For Section 4, the final exam is scheduled for Friday, May 13, 7:30 – 9:30 am.
For Section 6, the final exam is scheduled for Saturday, May 14, 10:00 am - 12:00 noon.

Final exams will take place in our usual classrooms. The final will be comprehensive. I anticipate being available for questions in my office in the mornings Monday through Thursday of finals week. As you know by now, I am also willing to (try to) answer questions by email at essentially any time. You may also make use of, e.g., the free tutoring available in room 141B Middleton Library (hours: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available, or these hours will be valid during finals week. As usual, books, notes, calculators, etc. may not be used on the exam. Some remarks and problems concerning the material covered since the third in-class exam are included below.

In addition to the material covered on the in-class exams, the final will also cover material from sections 17.2 and 17.3. Some potentially relevant problems from the Chapter Review Exercises are:

Ch. 17 (p. 1043) #9, 10, 18, 19, 21, 22, 23, 25

This is **not** a comprehensive list. See the WeBWorK assignment and the section Exercises for more problems.

Chapter 17. The focus of this chapter is on the three main theorems of vector calculus, Green's Theorem, Stokes' Theorem, and the Divergence Theorem, which may be viewed as generalizations of the Fundamental Theorem of Calculus.

§17.2 Stokes' Theorem is discussed in this section. See page 1019 for the definition of the curl of a vector field, and see page 1020 for the precise statement of the theorem. The punchline is $\oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \bullet d\mathbf{S}$.

§17.3 The Divergence Theorem is discussed here. See page 1031 for the definition of the divergence of a vector field, and see page 1032 for the precise statement of the theorem. The punchline is $\iint_{\partial\mathcal{W}} \mathbf{F} \bullet d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV$.

Remarks. Regarding these sections, you should be able to compute the curl and divergence of a vector field, and be prepared to use Stokes' Theorem and the Divergence Theorem. I will not hold you responsible for the precise statements of these theorems on the final. However, these are important results, and problems from these sections and the corresponding WeBWorK assignment are additionally very useful for review, as they address line integrals, surface integrals, and triple integrals (sometimes two of these types of integrals at a time, e.g., "Verify that Stokes' Theorem is true for..." and "Verify that the Divergence Theorem is true for..."). Some of the Ch. 17 Review Exercises above are of this nature.

As usual, I will expect you to know the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponential and logarithm functions.

The in-class exams (including answers) and corresponding reviews are all posted. Additional problems may be found in the Review Exercises for Chapters 14–17. All WeBWorK assignments will remain open through the final. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, log in with Username: `aaaaaa` and Password: `123456789`.

Review Problems.

1. Consider the vector field $\mathbf{F} = \langle yz(2x+y), xz(x+2y), xy(x+y)\rangle$ defined on all of $\mathbb{R}^3$.
   (a) Calculate the curl of $\mathbf{F}$.
   (b) Calculate the divergence of $\mathbf{F}$.
   (c) Can one of the two prior calculations be used to determine if $\mathbf{F}$ is conservative on $\mathbb{R}^3$? Explain

2. Let $\mathcal{S}$ be the portion of the hemisphere $x^2+y^2+z^2=4$, $z \geq 0$, inside the cylinder $x^2+y^2=1$, oriented up, and let $\mathbf{F} = \langle xz, yz, xy\rangle$. Verify Stokes' Theorem for this oriented surface and this vector field.

3. Let $\mathcal{W}$ be the solid bounded below by the paraboloid $z = x^2+y^2$ and bounded above by the plane $z=4$, and let $\mathbf{F} = \langle x^2, xy, z\rangle$. Verify the Divergence Theorem for this solid and this vector field.

4. Use either Stokes' Theorem or the Divergence Theorem to evaluate the following:
   (a) $\displaystyle\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \bullet d\mathbf{S}$ where $\mathbf{F} = \langle yz^3, \sin(xyz), x^3\rangle$ and $\mathcal{S}$ is the part of the paraboloid $y = 1-x^2-z^2$ that lies to the right of the xz-plane, oriented towards the xz-plane.
   (b) $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s}$ where $\mathbf{F} = \langle -y^2, x, z^2\rangle$ and $\mathcal{C}$ is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$, oriented counterclockwise when viewed from above.
   (c) $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$ where $\mathbf{F} = \langle 3xy, y^2, -x^2y^4\rangle$ and $\mathcal{S}$ is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented outward.

Selected Review Problem Answers.

1. (a) $\operatorname{curl}(\mathbf{F}) = \mathbf{0} = \langle 0,0,0\rangle$
   (b) $\operatorname{div}(\mathbf{F}) = 2yz + 2xz$
   (c) Part (a) implies that $\mathbf{F}$ **is** conservative on $\mathbb{R}^3$. Explain why, and find a potential function for $\mathbf{F}$.

