

On representations and K -theory of the braid groups

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Received: 6 February 2002 / Revised version: 19 September 2002 /
Published online: 8 April 2003 – © Springer-Verlag 2003

Abstract. Let Γ be the fundamental group of the complement of a $K(\Gamma, 1)$ hyperplane arrangement (such as Artin's pure braid group) or more generally a *homologically toroidal* group as defined below.

The triviality of bundles arising from orthogonal representations of Γ is characterized completely as follows. An orthogonal representation gives rise to a trivial bundle if and only if the representation factors through the spinor groups. Furthermore, the subgroup of elements in the complex K -theory of $B\Gamma$ which arises from complex unitary representations of Γ is shown to be trivial. In the case of real K -theory, the subgroup of elements which arises from real orthogonal representations of Γ is an elementary abelian 2-group, which is characterized completely in terms of the first two Stiefel-Whitney classes of the representation.

In addition, quadratic relations in the cohomology algebra of the pure braid groups which correspond precisely to the Jacobi identity for certain choices of Poisson algebras are shown to give the existence of certain homomorphisms from the pure braid group to generalized Heisenberg groups. These cohomology relations correspond to non-trivial Spin representations of the pure braid groups which give rise to trivial bundles.

Mathematics Subject Classification (2000): 20F36, 32S22, 55N15, 55R50

1. Introduction

Given a discrete group Γ , consider the set of homomorphisms $\text{Rep}(\Gamma, G)$ where G denotes either the real orthogonal group $O(n)$ or the complex unitary group $U(n)$. There is a natural map of sets $\text{Rep}(\Gamma, G) \rightarrow [B\Gamma, BG]$, where the target

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*Partially supported by the NSF

**Partially supported by grant LEQSF(1999-02)-RD-A-01 from the Louisiana Board of Regents, and by grant MDA904-00-1-0038 from the National Security Agency

is the set of *pointed* homotopy classes of maps from one classifying space to the other. It is natural to ask if an element in the image of this map is trivial, or equivalently, if the natural vector bundle associated to such a representation is a trivial bundle. This question is addressed here for a certain class of discrete groups, which includes Artin's pure braid groups. For these groups, the bundle associated to an orthogonal representation is trivial if and only if the representation factors through the natural composite $Spin(n) \rightarrow SO(n) \rightarrow O(n)$ as given in Theorem 1.4. Such a representation will be referred to as a *Spin representation*.

More generally, it seems natural to compare bundles arising from representations, regardless of whether they are trivial or not. However, there is not a natural way to form differences of bundles, as the set $[B\Gamma, BG]$ is not, in general, a group. Thus the problem of comparing bundles arising from representations requires additional structure; the approach adopted here will be to use classical topological K -theory.

If G is the complex unitary group $U(n)$, there is a stabilization map $[B\Gamma, BU(n)] \rightarrow [B\Gamma, BU]$, which gives rise to a natural map $\text{Rep}(\Gamma, U(n)) \rightarrow [B\Gamma, BU] = KU^0(B\Gamma)$. Similarly, there is a natural map $\text{Rep}(\Gamma, O(n)) \rightarrow [B\Gamma, BO] = KO^0(B\Gamma)$. Note that *reduced* K -theory is used here.

These natural maps motivate the following.

Definition 1.1. *The groups $KU_{\text{rep}}^0(B\Gamma)$ and $KO_{\text{rep}}^0(B\Gamma)$ are defined to be the subgroups of $KU^0(B\Gamma)$ and $KO^0(B\Gamma)$ generated by the images, for all $n \geq 1$, of the maps*

$$\text{Rep}(\Gamma, U(n)) \rightarrow [B\Gamma, BU] \quad \text{and} \quad \text{Rep}(\Gamma, O(n)) \rightarrow [B\Gamma, BO].$$

The purpose of this article is to study these maps, together with related constructions, for the following class of discrete groups.

Definition 1.2. *A discrete group Γ is said to be homologically toroidal if there is a homomorphism $\mathcal{F} \rightarrow \Gamma$ inducing a split epimorphism in integral homology, where \mathcal{F} is a finite free product of free abelian groups of finite rank.*

Similarly, a topological space X is said to be homologically toroidal if there is a continuous map $\mathcal{T} \rightarrow X$ inducing a split epimorphism in integral homology, where \mathcal{T} is a finite bouquet of finite dimensional tori.

