LECTURE 1
Equivariant Homology and Intersection Homology

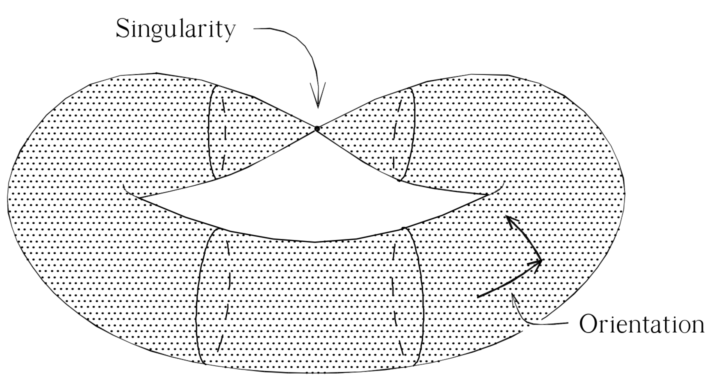
(Geometry of Pseudomanifolds)

1.1. Introduction

⟨**1.1**⟩ In this lecture, we will give a geometric way of defining equivariant homology and equivariant intersection homology. The standard definitions of these homology theories, as found in the literature, are good for proving properties, but are perhaps not so intuitive. In this lecture, we will consider $G \circlearrowright X$: an action of a general Lie group G on a space X, although in the other lectures we are interested mainly in the case that G is a torus T.

⟨**1.2**⟩ The definitions we present are based on the notion of a pseudomanifold. A k-dimensional manifold is a space that looks locally like k-dimensional Euclidean space near every point. A k-dimensional pseudomanifold P is allowed to have *singularities*, i.e. points where it doesn't locally look like Euclidean space. However, it must satisfy two properties:

(1) The part of P where it is a k-manifold is open and dense in P and it must be oriented.
(2) The set of singularities has dimension at most $k-2$ (i.e. codimension at least 2).



A pseudomanifold (the pinched torus)

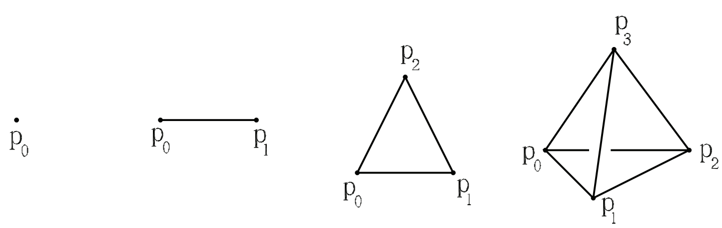
There are several ways to make this intuitive notion of a pseudomanifold rigorous. We will use simplicial complexes, because that is the one most in keeping with the spirit of these notes. Readers who are comfortable with pseudomanifolds can skip directly to §1.4

⟨**1.3**⟩ Equivariant homology theories are difficult to compute directly from the definitions as given in this Lecture. However, the methods of Lectures 3 to 5 provide effective computations in many interesting cases.

1.2. Simplicial Complexes

Readers familiar with simplicial complexes and orientations can skip this section.

⟨**2.1**⟩ A *k-simplex* Δ is the convex hull of $k+1$ points $p_0, \dots, p_k$ in general position in some Euclidean space. Here general position just means that the points don't all lie in any $(k-1)$-dimensional Euclidean subspace. The k-simplex is a polyhedron. Its faces are themselves simplices; they are the convex hulls of subsets of the points p_i. The points p_i are the vertices of Δ.



0-simplex 1-simplex 2-simplex 3-simplex

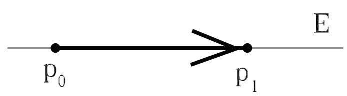
⟨**2.2**⟩ **Definition.** A *simplicial complex* is a set S of simplices in some Euclidean space with the properties

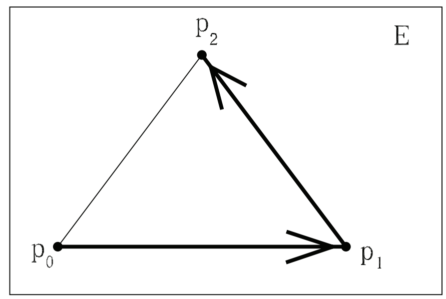
(1) Any two simplices in S are either disjoint or intersect in a set that is a face of each of them.
(2) Any face of a simplex in S is itself in S.

