Exam 3 will take place on Thursday, April 28. It will cover §15.5, Chapter 16, and §17.1. Some remarks concerning this material are included below. Books, notes, calculators, etc. may **not** be used on the exam.

If you have questions regarding this material, be ready to ask them in class in the next week. You may also make use of my office hours, and the free tutoring available in 141B Middleton Library (hours: M–Th 10:00–7:00, F 10:00–3:00). I don't know when people capable of tutoring for MATH 2057 are available.

Some review problems are included on the next page. This is **not** a comprehensive list. Additional problems may be found in the Exercises of the sections we've covered, and in the WeBWorK assignments. Some potentially relevant problems from the Chapter Review Exercises are:

§15.5 (p. 937) #43, 45
Ch. 16 (p. 1001) #9, 13, 15, 19, 23, 25, 27, 37, 41, 43
§17.1 (p. 1043) #1–6

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§15.5 Change of variables. If $\Phi(u,v) = \big(x(u,v), y(u,v)\big)$ is a transformation that maps a region $D_0$ in the $uv$-plane to a region $D$ in the $xy$-plane and $f$ is continuous on $R$, then under certain conditions (see page 929), $\displaystyle\iint_D f(x,y)\,dA = \iint_{D_0} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$, where $\displaystyle\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}$ is the Jacobian of $\Phi$. Given a transformation, you should be able to use this to evaluate double integrals.

Chapter 16. The focus of this chapter is on the calculus of vector fields, including line integrals.

§16.1 This section introduces vector fields, functions that assign to each point in some domain a vector. The gradient of a function of two or three variables is a familiar example. As you should know, a vector field $\mathbf{F}$ is a gradient vector field if $\mathbf{F} = \nabla\phi$ for some function $\phi$. In this situation, $\phi$ is a "potential function" for $\mathbf{F}$.

§16.2 Line integrals in the plane and in space are introduced in this section. Given a parameterized curve and a function or vector field, you should be able to evaluate a relevant line integral by expressing it in terms of the parameter. (You should be able to parameterize certain curves, such as line segments, portions of circles, etc., yourself.) Applications of line integrals include mass, center of mass, work...

§16.3 This section addresses a number of conceptual aspects of line integrals. These include the fundamental theorem for line integrals involving gradient vector fields: Briefly, if $\mathbf{F} = \nabla\phi$ is a gradient vector field that is continuous on a neighborhood of a curve $C$ with initial point $P$ and terminal point $Q$, then $\int_C \mathbf{F} \bullet d\mathbf{s} = \phi(Q) - \phi(P)$. The notion of independence of path is introduced in this section, and conditions (on the vector field $\mathbf{F}$ and the region $D$ on which it is defined) which insure that that $\mathbf{F}$ is independent of path in $D$ (resp., conservative on $D$) are discussed. I will expect you to understand these notions and conditions; to be able to draw conclusions from them; to be able to find a potential function when appropriate; to understand when the fundamental theorem may be used to evaluate line integrals, etc.

§16.4 and §16.5 These sections introduce surface integrals of functions and vector fields over parameterized surfaces, see page 982 and page 992. You should be prepared to find parameterizations of surfaces that are graphs of functions, that are naturally expressed in cylindrical or spherical coordinates, etc., to determine the normal vector from the parameterization, and to compute surface integrals using these.

§17.1 Green's Theorem is discussed in this section. I will expect you to thoroughly know the statement(s) of this theorem (the precise conditions when it applies, and the conclusion), and be able to use this theorem in various ways. The punchlines of this theorem are: $\displaystyle\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ and $\displaystyle\oint_{\partial D} \mathbf{F} \bullet d\mathbf{s} = \iint_D \operatorname{curl}(\mathbf{F}) \bullet \mathbf{k}\,dA$ (you should state the necessary conditions on $P$, $Q$, $\partial D$, $D$, and $\mathbf{F}$, as well as identify $\operatorname{curl}(\mathbf{F}) \bullet \mathbf{k} = \operatorname{curl}_z(\mathbf{F})$). For instance, the first of these versions may be used to trade a line integral for double integral (and visa versa), to compute area using line integrals, etc.