Artin's pure braid groups are examples of homologically toroidal groups. More generally, this class of groups includes the fundamental groups of complements of complex hyperplane arrangements which are aspherical. Let \mathcal{A} be a hyperplane arrangement, a finite collection of codimension one affine subspaces in \mathbb{C}^ℓ , with complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. Call \mathcal{A} aspherical, or a $K(\Gamma, 1)$ arrangement, if $M(\mathcal{A})$ is an Eilenberg-Mac Lane space of type $K(\Gamma, 1)$. Examples include the orbit configuration spaces associated to free actions of finite cyclic groups on \mathbb{C}^* , see [9, 27].

Note that if Γ is a homologically toroidal group, then the classifying space $B\Gamma$ is a homologically toroidal space. However, the fundamental group of a homologically toroidal space need not be a homologically toroidal group. For instance, the complement of any complex hyperplane arrangement is a homologically toroidal space, see Proposition 2.6. But there are arrangements for which the homology of the fundamental group of the complement, unlike that of the complement itself, is not finitely generated, see [2, 24].

The relationships between representations of homologically toroidal groups, the bundles associated to them, and the real or complex K -theory of their classifying spaces are studied in this article. In addition to complements of $K(\Gamma, 1)$ arrangements, including the orbit configuration spaces mentioned previously, some of the results here apply to the fundamental groups of orbit configuration spaces associated to elliptic curves, although these groups (if non-trivial) are not homologically toroidal as their homology groups are not finitely generated. These orbit configuration spaces are studied in [12, 14].

The organization of this paper is as follows:

- (1) Theorem 1.3 gives information characterizing bundles which are stably trivial, and provides the starting point for the work presented here. The proof of Theorem 1.3 is given in §4, for which sections §2 and §3 serve as technical preparation.
- (2) Theorem 1.4 gives the unstable result that the bundle associated to an $O(n)$ representation is trivial if and only if the representation lifts to $Spin(n)$. This theorem may be regarded as a “destabilization” of Theorem 1.3. The proof of Theorem 1.4 is given in §4.
- (3) Theorem 1.5 assembles all of the stable bundles which arise from representations thus allowing a comparison of bundles arising from different representations, and identifies the entire group generated by representations in terms of characteristic classes. The proof of Theorem 1.5 is given in §4.
- (4) Theorem 1.6 gives an equivalent formulation for the *three term relations* in the cohomology algebra of the configuration space of points in the plane in terms of certain representations involving *generalized Heisenberg groups*. The proof of Theorem 1.6 is given in §5.
- (5) Propositions 1.7, and 1.8 provide some direct information about related bundles obtained from representations of the full braid groups. The proofs of these propositions are given in §6.

The first theorem below concerns the group structure as discussed above; a stronger result obtained for individual bundles is stated immediately after it.

Theorem 1.3. *Let Γ be a homologically toroidal group.*

1. If $g : \Gamma \rightarrow U(n)$ is a group homomorphism, then the composite map

$$B\Gamma \rightarrow BU(n) \rightarrow BU$$

is null-homotopic. Thus $KU_{\text{rep}}^0(B\Gamma)$ is the trivial group.

2. If $g : \Gamma \rightarrow O(n)$ is a group homomorphism, then the composite map

$$B\Gamma \rightarrow BO(n) \rightarrow BO$$

is essential (not null-homotopic) if and only if at least one of the first two Stiefel-Whitney classes of the representation is non-zero. If the first two Stiefel-Whitney classes of g vanish, then the natural real n -plane bundle over $B\Gamma$ associated to the representation g is trivial.

Theorem 1.4. *Let $\Gamma = \pi_1(M(\mathcal{A}))$ be the fundamental group of the complement of a complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^\ell$. If $g : \Gamma \rightarrow O(n)$ is a representation which lifts to $Spin(n)$, then the vector bundle over $M(\mathcal{A})$ associated to this representation is trivial.*

If $\Gamma \rightarrow U(n)$ is any unitary representation, then the vector bundle over $M(\mathcal{A})$ associated to this representation is trivial.