⟨**2.3**⟩ **Spaces of finite type.** We define a *space of finite type* to be a topological space homeomorphic to the difference $S - S'$ where S is a simplicial complex and S' is a sub simplicial complex. We will assume without further mention that all of our spaces are of finite type. Spaces generated by finite operations, such as real algebraic varieties, or their images under algebraic maps, are all of finite type (although proving it takes technology developed over many years). On the other hand, a Cantor set, or $\mathbb{Z}$ are not of finite type.

⟨**2.4**⟩ An *orientation* $\mathbb{O}$ of a simplex Δ is an ordering of the vertices of Δ, two orderings being considered equivalent if one is an even permutation of the other. (This definition doesn't work for a 0-simplex. An orientation of a 0-simplex is simply one of the symbols $+$ or $-$.) Any simplex has exactly two orientations, these two orientations are called opposite orientations of each other.

⟨**2.5**⟩ An orientation of a Euclidean space is an ordered set of basis vectors, two being considered equivalent if one is a continuous deformation of the other. We can draw an orientation by representing the basis vectors as arrows, and signaling the ordering by placing the tail of each arrow at the head of the previous one. An orientation $\mathbb{O}$ of k-simplex Δ determines an orientation of the k-dimensional Euclidean space E containing Δ as follows: Suppose $\mathbb{O} = \{p_0 < p_1 < \cdots < p_k\}$. Then $\{p_1 - p_0, p_2 - p_1, \ldots, p_k - p_{k-1}\}$ is the ordered basis.





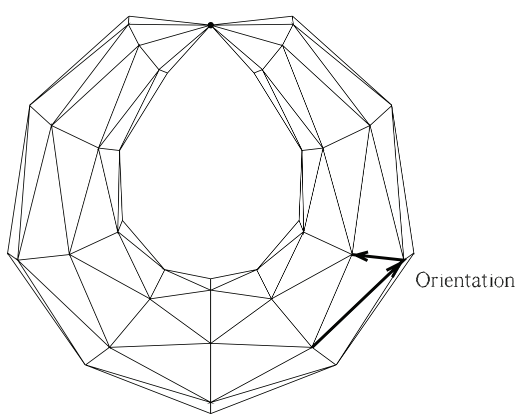
Exercise. Show that two orientations of Δ are equivalent if and only if they determine equivalent ordered bases of E.

⟨**2.6**⟩ If Δ is a k-simplex and Δ' is a $(k-1)$-simplex, an orientation $\mathbb{O}$ of Δ induces an orientation $\mathbb{O}'$ of Δ' as follows: Pick an equivalent ordering such that the unique vertex of Δ not in Δ' is the last one of the ordering. Then $\mathbb{O}'$ is the restriction of that ordering to Δ'. (This definition doesn't work if Δ is a 1-simplex. In this case, $\mathbb{O}'$ is $-$ if Δ' is the first vertex of the ordering, and it is $+$ if it is the second one.)

1.3. Pseudomanifolds

⟨**3.1**⟩ **Definition.** A *k-dimensional pseudomanifold* is a simplicial complex together with an orientation $\mathbb{O}(\Delta)$ of each of its k-simplices, with the following properties:

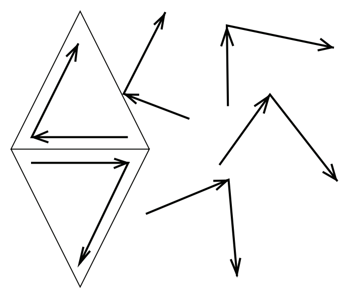
(1) Every simplex is a face of some k-simplex.
(2) Every $(k-1)$-simplex is the face of exactly two k-simplices.
(3) *(The continuity of orientation property)* If Δ' is a $(k-1)$-simplex and Δ and $\tilde{\Delta}$ are the two k-simplices that contain Δ' in their boundary, then the given orientations $\mathbb{O}(\Delta)$ and $\mathbb{O}(\tilde{\Delta})$ induce opposite orientations on Δ'.



A pseudomanifold (the simplicial pinched torus)

⟨**3.2**⟩ The following exercise shows why property 3 is called continuity of orientation.

Exercise. Suppose that Δ and $\tilde{\Delta}$ are two k-simplices in a Euclidean k-space, and that they intersect in a $(k-1)$-simplex Δ'. Show that the orientations $\mathbb{O}(\Delta)$ and $\mathbb{O}(\tilde{\Delta})$ induce opposite orientations on Δ' if and only if the ordered basis for E determined by Δ as in exercise 1.2.5 can be continuously deformed into the ordered basis for E determined by $\mathbb{O}(\tilde{\Delta})$.



