Exam 2 will take place on Tuesday, March 22. It will cover section 14.8 and the first four sections of Chapter 15. (Cylindrical and spherical coordinates, used in section 15.4, are introduced in section 12.7.) Some remarks concerning this material are included below. Books, notes, calculators, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class in the next week. You may also make use of my office hours, and the free tutoring available in 141B Middleton Library (hours: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available.

Some review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. Some potentially relevant problems from the Chapter Review Exercises are:

§14.8 (p. 871) #54, 55, 57, 60 Ch. 15 (p. 937) #5–17 odd, 19, 20, 22, 26, 27, 29, 31, 33–39

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§14.8 Lagrange multipliers. To find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, find all values of x, y, z, and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$. Then evaluate f at the points (x, y, z) you found, and identify the maximum and minimum values, if any. If f has an extremum subject to the constraint $g = 0$ at $P = (x_0, y_0, z_0)$ (and $\nabla g_P \neq 0$), there is a scalar λ_0 so that $\nabla f_P = \lambda_0 \nabla g_P$. There is an analogous 2-variable version of this method, see e.g., Example 1. I will not ask you to consider problems involving two constraints (see Example 4) on the exam.

Chapter 15. The focus of this chapter is on extending the basic notions of integral calculus to functions of two or more variables.

I will try to avoid lengthy techniques of integration problems on the exam (you were presumably tested on these in Calulus II). But standard techniques such as u-substitution and integration by parts are fair game.

§15.1 Double integrals over rectangles are introduced in this section, and Fubini's Theorem for calculating these via iterated integrals is discussed. I will not ask you to work directly with the definition of a double integral (involving Riemann sums) on the exam. But you should be aware of this definition, properties of double integrals, and the geometric interpretations, related notions, and applications discussed in this chapter.

§15.2 Double integrals over simple regions may also be calculated using iterated integrals, see Theorem 2 on page 888. These are our main computational tools for dealing with double integrals, and you should be comfortable with them. Be able to (quickly) decide what is the best way to express a given region, to set up iterated integrals over vertically and horizontally simple regions, to change the order of integration if necessary (and to recognize when this is necessary), and of course to calculate iterated integrals.

§15.3 Triple integrals over boxes and more generally simple solids may also be calculated via iterated integrals, see for instance Theorem 2 on page 901. Again, be able to decide what is the best way to express a given solid, to set up and evaluate iterated integrals over simple solids. Applications of triple integrals include volume, average value (of a function of three variables), mass, center of mass...

§15.4 Integration in polar, cylindrical and spherical coordinates. If $D = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2,\ \alpha(\theta) \leq r \leq \beta(\theta)\}$ is a polar region, and f is continuous on D, you should be able to calculate $\iint_D f(x, y)\, dA$ by switching to polar coordinates, see page 912. When carrying this out, do not forget that the "polar element of area" is $r\, dr\, d\theta$. If W is a solid of the form $\theta_1 \leq \theta \leq \theta_2,\ \alpha(\theta) \leq r \leq \beta(\theta),\ z_1(r, \theta) \leq z \leq z_2(r, \theta)$, and f is continuous on W, the triple integral $\iiint_W f(x, y, z)\, dV$ may be calculated by switching to cylindrical coordinates, see page 914. In particular, the "element of volume" is $r\, dz\, dr\, d\theta$. If W is of the form $\theta_1 \leq \theta \leq \theta_2,\ \phi_1 \leq \phi \leq \phi_2,\ \rho_1(\phi, \theta) \leq \rho \leq \rho_2(\phi, \theta)$, and f is continuous on W, the triple integral $\iiint_W f(x, y, z)\, dV$ may be calculated by switching to spherical coordinates, see page 917. In particular, the "element of volume" is $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$. Be prepared to set up and evaluate multiple integrals by changing to one of these coordinate systems. For this, you will need to be able to make translations between rectangular and polar, cylindrical, and spherical coordinates. You should also develop enough experience with these regions and solids and multiple integrals to recognize when switching to polar, cylindrical, or spherical coordinates is appropriate.

Remarks.

I will expect you to know the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponentials and logarithms.

Review Problems.

1. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $z^2 - xy = 1$.
2. Let D be the region in the first quadrant bounded by $x = y^2$, $y = 0$, and $x = 1$. Evaluate $\displaystyle\iint_D y\sin(\pi x^2)\,dA$.
3. Use a double integral to find the area of the region inside one loop of the three-leaved rose $r = \cos(3\theta)$.
4. A plane lamina occupies the region D in the xy-plane bounded by the $x = 1$, $y = 1$, and $xy = 2$. The density at the point (x, y) in this region is given by $\rho(x, y) = 4xy$. Find the center of mass of the lamina.
5. Consider the solid bounded by the surface $z = 1 - y^2$ and the planes $x = 0$, $z = 0$, $y = 1$, and $y = x$.
   (a) Make a beautiful sketch of this solid.
   (b) Use a double integral to find the volume of this solid.
6. Use cylindrical or spherical coordinates, whichever is most appropriate.
   (a) Let E be the solid in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, and the plane $z = 2$. Evaluate $\iiint_E xyz\,dV$.
   (b) Find the volume common to the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + (z - 2)^2 = 4$

Selected Review Problem Answers.

1. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $z^2 - xy = 1$.