Trivial vector bundles over complements of arrangements arise in a number of contexts. For instance, Kohno [21] develops Vassiliev invariants of pure braids using a flat connection on such a bundle over the configuration space $F(\mathbb{C}, n)$ of n ordered points in \mathbb{C} , the complement of the braid arrangement. These structures also arise in mathematical physics, for example in work of Drinfel'd [15] and Kohno [20] in the context of quasi-Hopf algebras and the Yang-Baxter equations, and in the Drinfel'd-Kohno monodromy theorem relating the universal R -matrix representation of the braid group to the monodromy of the Knizhnik-Zamolodchikov differential equations, see [8]. Under certain conditions on the corresponding flat connection, Schechtman and Varchenko [26] give solutions of these equations in terms of generalized hypergeometric integrals defined on $F(\mathbb{C}, n)$.

The fundamental group of the complement of any $K(\Gamma, 1)$ arrangement is homologically toroidal, see Proposition 2.6. For such a group Γ , the group $KO_{\text{rep}}^0(B\Gamma)$ may be completely computed as follows.

Theorem 1.5. *Let Γ be the fundamental group of the complement of a $K(\Gamma, 1)$ arrangement and let ζ_1 and ζ_2 be arbitrary classes in $H^1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ and $H^2(\Gamma; \mathbb{Z}/2\mathbb{Z})$. Then there is a finite dimensional orthogonal representation of Γ which factors through the abelianization of Γ with first and second Stiefel-Whitney classes given by ζ_1 and ζ_2 respectively. Moreover for these groups the Stiefel-Whitney classes induce an isomorphism*

$$KO_{\text{rep}}^0(B\Gamma) \cong H^1(\Gamma, \mathbb{Z}/2) \oplus H^2(\Gamma, \mathbb{Z}/2).$$

This result admits the following specialization to the pure braid groups. The abelianization of the pure braid group P_n on n strands is free abelian of rank $m = \binom{n}{2}$. The mod-2 reduction of this abelianization takes values in an elementary abelian 2-group which can be regarded as the natural $\mathbb{Z}/2\mathbb{Z}$ -torus in

$O(m)$. In Section 4, representations of the pure braid group which factor through a representation of some $\mathbb{Z}/2\mathbb{Z}$ -torus, and which give rise to any values of the first two Stiefel-Whitney classes are explicitly constructed. Thus the elements in the K -theory of the classifying space for the pure braid group which arise from representations are determined completely, and are very restricted.

The previous proposition admits a natural extension to the level of classifying spaces which is given in Section 7. Namely, the second stage of the Postnikov tower for BO captures the contribution to K -theory arising from representations of homologically toroidal groups, and there is a classifying space which captures $KO_{\text{rep}}^0(B\Gamma)$ for a homologically toroidal group Γ .

On the other hand, the entire K -theory of a homologically toroidal group can be computed easily. Indeed, if X is a CW -complex whose suspension is homotopy equivalent to a wedge of spheres, then for any double loop space Y , there are group isomorphisms

$$[X, Y] \rightarrow \bigoplus_{q>0} \text{Hom}(H_q(X, \mathbb{Z}), \pi_q Y).$$

If Γ is homologically toroidal and of finite cohomological dimension, the suspension $\Sigma(B\Gamma)$ has the homotopy type of a wedge of spheres (see Section 2). Hence there are isomorphisms

$$[B\Gamma, BU] \cong \bigoplus_{q>0} \text{Hom}(H_q(\Gamma, \mathbb{Z}), \pi_q BU) \quad \text{and}$$

$$[B\Gamma, BO] \cong \bigoplus_{q>0} \text{Hom}(H_q(\Gamma, \mathbb{Z}), \pi_q BO).$$

Properties of Spin representations of the pure braid group are summarized in the next theorem. One noteworthy feature is that this result supplies Spin representations arising from the structure of the cohomology algebra for the pure braid group. The theorem also gives the (redundant) statement that Spin representations of the pure braid group give trivial stable vector bundles.