A path in the space of ordered bases of the plane

⟨**3.3**⟩ **Definition.** A *k-dimensional pseudomanifold with boundary* is a simplicial complex S, an orientation $\mathbb{O}(\Delta)$ of each of its k-simplices, and a sub simplicial complex B called the *boundary*, with the following properties:

(1) The boundary B is a $(k-1)$-dimensional pseudomanifold
(2) Every simplex of S is a face of some k-simplex.
(3) Every $(k-1)$-simplex Δ' that not in B is the face of exactly two k-simplices, and the continuity of orientation property holds for Δ'.
(4) Every $(k-1)$-simplex Δ' in B is the face of exactly one k-simplex Δ in S. The orientation of Δ' induced from $\mathbb{O}(\Delta)$ coincides with the orientation $\mathbb{O}(\Delta')$ of Δ' from the pseudomanifold structure on B.

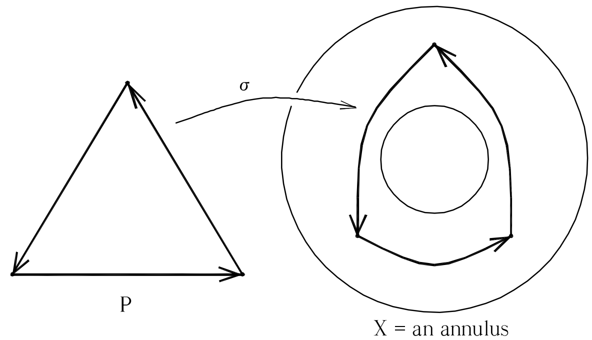
⟨**3.4**⟩ **Exercise.** Show that the continuity of orientation property for the boundary B of a pseudomanifold with boundary follows from the other properties in the definition.

1.4. Ordinary Homology Theory

As a warm up, we will give a definition of ordinary homology theory in the spirit of the definitions of more complicated theories to come. This definition of ordinary homology has roots going back to Poincaré and Veblen and the earliest days of homology theory.

⟨**4.1**⟩ **Definition.** Let X be a topological space. An *i-cycle* is an i-dimensional pseudomanifold P together with a map $\sigma : P \longrightarrow X$.

The idea is that an i-cycle captures the "holes" in a topological space by surrounding them. For example, the following 1-cycle surrounds the hole in the annulus:

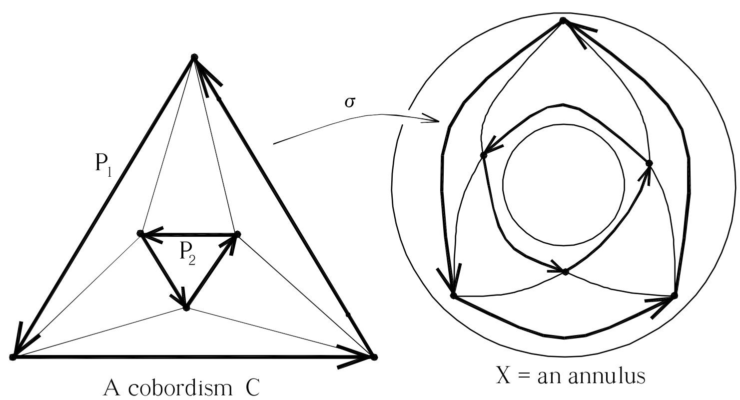


We will refer to the i-cycle $\sigma : P \longrightarrow X$ by the symbol P when there's no confusion about what σ is.

⟨4.2⟩ Definition. If $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ are two i-cycles, then the *sum* $P_1 + P_2$ is their union $\sigma : P_1 \cup P_2 \longrightarrow X$ where $\sigma|P_1 = \sigma_1$ and $\sigma|P_2 = \sigma_2$.

The *negative* $-P$ of an i-cycle is the same i-cycle P with the opposite orientation for every i-simplex. As usual, $P_1 - P_2$ is $P_1 + (-P_2)$.

⟨4.3⟩ Definition. A *cobordism* between two i-cycles $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ is a $(i+1)$-dimensional pseudomanifold with boundary C, and a map $\sigma : C \longrightarrow X$ such that the boundary B of C is $P_1 - P_2$ and the restriction of σ to B coincides with σ_1 and σ_2. Two i-cycles $\sigma_1 : P_1 \longrightarrow X$ and $\sigma_2 : P_2 \longrightarrow X$ are said to be *cobordant* if there is a cobordism between them.