   The system of equations one must solve, $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$, $g(x, y, z) = k$, is:
   $$2x = -\lambda y, \quad 2y = -\lambda x, \quad 2z = 2\lambda z, \quad z^2 - xy = 1.$$
   From the third of these equations, either $z = 0$ or $\lambda = 1$.

   If $z = 0$, then $y = -1/x$ (from the fourth equation). Use this in the first two equations to see that $x^4 = 1$, so $x = 1$ or $x = -1$. If $x = 1$, then $y = -1$. If $x = -1$, then $y = 1$. So we must consider the points $(1, -1, 0)$ and $(-1, 1, 0)$.

   If $\lambda = 1$, use the first two equations to show that $x = 0$ and $y = 0$. Then the fourth equation reads $z^2 = 1$. So $z = 1$ or $z = -1$, and we must consider the points $(0, 0, 1)$ and $(0, 0, -1)$.

   $f(1, -1, 0) = f(-1, 1, 0) = 2$ is the maximum, and $f(0, 0, 1) = f(0, 0, -1) = 1$ is the minimum

2. Let D be the region in the first quadrant bounded by $x = y^2$, $y = 0$, and $x = 1$. Evaluate $\displaystyle\iint_D y\sin(\pi x^2)\,dA$.

$D = \{(x, y) \mid y^2 \le x \le 1,\ 0 \le y \le 1\} = \{(x, y) \mid 0 \le y \le \sqrt{x},\ 0 \le x \le 1\}$

$$\iint_D y\sin(\pi x^2)\,dA = \int_0^1 \int_{y^2}^1 y\sin(\pi x^2)\,dx\,dy = \int_0^1 \int_0^{\sqrt{x}} y\sin(\pi x^2)\,dy\,dx = \frac{1}{2\pi}$$

Evaluate the second of the two iterated integrals above (the one on the right).

3. Use a double integral to find the area of the region inside one loop of the three-leaved rose $r = \cos(3\theta)$.
Let D be the region inside the loop of $r = \cos(3\theta)$ that lies in the first and fourth quadrants (see below).
This region is given by $D = \{(r, \theta) \mid 0 \le r \le \cos(3\theta), -\pi/6 \le \theta \le \pi/6\}$. The area of D is $A = \displaystyle\iint_D 1\,dA$.

Using polar coordinates, we have $A = \displaystyle\int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r\,dr\,d\theta = \frac{\pi}{12}$

4. A plane lamina occupies the region D in the xy-plane bounded by the $x = 1$, $y = 1$, and $xy = 2$. The density at the point (x, y) in this region is given by $\rho(x, y) = 4xy$. Find the center of mass of the lamina.

The mass is $m = \displaystyle\int_1^2 \int_1^{2/x} 4xy\,dy\,dx = 8\ln(2) - 3$. The center of mass is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_1^2 \int_1^{2/x} 4x^2y\,dy\,dx = \frac{10}{3m} \text{ and } \bar{y} = \frac{1}{m}\int_1^2 \int_1^{2/x} 4xy^2\,dy\,dx = \frac{10}{3m}$$

5. Consider the solid bounded by the surface $z = 1 - y^2$ and the planes $x = 0$, $z = 0$, and $y = x$.
   (a) Make a beautiful sketch of this solid. See below.
   (b) Use a double integral to find the volume of this solid.

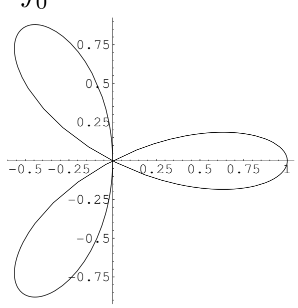
$$V = \int_0^1 \int_x^1 (1 - y^2)\,dy\,dx = \int_0^1 \int_0^y (1 - y^2)\,dx\,dy = \frac{1}{4}$$

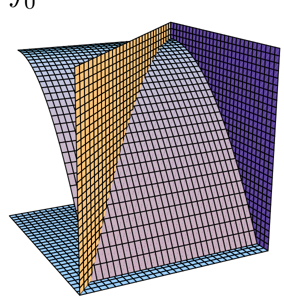
6. Use cylindrical or spherical coordinates...

(a) $$\int_0^{\pi/2} \int_0^2 \int_r^2 r^3\, z\, \sin\theta\, \cos\theta\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2/\cos\phi} \rho^5\, \sin^3\phi\, \cos\phi\, \sin\theta\, \cos\theta\,d\rho\,d\phi\,d\theta = \frac{4}{3}$$

(b) $$\int_0^{2\pi} \int_0^{3\sqrt{7}/4} \int_{2-\sqrt{4-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_{3\sqrt{7}/4}^2 \int_{2-\sqrt{4-r^2}}^{2+\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \frac{63\pi}{8}$$

$$\int_0^{2\pi} \int_0^{\cos^{-1}(3/4)} \int_0^3 \rho^2\, \sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi} \int_{\cos^{-1}(3/4)}^{\pi/2} \int_0^{4\cos\phi} \rho^2\, \sin\phi\,d\rho\,d\phi\,d\theta = \frac{63\pi}{8}$$





Many additional problems may be found in the Chapter 14 and 15 Review Exercises, and in the WeBWorK assignments. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, log in with Username: `aaaaaa` and Password: `123456789`.