Recall [1, 10] that the cohomology ring of the configuration space $F(\mathbb{R}^k, n)$ of n ordered points in \mathbb{R}^k is the quotient of the exterior algebra generated by elements $A_{i,j}$ of degree $k - 1$ for $1 \leq j < i \leq n$ by the ideal generated by the relations

$$A_{i,j} \cdot A_{i,t} - A_{t,j} \cdot A_{i,t} + A_{t,j} \cdot A_{i,j} \quad \text{for } 1 \leq j < t < i \leq n.$$

The relation $A_{i,j} \cdot A_{i,t} - A_{t,j} \cdot A_{i,t} + A_{t,j} \cdot A_{i,j}$ will be called the *three term relation* below, and is the dual of the Jacobi identity for a certain choice of Poisson algebra [10]. This relation is used in the next theorem to construct non-trivial Spin representations of P_n .

Theorem 1.6. *Each three term relation in the cohomology algebra of the pure braid group corresponds to a choice of lifting of the abelianization homomorphism to a product of generalized Heisenberg groups, and thus corresponds to a Spin representation of P_n . This representation is non-trivial, but gives a trivial vector bundle over the configuration space $F(\mathbb{C}, n)$.*

There are related results for bundles obtained from certain representations for Artin’s full braid group B_n . The set of irreducible complex representations, $\text{Irr}(B_n, GL(m, \mathbb{C}))$, was studied intensively by Formanek and Procesi in [17, 18].

Some relatively direct consequences for bundles arising from representations are given next. To state these results, recall that the Burau representation is a homomorphism

$$b : B_n \rightarrow GL(n, \mathbb{Z}[t, t^{-1}])$$

where $\mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials over \mathbb{Z} in the indeterminate t .

Regarding $GL(n, \mathbb{Z}[t, t^{-1}])$ as a subgroup of $GL(n, \mathbb{C}[t, t^{-1}])$, and then evaluating t at a unit in \mathbb{C} gives rise to a homomorphism $B_n \rightarrow GL(n, \mathbb{C})$. Analogously, evaluating t at a unit in \mathbb{R} gives rise to a homomorphism $B_n \rightarrow GL(n, \mathbb{R})$.

For $n > 6$, all irreducible representations in low degrees, $m < n$, are obtained from specializations of the reduced Burau representation, possibly tensored with a one-dimensional representation [17]. It should be pointed out that there are complex representations of the braid groups which are neither irreducible, nor sums of irreducible representations.

The Burau representation is non-trivial, but induces the trivial element in complex K -theory (but not real K -theory) when specialized at $t = 1$. Since \mathbb{C}^* is path-connected, any specialization of the Burau representation behaves in an analogous manner.

These facts may be used to determine the maps in K -theory induced by maps of the stable braid group to $U(n)$.

Proposition 1.7. *Let $b_n : B_n \rightarrow GL(n, \mathbb{C})$ be given by evaluation of the Burau representation at a point in \mathbb{C}^* .*

1. *The induced map $BB_n \rightarrow BGL(n, \mathbb{C})$ is null-homotopic. Thus any element in complex K -theory induced by the Burau representation by evaluation at a unit is trivial. Furthermore, the tensor product of this specialization of the Burau representation with any other representation gives a trivial element in complex K -theory.*
2. *For $m < n$ and $n > 6$, the natural map $\text{Irr}(B_n, GL(m, \mathbb{C})) \rightarrow KU^0(BB_n)$ is trivial.*
3. *Analogous evaluation of the Burau representation at $t = 1$ over the real numbers \mathbb{R} gives a map $BB_n \rightarrow BGL(n, \mathbb{R})$ which has order 2, is injective in mod-2 homology, and has non-vanishing i -th Stiefel-Whitney class for $2i \leq n$.*

As shown by Formanek [17, Lemma 9], for $n > 2$, an irreducible complex representation of B_n of dimension $n - 1$ does not extend to B_{n+2} . The following is a consequence of this result.

Proposition 1.8. *Let $\rho : B_\infty \rightarrow GL(n, \mathbb{C})$ be a finite dimensional complex representation of the stable braid group. If the restriction of ρ to some B_m is an irreducible representation of dimension at least 2, then ρ is the trivial representation. Thus the induced map in complex K -theory is also trivial. Furthermore, if ρ is any unitary representation, then ρ factors through \mathbb{Z} , the abelianization of B_∞ , and the induced element in complex K -theory is also trivial.*

Throughout most of this paper, orthogonal or unitary representations of homologically toroidal groups will be considered. The final section contains brief remarks about representations into $GL(n, R)$, where $R = \mathbb{R}$ or \mathbb{C} . Although there are many more representations, the K -theoretic analysis for these more general representations will not be addressed directly in this paper.