The 1-cycles P_1 and P_2 are cobordant

The idea behind this definition is that if P_1 and P_2 are cobordant, they surround the same holes in the same way. For example, if σ has appropriate differentiability assumptions so that it makes sense, any closed differential i-form will have the same integral on P_1 and on P_2, by Stokes' Theorem.

⟨4.4⟩ Proposition. Cobordism is an equivalence relation among i-cycles.

Exercise Prove this. For example, if S_1 is a cobordism between P_1 and P_2, and S_2 is a cobordism between P_2 and P_3, then S_1 and S_2 can be glued together to provide a cobordism between P_1 and P_3.

⟨**4.5**⟩ **Definition – Proposition.** The i-th homology, notated $H_i(X)$, is the set of cobordism classes of i-cycles. The operations $+$ and $-$ induce the structure of an Abelian group on this set. The identity element is represented by the empty pseudomanifold.

⟨**4.6**⟩ For example, if X is the annulus, $H_0(X)$ is $\mathbb{Z}$ generated by a point, and $H_1(X)$ is $\mathbb{Z}$ generated by the cycle P_1 or P_2 as in the pictures above.

⟨**4.7**⟩ **Exercise.** Show for any X that $H_0(X)$ is $\mathbb{Z}^k$ where k is the number of path connected components of X.

⟨**4.8**⟩ **Convention.** We write $H_*(X)$ for $\bigoplus_i H_i(X)$. It's a summation convention: wherever a star appears, it means a direct sum over the possible indices i that might appear there.

1.5. Basic Definitions of Equivariant Topology

⟨**5.1**⟩ A *topological group* G is a set that is simultaneously a group and a topological space, with the property that the multiplication operation $G \times G \longrightarrow G$ and the inverse operation $G \longrightarrow G$ are both continuous. G is a Lie group if it is one of our spaces of finite type §1.2.3 (or, what turns out to be the same thing for topological groups, if it's a topological manifold with finitely many connected components.)

⟨**5.2**⟩ A *space with a group action* $G \circlearrowright X$ is a topological space X (which for us will always be of finite type), and an a map $G \times X \longrightarrow X$ that is continuous, such that $(g \cdot g')x = g(g'(x))$ (§0.1.5).

⟨**5.3**⟩ **The equivariant category.** Suppose $G \circlearrowright X$ and $G' \circlearrowright X'$ are two topological spaces with a group action. A morphism $G \circlearrowright X \Longrightarrow G' \circlearrowright X'$ is a continuous group homomorphism $\phi : G \longrightarrow G'$ together with a continuous map $\psi : X \longrightarrow X'$ such that $\psi(gx) = \phi(g)\psi(x)$.

For example, for any $G \circlearrowright X$ there is a canonical morphism $G \circlearrowright X \Longrightarrow 1 \circlearrowright X/G$. Here 1 is the one element group; X/G is the quotient space $X/\sim$ where $\sim$ is the equivalence relation $x \sim x'$ if there is a $g \in G$ such that $gx = x'$; $\phi : G \longrightarrow 1$ is the only thing it could be; and $\phi : X \longrightarrow X/G$ is the quotient map.

⟨**5.4**⟩ G **equivariant maps.** If G is a fixed group, then *the category of G-spaces* is the sub category of the equivariant category where the map ϕ on G is the identity. The maps in this category are called G *equivariant maps*. In other words, if X and X' are both G-spaces, then an equivariant map from X to X' is a continuous map $\psi : X \longrightarrow X'$ such that $\psi(gx) = g\psi(x)$ for all $g \in G$ and all $x \in X$. We consider two G spaces *equivalent* if they are isomorphic in this category. This means that the map ψ is a homeomorphism.

⟨**5.5**⟩ **Free actions.** An action of G on X is *free* if no element of G except the identity fixes any point in X, i.e. $gx = x$ implies g is the identity.

Another commonly used terminology for the same thing is this: The map $\pi : X \longrightarrow X/G$ is called a *principal G-bundle* if and only if the action $G \circlearrowright X$ is free. In this terminology, X/G is called the *base* of the principal bundle; X is called the total space; and π is called the projection.

Yet another popular terminology is to say that X is a *G-torsor* over X/G.