2. We can use spherical coordinates to parameterize the surface $\mathcal{S}$ in this problem.
   $\Phi(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $0 \leq \phi \leq \pi/6$, $0 \leq \theta \leq 2\pi$.
   For this parameterization, the normal $\mathbf{n} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$ points up.
   Check that $\operatorname{curl}(\mathbf{F}) = \langle x-y, x-y, 0\rangle = \langle 2\sin\phi(\cos\theta-\sin\theta), 2\sin\phi(\cos\theta-\sin\theta), 0\rangle$
   $\operatorname{curl}(\mathbf{F}) \bullet \mathbf{n} = \langle 2\sin\phi(\cos\theta-\sin\theta), 2\sin\phi(\cos\theta-\sin\theta), 0\rangle \bullet \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$
   $= 8\sin^3\phi(\cos^2\theta - \sin^2\theta)$

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \bullet d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/6} 8\sin^3\phi\,(\cos^2\theta-\sin^2\theta)\,d\phi\,d\theta = 0$$

   The boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is the circle $x^2+y^2=1$, $z=\sqrt{3}$, oriented counterclockwise when viewed from above. This (oriented) circle may be parameterized by $\mathbf{c}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle$, $0 \leq t \leq 2\pi$. Note that $\mathbf{c}'(t) = \langle -\sin t, \cos t, 0\rangle$, and that $\mathbf{F}(\mathbf{c}(t)) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t\rangle$.

$$\oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t\rangle \bullet \langle -\sin t, \cos t, 0\rangle\,dt = 0$$

3. The boundary of the solid $\mathcal{W}$ is the union of two surfaces, $\partial\mathcal{W} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ is the portion of the paraboloid $z = x^2 + y^2$ where $z \le 4$ and $\mathcal{S}_2$ is the disk $x^2 + y^2 \le 4$, $z = 4$. The outward pointing normal is $\mathbf{n}_1 = \langle 2x, 2y, -1\rangle$ on $\mathcal{S}_1$, and is $\mathbf{n}_2 = \langle 0, 0, 1\rangle$ on $\mathcal{S}_2$. Let $D = \{(x, y) \mid x^2 + y^2 \le 4\}$. Then

$$\iint\limits_{\partial\mathcal{W}} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{\mathcal{S}_1} \mathbf{F} \bullet d\mathbf{S} + \iint\limits_{\mathcal{S}_2} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{D} \mathbf{F} \bullet \mathbf{n}_1 \, dA + \iint\limits_{D} \mathbf{F} \bullet \mathbf{n}_2 \, dA$$
$$= \iint\limits_{D} (2x^3 + 2xy^2 - x^2 - y^2)\, dA + \iint\limits_{D} 4\, dA = -8\pi + 16\pi = 8\pi$$

Check that $\text{div}(\mathbf{F}) = 3x + 1$. Then

$$\iiint\limits_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = \iint\limits_{D} \int_{x^2+y^2}^{4} (3x + 1)\, dz\, dA = \iint\limits_{D} (3x + 1)(4 - x^2 - y^2)\, dA = 8\pi$$

4. (a) The boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is the unit circle in the xz-plane, oriented counterclockwise. This circle may be parameterized by $\mathbf{c}(t) = (\cos(t), 0, \sin(t))$, $0 \le t \le 2\pi$. Using Stokes' Theorem,

$$\iint\limits_{\mathcal{S}} \text{curl}(\mathbf{F}) \bullet d\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \bullet \mathbf{c}'(t)\, dt = \int_0^{2\pi} \langle 0, 0, \cos^3(t)\rangle \bullet \langle -\sin(t), 0, \cos(t)\rangle\, dt$$
$$= \int_0^{2\pi} \cos^4(t)\, dt = \frac{3\pi}{4}$$

(b) The curve $\mathcal{C}$ is the boundary $\partial\mathcal{S}$ of the surface $\mathcal{S}$ given by the portion of the plane $y + z = 2$ inside the cylinder $x^2 + y^2 = 1$, oriented up. We can use, for instance, cylindrical coordinates to parameterize $\mathcal{S}$: $\Phi(r, \theta) = (r\cos\theta, r\sin\theta, 2 - r\sin\theta)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Check that this parameterization yields the upward pointing normal $\mathbf{n} = \mathbf{T}_r \times \mathbf{T}_\theta = \langle 0, r, r\rangle$, and check that for $\mathbf{F} = \langle -y^2, x, z^2\rangle$, we have $\text{curl}(\mathbf{F}) = \langle 0, 0, 1 + 2y\rangle$. Using Stokes' Theorem,

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s} = \iint\limits_{\mathcal{S}} \text{curl}(\mathbf{F}) \bullet d\mathbf{S} = \int_0^{2\pi} \int_0^1 \text{curl}(\mathbf{F})(\Phi(r, \theta)) \bullet \mathbf{n}\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 \langle 0, 0, 1 + 2r\sin\theta\rangle \bullet \langle 0, r, r\rangle\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r + 2r^2\sin\theta)\, dr\, d\theta = \pi$$

(c) The surface $\mathcal{S}$ is the boundary $\partial\mathcal{W}$ of the solid $\mathcal{W}$ given by $0 \le z \le 1 - x - y$, $0 \le x \le 1 - y$, $0 \le y \le 1$. For $\mathbf{F} = \langle 3xy, y^2, -x^2y^4\rangle$, check that $\text{div}(\mathbf{F}) = 5y$. Using the Divergence Theorem,

$$\iint\limits_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iiint\limits_{\mathcal{W}} \text{div}(\mathbf{F})\, dV = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} 5y\, dz\, dx\, dy = \frac{5}{24}$$