2. K -theory of homologically toroidal groups

This section addresses the K -theory of certain spaces which include the classifying spaces of homologically toroidal groups. For brevity, denote homology with integer coefficients by $H_*X := H_*(X; \mathbb{Z})$.

Proposition 2.1. *Let X be a finite dimensional CW-complex such that the suspension ΣX is homotopy equivalent to a bouquet of spheres. Consider the group of pointed homotopy classes of maps $[X, \Omega(Y)]$.*

1. *The group of pointed homotopy classes of maps $[X, \Omega(Y)]$ is isomorphic, as a set, to the set $\bigoplus_{q>0} \text{Hom}(H_q X, \pi_q \Omega(Y))$.*
2. *The group of pointed homotopy classes of maps $[X, \Omega^2(Y)]$ is isomorphic, as a group, to $\bigoplus_{q>0} \text{Hom}(H_q X, \pi_q \Omega^2(Y))$.*

Proof. Notice that the set $[X, \Omega(Y)]$ is isomorphic as a group to $[\Sigma X, Y]$. Since $\Sigma(X)$ is homotopy equivalent to a bouquet of spheres, the underlying set of the group $[\Sigma(X), Y]$ is isomorphic to $\bigoplus_{q>0} \text{Hom}(H_q X, \pi_q \Omega(Y))$. In addition, $[X, \Omega^2(Y)]$ and $[\Sigma(X), \Omega(Y)]$ are isomorphic as groups. □

Corollary 2.2. *If X is a space which satisfies the above hypotheses, then*

$$KO^0(X) = \bigoplus_{q>0} \text{Hom}(H_q X, \pi_q BO) \quad \text{and} \quad KU^0(X) = \bigoplus_{q>0} \text{Hom}(H_q X, \pi_q BU).$$

Note that these can be made explicit using Bott periodicity.

Example 2.3. For the pure braid group P_n , it is well known that the suspension of $K(P_n, 1)$ is homotopy equivalent to a bouquet of spheres. Thus the K -theory of this space is obtained from the tensor product of the integral cohomology for $K(P_n, 1)$ with the K -theory of a point. The K -theory of P_n is then given in terms of integer cohomology which is torsion free and has Euler-Poincaré series $(1+t)(1+2t)\cdots(1+[n-1]t)$. A similar assertion holds for any complex hyperplane arrangement, since the single suspension of the complement is homotopy equivalent to a bouquet of spheres, see [25].

Now let Γ be a homologically toroidal group. Then, by definition, there is a homomorphism $w : \mathcal{F} \rightarrow \Gamma$ inducing a split surjection in integral homology, where $\mathcal{F} \cong \coprod_{1 \leq i \leq m} G_i$ is a finite free product of free abelian groups G_i , each of finite rank. It follows that the homology of Γ with any trivial coefficients is a split summand of that of \mathcal{F} . Moreover, for any cohomology theory E^* there is an induced map $w^* : E^*(B\Gamma) \rightarrow E^*(B\mathcal{F})$ which is a split monomorphism. There is also a natural isomorphism $\bigoplus_{1 \leq i \leq m} \bar{E}^*(G_i) \rightarrow \bar{E}^*(\mathcal{F})$. Note that if $E^*(B\Gamma) \rightarrow E^*(BG_i)$ is trivial for every i , then $E^*(B\Gamma)$ must be trivial.

If Γ is of finite cohomological dimension, then the classifying space $B\Gamma$ has the homotopy type of a finite dimensional CW -complex. Thus, to apply the above corollary to compute the K -theory of a homologically toroidal group, it suffices to establish the following.

Lemma 2.4. *If Γ is a homologically toroidal group, then the suspension of the classifying space $B\Gamma$ has the homotopy type of a bouquet of spheres.*

Proof. Since Γ is homologically toroidal, the homology of Γ is a split summand of the homology of $\mathcal{F} = \coprod_{1 \leq i \leq m} G_i$, where each G_i is free abelian of finite rank. The suspension $\Sigma(B\Gamma)$ is a retract of $\Sigma(B\mathcal{F})$, which is a finite bouquet of spheres. So, in each degree, the homology of $\Sigma(B\Gamma)$ is a finitely generated free abelian group.

