y < \pi\}$.

(a) Sketch the set S . Is S a domain?

S is the shaded region:

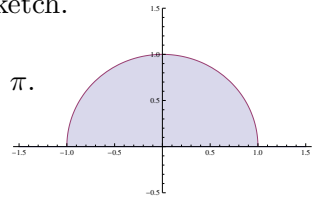
S is nonempty, open and connected, so S is a domain.



(b) Find the image of S under the mapping $w = f(z) = \exp(z)$. Include a sketch.

$w = f(z) = \exp(z) = e^x \cos(y) + i e^x \sin(y)$. Since $x < 0$, $0 < e^x < 1$.

So $0 < |w| < 1$, and the argument of w ranges from 0 to π since $0 < y < \pi$.



7. Suppose that $f(z)$ is entire, and is real-valued. Show that $f(z)$ must be a constant.

Write $f(z) = f(x + iy) = u(x, y) + i v(x, y)$. Since $f(z)$ is real-valued, we have $v(x, y) = 0$. Since $f(z)$ is entire, the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold for all (x, y) . So $u_x = v_y = 0$ and $u_y = -v_x = 0$. It follows that $u(x, y) = c$ is constant, so $f(z) = f(x + iy) = u(x, y) = c$ is constant. (Or it follows that $f'(z) = u_x + i v_x = 0$, which implies that $f(z) = c$ is constant on \mathbb{C} since $f(z)$ is analytic on \mathbb{C} .)

8. Recall that $\text{Log}(z)$ is the principal branch of $\log(z)$. $\text{Log}(z) = \ln |z| + i \text{Arg}(z)$, where $-\pi < \text{Arg}(z) < \pi$.

(a) Find all values of $\log(i^{1/2})$.

$$i^{1/2} = \exp\left(\frac{1}{2} \log(i)\right) = \exp\left(\frac{1}{2}(\ln |i| + i(\frac{\pi}{2} + 2k\pi))\right) = \exp\left(i\left(\frac{\pi}{4} + k\pi\right)\right)$$

So $\log(i^{1/2}) = \frac{\pi}{4} + k\pi$, where $k \in \mathbb{Z}$ is an integer.

(b) Find a complex number z_1 such that $\text{Log}(z_1^2) = 2 \text{Log}(z_1)$.

If $-\frac{\pi}{2} < \text{Arg}(z_1) < \frac{\pi}{2}$, then $-\pi < \text{Arg}(z_1^2) < \pi$ and $\text{Log}(z_1^2) = 2 \text{Log}(z_1)$.

For instance if $z_1 = 1 + i = \sqrt{2}e^{\pi i/4}$, then $z_1^2 = 2e^{\pi i/2}$ and

$$\text{Log}(z_1^2) = \ln(2) + \pi i/2 = 2(\ln(2^{1/2}) + \pi i/4) = 2 \text{Log}(z_1).$$

(c) Find a complex number z_2 such that $\text{Log}(z_2^2) \neq 2 \text{Log}(z_2)$.

If, for example $\frac{\pi}{2} < \text{Arg}(z_2) < \pi$, then $\pi < 2 \text{Arg}(z_2) < 2\pi$.

For instance if $z_2 = -1 + i = \sqrt{2}e^{3\pi i/4}$, then $z_2^2 = 2e^{3\pi i/2} = 2e^{-\pi i/2}$ and

$$\text{Log}(z_2^2) = \ln(2) - \pi i/2 \neq 2(\ln(2^{1/2}) + 3\pi i/4) = 2 \text{Log}(z_2).$$

9. Evaluate the contour integral $\int_C |z|^2 dz$, where C is the line segment from 0 to $2 + i$.

Note that $|z|^2 = |x + iy|^2 = x^2 + y^2$. Let $z(t) = (2 + i)t = 2t + it$, $0 \leq t \leq 1$. Then $z'(t) = 2 + i$ and

$$\int_C f(z) dz = \int_0^1 f(z(t))z'(t) dt = \int_0^1 [(2t)^2 + t^2](2 + i) dt = (2 + i) \int_0^1 5t^2 dt = 5(2 + i)/3.$$

Score	Number
90 - 100	3
80 - 89	4
70 - 79	3
60 - 69	1
50 - 59	3
0 - 49	7
total	21

average: 64

median: 68