1. If C is the arc of the circle $|z| = 2$ from 2 to $2i$ in the first quadrant, show that $\left|\int_C \frac{1}{z^2-1}\,dz\right| \leq \frac{\pi}{3}$.
$\left|\int_C f(z)\,dz\right| \leq ML$, where $f(z) \leq M$ for z on C and L is the length of C. Note that the length of the given curve is $\pi = L$. Since $|z^2-1| \geq ||z^2|-1| = 3$ if $|z| = 2$, we have $\left|\frac{1}{z^2-1}\right| \leq \frac{1}{3} = M$.

2. If C a contour from the point 1 to the point i, evaluate the contour integral $\int_C \frac{1}{z^2}\,dz$.
$f(z) = z^{-2}$ has antiderivative $F(z) = -z^{-1}$ on $D = \mathbb{C}\setminus\{0\}$. So contour integrals involving $f(z)$ are independent of path in D. In particular, if C is a contour from 1 to i, we have $\int_C z^{-2}\,dz = F(i) - F(1) = i + 1$.

3. Consider the function $f(z) = \frac{z^2}{(z-2)(z-4)^2}$. Note that $f(z)$ is analytic on $\mathbb{C}\setminus\{2,4\}$.
   (a) If C_1 is the circle of radius 1 centered at $z_0 = 0$, positively oriented, evaluate $\int_{C_1} f(z)\,dz$
   $f(z)$ is analytic inside and on C_1, so $\int_{C_1} f(z)\,dz = 0$ by the Cauchy-Goursat Theorem.
   (b) If C_2 is the circle of radius 3 centered at $z_0 = 0$, positively oriented, evaluate $\int_{C_2} f(z)\,dz$
   $\int_{C_2} f(z)\,dz = \int_{C_2} \frac{g(z)}{z-2}\,dz$, where $g(z) = \frac{z^2}{(z-4)^2}$ is analytic inside and on C_2. So, by the Cauchy Integral Formula, $\int_{C_2} f(z)\,dz = \int_{C_2} \frac{g(z)}{z-2}\,dz = 2\pi i g(2) = 2\pi i$
   (c) If C_3 is the circle of radius 1 centered at $z_0 = 4$, positively oriented, evaluate $\int_{C_3} f(z)\,dz$
   $\int_{C_3} f(z)\,dz = \int_{C_3} \frac{h(z)}{(z-4)^2}\,dz$, where $h(z) = \frac{z^2}{z-2}$ is analytic inside and on C_3. So, by the extension of the Cauchy Integral Formula, $\int_{C_2} f(z)\,dz = \int_{C_2} \frac{h(z)}{(z-4)^2}\,dz = 2\pi i h'(4) = 0$

4. Let C_1 be the line segment from $z_1 = 1$ to $z_2 = i$, and let C_2 be the portion of the circle $|z| = 1$ from $z_1 = 1$ to $z_2 = i$ (oriented counterclockwise). If $f(z) = \mathrm{Log}(z)$ is the principal branch of the logarithmic function $\log(z)$, is it true that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$? Explain.
There are many ways to see that this is true. Here is one: Let $C = C_2 - C_1$ (first traverse C_2 with its orientation, then traverse C_1 in the direction opposite to its orientation. The function $f(z) = \mathrm{Log}(z)$ is analytic inside and on the positively oriented simple closed contour C (the branch cut is the negative real axis), so by the Cauchy-Goursat Theorem, $\int_C f(z)\,dz = 0$. Since $C = C_2 - C_1$, we have $0 = \int_C f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz$. So $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

5. If $f(z) = u(x,y) + i\,v(x,y)$ and the imaginary part $v(x,y)$ of f is bounded, can you conclude that the function $v(x,y)$ is constant? Explain. Hint: Consider the function $-i\,f(z)$.
If f is not entire (or analytic where it is defined), $v(x,y)$ need not be constant. For instance, let $f(z) = f(x+i\,y) = i\,\sin(x)$. Here $v(x,y) = \sin(x)$ is bounded, but not constant (and f is not entire).