This, together with the fact that every element in the homology of $\Sigma(B\mathcal{F})$ is spherical, implies that there is a bouquet of spheres \mathcal{S} which maps to the suspension of $B\Gamma$, giving a homology isomorphism.

The map $\mathcal{S} \rightarrow \Sigma(B\Gamma)$ is a homotopy equivalence provided that the suspension of $B\Gamma$ has the homotopy type of a CW -complex. Since Γ is a discrete group, the classifying space is naturally a CW -complex. Thus so is the suspension. \square

The above observations yield

Proposition 2.5. *If Γ is a homologically toroidal group of finite cohomological dimension, then*

$$KU^0(B\Gamma) \cong \bigoplus_{q>0} H^{2q}(\Gamma, \mathbb{Z}).$$

An analogous calculation can be made for $KO^0(B\Gamma)$.

Next, it is shown that the fundamental group of the complement of a $K(\Gamma, 1)$ arrangement is homologically toroidal; this provides a useful source of examples. Indeed, all examples of homologically toroidal groups mentioned above may be realized as fundamental groups of complements of $K(\Gamma, 1)$ arrangements. Let \mathcal{A} be a hyperplane arrangement in \mathbb{C}^ℓ , with complement $M(\mathcal{A})$. Note that $M(\mathcal{A})$ has the homotopy type of a finite dimensional CW -complex.

Recall from Definition 1.2 that a topological space is homologically toroidal if it admits a map from a bouquet of tori which induces a split surjection in integral homology. The above assertion concerning $K(\Gamma, 1)$ arrangements follows from the fact that the complement of any arrangement is homologically toroidal. This result may be derived from (co)homological properties of the complement of an arrangement known from work of Brieskorn [6], Falk [16], and others (see [23]). An alternative argument is given below.

Proposition 2.6. *For any complex hyperplane arrangement \mathcal{A} , the complement $M(\mathcal{A})$ is a homologically toroidal space.*

Proof. Let \mathcal{A} be an arrangement in \mathbb{C}^ℓ and, without loss of generality, assume that \mathcal{A} contains ℓ linearly independent hyperplanes. The proof is by induction on ℓ .

In the case $\ell = 1$, an arrangement $\mathcal{A} \subset \mathbb{C}$ is a finite collection of points, and the complement is a bouquet of circles.

For general ℓ , let α be a homology class in $H_q(M(\mathcal{A}); \mathbb{Z})$. It is enough to show that there are maps $\beta : (S^1)^q \rightarrow M(\mathcal{A})$ such that $\alpha = \sum_{\beta \in J} \beta_*([T])$, where $[T]$ denotes a choice of fundamental class for the manifold $T = (S^1)^q$ and J is some (finite) indexing set.

If $q < \ell$, let W be a q -dimensional subspace of \mathbb{C}^ℓ that is transverse to \mathcal{A} . The intersection $W \cap M(\mathcal{A})$ may itself be realized as the complement of a hyperplane arrangement $W \cap \mathcal{A}$ in $W \cong \mathbb{C}^q$, so is homologically toroidal by induction. Since W is transverse to \mathcal{A} and is q -dimensional, the inclusion $i : W \cap M(\mathcal{A}) \hookrightarrow M(\mathcal{A})$ induces an isomorphism $i_* : H_j(W \cap M(\mathcal{A}); \mathbb{Z}) \rightarrow H_j(M(\mathcal{A}); \mathbb{Z})$ in integral homology for each j , $1 \leq j \leq q - 1$, and a surjection $i_* : H_q(W \cap M(\mathcal{A}); \mathbb{Z}) \rightarrow H_q(M(\mathcal{A}); \mathbb{Z})$ by a Lefschetz-type theorem (cf. [19]).