Remarks.

I will expect you to know the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, and related identities. If necessary, I will give you other identities, such as half-angle and double-angle formulas. I will also expect you to know the values of the trigonometric functions at "standard angles" such as those given below (in radians).

| $\theta$ | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $1/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | $1$ |
| $\cos\theta$ | $1$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $0$ |

From these, and the graphs of the sine and cosine functions, with which you should also be intimately acquainted, you can determine the values of the trigonometric functions at many other "standard angles."

I will also expect you to know basic properties of exponential and logarithm functions.

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Review Problems.

1. Let $S$ be the square in the $uv$-plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Let $\Phi$ be the transformation given by $x = 3u - v$ and $y = u + v$. Let $R$ be the image in the $xy$-plane of $S$ under this transformation. Sketch the region $R$, compute the Jacobian of $\Phi$, and use this transformation to evaluate the integral $\iint_R (x^2 - y^2)\, dA$.

2. Sketch the vector field $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j} = \langle y, x^2\rangle$. Is $\mathbf{F}$ a gradient vector field? Explain.

3. Let $C$ be the curve parameterized by $\mathbf{c}(t) = 6t\,\mathbf{i} + 3\sqrt{2}t^2\,\mathbf{j} + 2t^3\,\mathbf{k}$, $0 \le t \le 1$.
   (a) Find the mass of a thin wire in the shape of $C$ if $\rho(x,y,z) = xz$ is the density at the point $(x,y,z)$.
   (b) Find the work done by the vector (force) field $\mathbf{F} = z\,\mathbf{i} + (y/x)\,\mathbf{j} - \mathbf{k}$ in moving an object along $C$.

4. Consider the vector field $\mathbf{F} = \langle yz(2x+y), xz(x+2y), xy(x+y)\rangle$ defined on all of $\mathbb{R}^3$.
   (a) What does "$\mathbf{F}$ is conservative" mean? Is the vector field $\mathbf{F}$ given above conservative (on $\mathbb{R}^3$)?
   (b) What does "$\int_C \mathbf{F} \bullet d\mathbf{s}$ is independent of path" mean? Are line integrals involving the vector field $\mathbf{F}$ given above independent of path (on $\mathbb{R}^3$)?
   (c) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{s}$ for the curve $C$ given by $\mathbf{c}(t) = \langle 1+t, 1+2t^2, 1+3t^2\rangle$, $0 \le t \le 1$.

5. Let $\mathcal{S}$ be the portion of the cone $z = \sqrt{x^2+y^2}$ which lies below the plane $z = 3$.
   (a) If $f(x,y,z) = x^2z^2$, evaluate the surface integral $\displaystyle\iint_{\mathcal{S}} f(x,y,z)\, dS$.
   (b) If $\mathbf{F} = \langle x, y, z^4\rangle$ and $S$ is oriented down, evaluate the surface integral $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$.

6. State Green's Theorem.
   (a) Use Green's Theorem to find the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$,
   (b) Use Green's Theorem to show that the area of a domain $D$ is equal to the line integral $\oint_{\partial D} x\, dy$. Use this to find the area inside the hypocycloid parameterized by $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le 2\pi$.
   Evaluate the following in two ways: (i) directly; and (ii) using Green's Theorem
   (c) $\oint_C x^2y\, dx + x\, dy$, where $C$ is the triangle with vertices $(0,0)$, $(1,2)$, $(0,2)$ oriented counterclockwise.
   (d) $\oint_C x^2y\, dx + x\, dy$, where $C$ is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Many additional problems may be found in the Chapters 15, 16, and 17 Review Exercises, and in the WeBWorK assignments. If you would like to try problems that are numerically different, but conceptually the same as the problems you've done in WeBWorK, log in with Username: `aaabbb` and Password: `123456789`.

Selected Review Problem Answers.

1. The image of the square $S$ under the transformation $\Phi$ is the rectangle $R$ in the $xy$-plane with vertices $(0,0)$, $(3,1)$, $(2,2)$, and $(-1,1)$.