For f entire, note that $-i\,f(z) = v(x,y) - i\,u(x,y)$. Then $g(z) = \exp(-i\,f(z)) = e^{v(x,y)}e^{-i\,u(x,y)}$ is entire, and $|g(z)| = e^{v(x,y)}$ is bounded since $v(x,y)$ is bounded. So, by Liouville's Theorem, $g(z) = c$ is constant. So $|g(z)| = e^{v(x,y)} = |c| > 0$, and $v(x,y) = \ln(|c|)$ is constant.

6. If $\displaystyle\sum_{n=1}^{\infty} z_n = S$, show that $\displaystyle\sum_{n=1}^{\infty} i\,z_n = i\,S$.

   Let $z_n = x_n + i\,y_n$, $S = X + i\,Y$, and $S_N = z_1 + \cdots + z_N = x_1 + \cdots + x_N + i(y_1 + \cdots + y_N) = X_N + i\,Y_N$. Since $\displaystyle\sum_{n=1}^{\infty} z_n = S$, we have $\displaystyle\lim_{N\to\infty} S_N = S$, which implies that $\displaystyle\lim_{N\to\infty} X_N = X$ and $\displaystyle\lim_{N\to\infty} Y_N = Y$. It follows that $\displaystyle\lim_{N\to\infty} -Y_N = -Y$, so $\displaystyle\lim_{N\to\infty} -Y_N + i\,X_N = -Y + i\,X$. Since $-Y_N + i\,X_N = S'_N$ is the N-th partial sum of the series $\displaystyle\sum_{n=1}^{\infty} i\,z_n$, we have $\displaystyle\lim_{N\to\infty} S'_N = -Y + i\,X = i\,S$, and $\displaystyle\sum_{n=1}^{\infty} i\,z_n = i\,S$.

7. For $f(z) = \sin(z)$, find the Macaurin series of $f(z)$ …
   (a) by using the definition of the Maclaurin series (computing derivatives, etc.);
   (b) by using the fact that $\sin(z) = (e^{iz} - e^{-iz})/2i$ and the Maclaurin series for e^z.

   We have $f^{(n)}(z) = \begin{cases} \sin(z) & \text{if } n = 0, 4, 8, 12, \ldots \\ \cos(z) & \text{if } n = 1, 5, 9, 13, \ldots \\ -\sin(z) & \text{if } n = 2, 6, 10, 14, \ldots \\ -\cos(z) & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}$ and $f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 0, 4, 8, 12, \ldots \\ 1 & \text{if } n = 1, 5, 9, 13, \ldots \\ 0 & \text{if } n = 2, 6, 10, 14, \ldots \\ -1 & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}$

   so the Maclaurin series of $f(z) = \sin(z)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

   Since $e^w = \displaystyle\sum_{n=0}^{\infty} \frac{w^n}{n!}$, we have $\displaystyle\frac{e^{i\,z} - e^{-i\,z}}{2i} = \frac{1}{2i}\sum_{n=0}^{\infty}\left[\frac{i^n z^n}{n!} - \frac{(-i)^n z^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}\left[i^n(1 - (-1)^n)\frac{z^n}{n!}\right]$.

   Since $1 - (-1)^n = 0$ for n even, $1 - (-1)^n = 2$ for n odd, and $i^n = \begin{cases} 1 & \text{if } n = 0, 4, 8, 12, \ldots \\ i & \text{if } n = 1, 5, 9, 13, \ldots \\ -1 & \text{if } n = 2, 6, 10, 14, \ldots \\ -i & \text{if } n = 3, 7, 11, 15, \ldots \end{cases}$

   this yields the same power series as above.

8. Find the Laurent series of the function $f(z) = \dfrac{z^2}{4+z}$ that is valid when $4 < |z| < \infty$.

$$f(z) = \frac{z}{1 + 4/z} = z\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{4^n}{z^{n-1}} = z - 4 + \frac{4^2}{z} - \frac{4^3}{z^2} + \ldots \text{ for } \frac{4}{|z|} < 1, \text{ i.e., } 4 < |z|$$

Extra Credit.

If C is the circle of radius 10 centered at $z_0 = 0$, positively oriented, evaluate $\displaystyle\int_C f(z)\,dz$, where $f(z) = \dfrac{z^2}{(z-2)(z-4)^2}$ is the function from problem 3.

$$\int_C f(z)\,dz = \int_{C_2 + C_3} f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = 2\pi i$$