Now, as is well known, the homology of the complement of an arrangement \mathcal{A} is torsion free. Furthermore, the Betti numbers are determined by the *intersection poset* $L(\mathcal{A})$, the partially ordered set of multi-intersections of elements of \mathcal{A} , (typically) ordered by reverse inclusion, with rank function $L(\mathcal{A}) \rightarrow \mathbb{Z}$ given by codimension, see [23]. Since W is transverse to \mathcal{A} , the posets $L(\mathcal{A})$ and $L(W \cap \mathcal{A})$ are identical through rank q . Consequently, the Betti numbers of the complements $M(\mathcal{A})$ and $W \cap M(\mathcal{A})$ are equal in dimensions 0 through q . Thus the surjection $i_* : H_q(W \cap M(\mathcal{A}); \mathbb{Z}) \rightarrow H_q(M(\mathcal{A}); \mathbb{Z})$ is, in fact, an isomorphism. This yields the result in case $q < \ell$.

It remains to consider the case $q = \ell$. Assume first that \mathcal{A} is a central arrangement in \mathbb{C}^ℓ , that is, $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. It is well known that the complement of such

an arrangement is homeomorphic to a product, $M(\mathcal{A}) \cong \mathbb{C}^* \times M(d\mathcal{A})$, where $M(d\mathcal{A})$ is the complement of a “decone” of \mathcal{A} , an arrangement in $\mathbb{C}^{\ell-1}$, see [23]. Since $M(d\mathcal{A})$ is homologically toroidal by induction, it follows immediately that $M(\mathcal{A})$ is as well.

If \mathcal{A} is a non-central arrangement in \mathbb{C}^ℓ , let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be the central subarrangements of \mathcal{A} which contain ℓ linearly independent hyperplanes. Then for each i , $1 \leq i \leq k$, the intersection $\bigcap_{H \in \mathcal{A}_i} H = z_i$ is a point in \mathbb{C}^ℓ . Let B_i be an open ball of radius ϵ about z_i in \mathbb{C}^ℓ . For ϵ sufficiently small, the intersection $B_i \cap M(\mathcal{A})$ is homeomorphic to $M(\mathcal{A}_i)$, the complement of the central subarrangement \mathcal{A}_i , so is homologically toroidal.

Finally, it is known that the top homology of $M(\mathcal{A})$ is isomorphic to the direct sum

$$H_\ell(M(\mathcal{A}); \mathbb{Z}) \cong \bigoplus_{i=1}^k H_\ell(B_i \cap M(\mathcal{A}); \mathbb{Z}) \cong \bigoplus_{i=1}^k H_\ell(M(\mathcal{A}_i); \mathbb{Z}),$$

see [23] or [19]. Since $M(\mathcal{A}_i)$ is homologically toroidal for each i , the result follows. □

In particular, the pure braid group P_n , the fundamental group of the complement of the braid arrangement $\mathcal{A} = \{\ker(z_i - z_j), 1 \leq i < j \leq n\}$ in \mathbb{C}^n , is homologically toroidal.

Corollary 2.7. *Let \mathcal{A} be a complex hyperplane arrangement with complement $M(\mathcal{A})$. There are isomorphisms*

$$[M(\mathcal{A}), BU] \rightarrow \bigoplus_{2j>0} H^{2j}(M(\mathcal{A}), \mathbb{Z}).$$

Remark. It is well known that the cohomology of the complement of an arrangement \mathcal{A} is determined by the combinatorial data recorded in the intersection poset $L(\mathcal{A})$, see [23]. The above result shows that the complex K -theory of the complement of any complex hyperplane arrangement is combinatorially determined as well. Similarly, the real K -theory of the complement of an arrangement is combinatorially determined, see Corollary 2.2.

Determining which elements in the K -theory of a homologically toroidal group are “realized” by representations hinges upon deciding which elements in the K -theory of a free abelian group of finite rank arise from representations. In the case of complex K -theory, one has the following result.

Proposition 2.8. *Let A be a free abelian group of finite rank. Then any homomorphism $A \rightarrow U(n)$ induces a trivial map on complex K -theory.*

Proof. Assume that A has rank q , and write $A \cong \mathbb{Z}^q$. The image of a representation of A is abelian. Since an abelian subgroup of $U(n)$ is conjugate to a subgroup of the group of diagonal matrices, any homomorphism $A \rightarrow U(n)$ factors through a product of n maps $\mathbb{Z} \rightarrow S^1$. So consider a homomorphism $A \rightarrow S^1$. Since the target is an abelian group, this map factors as

$$A \xrightarrow{\sim} \mathbb{Z}^q \rightarrow (S^1)^q \rightarrow S^1.$$

On the other hand, any homomorphism $\mathbb{Z} \rightarrow S^1$ is null-homotopic after passage to classifying spaces. This suffices. \square

3. On flat real bundles over a product of circles

This section is certainly well known to experts. The “bare-hands” results here are useful in what follows, and are included for convenience, as well as completeness.