   The Jacobian of $\Phi$ is 4.

   $$\iint\limits_R (x^2-y^2)dx\,dy = \iint\limits_S \big((3u-v)^2-(u+v)^2\big)4\,du\,dv = 4\int_0^1\int_0^1 (8u^2-8uv)du\,dv = \frac{8}{3}$$

2. $\mathbf{F} = \langle F_1, F_2\rangle = \langle y, x^2\rangle$ is not a gradient vector field since $\dfrac{\partial F_1}{\partial y} = 1 \neq 2x = \dfrac{\partial F_2}{\partial x}$.

3. If $\mathbf{c}(t) = 6t\,\mathbf{i} + 3\sqrt{2}t^2\,\mathbf{j} + 2t^3\,\mathbf{k}$, then $\mathbf{c}'(t) = 6\mathbf{i} + 6\sqrt{2}t\mathbf{j} + 6t^2\mathbf{k}$ and $|\mathbf{c}'(t)| = \sqrt{36+72t^2+36t^4} = 6(1+t^2)$.

   (a) $$m = \int_C \rho(x,y,z)\,ds = \int_C \rho(\mathbf{c}(t))|\mathbf{c}'(t)|\,dt = \int_0^1 (6t)(2t^3)\big(6(1+t^2)\big)dt = 72\int_0^1 (t^4+t^6) = \frac{72}{5}+\frac{72}{7}$$

   (b) $$W = \int_C \mathbf{F}\bullet d\mathbf{s} = \int_0^1 \langle 2t^3, (3\sqrt{2}t^2)/(6t), -1\rangle \bullet \langle 6, 6\sqrt{2}t, 6t^2\rangle dt = \int_0^1 12t^3\,dt = 3$$

4. (a) This vector field is conservative (on $\mathbb{R}^3$): for $\phi(x,y,z) = x^2yz + xy^2z$, $\mathbf{F} = \nabla\phi$, so $\mathbf{F}$ is a gradient field
   (b) Line integrals involving this vector field **are** independent of path (on $\mathbb{R}^3$)
   (c) $\int_C \mathbf{F}\bullet d\mathbf{s} = \phi(\mathbf{c}(1)) - \phi(\mathbf{c}(0)) = 118$

5. One parameterization of $\mathcal{S}$ is $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$. For this parameterization, the normal vector $\mathbf{n} = T_\theta \times T_r = \langle r\cos\theta, r\sin\theta, -r\rangle$ points down, and $|\mathbf{n}| = \sqrt{2}r$.

   (a) $$\iint\limits_{\mathcal{S}} x^2z^2\,dS = \int_0^{2\pi}\int_0^3 r^2\cos^2\theta\cdot r^2\cdot\sqrt{2}r\,dr\,d\theta = \frac{243\sqrt{2}\pi}{2}$$

   (b) Note that $\mathbf{F}\bullet\mathbf{n} = \langle r\cos\theta, r\sin\theta, r^4\rangle \bullet \langle r\cos\theta, r\sin\theta, -r\rangle = r^2 - r^5$.
   $$\iint\limits_{\mathcal{S}} \mathbf{F}\bullet d\mathbf{S} = \int_0^{2\pi}\int_0^3 (r^2-r^5)\,dr\,d\theta = -225\pi$$

6. (a) The ellipse $x^2/a^2 + y^2/b^2 = 1$ can be parameterized by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.
   If the region inside the ellipse is $D$, then
   $$\text{area}(D) = \iint\limits_D 1\,dA = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi$$

   (b) If $D$ is the region inside the hypocycloid, then
   $$\text{area}(D) = \iint\limits_D 1\,dA = \oint_{\partial D} x\,dy = \int_0^{2\pi} a\cos^3 t\cdot 3a\sin^2 t\cos t\,dt = \frac{3a^2\pi}{8}$$

   (c) $$\oint_C x^2y\,dx + x\,dy = \iint\limits_D (1-x^2)\,dA = \frac{5}{6} \quad \text{here } D = \{(x,y) \mid 2x \le y \le 2,\ 0 \le x \le 1\}$$

   (d) $$\oint_C x^2y\,dx + x\,dy = \iint\limits_D (1-x^2)\,dA = 0 \quad \text{here } D = \{(x,y) \mid x^2+y^2 \le 4\}$$