Recall that any real orthogonal representation of a finitely generated abelian group, not necessarily finite such as $A_n \cong \mathbb{Z}^n$, is conjugate to a Whitney sum of one and/or two-dimensional representations. Since commuting unitary transformations are simultaneously diagonalizable and have eigenvalues of length one, any finite dimensional unitary representation of a finitely generated abelian group is a sum of one-dimensional unitary representations. The analogous statement for finite dimensional orthogonal representations follows by an extension of scalars argument.

Let $\Theta : A_n \rightarrow O(2)$ be a two-dimensional representation. The Stiefel-Whitney classes as well as associated bundles for such a representation will be addressed below.

Lemma 3.1. *If there is an element e of A_n for which the determinant of $\Theta(e)$ is -1 , then for any choice of generators e_i of A_n , $\Theta(e_i)$ is given by either*

$$M(\lambda) = \pm \begin{pmatrix} -\cos(\lambda) & \sin(\lambda) \\ \sin(\lambda) & \cos(\lambda) \end{pmatrix} \text{ for a fixed real number } \lambda, \text{ or } \pm \mathbb{I}_2 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the representation Θ is homotopic through representations to a Whitney sum of one-dimensional representations. Furthermore, twice any one-dimensional representation of A_n gives an $SO(2)$ representation, and thus the associated bundle over BA_n is trivial.

Proof. If there is an element e in A_n for which the determinant of $\Theta(e)$ is not 1, then there is a basis element, say e_1 , such that $\Theta(e_1) = M(\lambda)$ for some $\lambda \in \mathbb{R}$.

Note that $M(\lambda)$ has order 2. Suppose $N \in O(2)$ commutes with $M(\lambda)$. If the determinant of N is -1 , a calculation reveals that $N = \pm M(\lambda)$. If the determinant of N is $+1$, a similar calculation shows that $N = \pm \mathbb{I}_2$. Thus, for all i , either $\Theta(e_i) = \pm M(\lambda)$ or $\Theta(e_i) = \pm \mathbb{I}_2$.

Now define $H : [0, 1] \times A_n \rightarrow O(2)$ by the formula

$$H(t, e_i) = \begin{cases} \pm \mathbf{M}(t\lambda) & \text{if } \Theta(e_i) = \pm \mathbf{M}(\lambda), \\ \pm \mathbb{I}_2 & \text{if } \Theta(e_i) = \pm \mathbb{I}_2. \end{cases}$$

Notice that the elements appearing as values of $H(t, e_i)$ all commute for any fixed value of t . Thus there is a unique extension of H such that $H(t, -)$ is a group homomorphism. Consequently, Θ is homotopic through homomorphisms to a map $\rho : A_n \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the subgroup of $O(2)$ consisting of diagonal matrices with diagonal entries ± 1 . Hence $B\rho$ is a sum of two one-dimensional representations. The first part of the lemma follows.

Finally, observe that twice any one-dimensional representation of A_n is an $SO(2)$ representation. Thus the associated bundle is trivial by Proposition 2.8. □

Remark. The homotopy in the above lemma is a special property of free abelian groups. For example, define $\mathbb{Z} \rightarrow O(2)$ by the matrix $-\mathbb{I}_2$. This representation factors through $\mathbb{Z}/2\mathbb{Z}$. There is a homotopy obtained by

$$H(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Notice that this is a homotopy of the map $\mathbb{Z} \rightarrow O(2)$ through representations, but is NOT a homotopy through representations of $\mathbb{Z}/2\mathbb{Z}$. Namely the matrix $H(t)$ does not have order 2 if $\cos(t)\sin(t) \neq 0$. Thus the null-homotopy of representations above is a property which depends on the source group being free abelian in order for the homotopy to be a group homomorphism at each level t .

To continue to investigate the homotopy class of the map $B\Theta$, notice that by an argument as above, it suffices to consider those that are all -1 's. Namely, an $SO(2)$ representation of A_n induces a map which is null-homotopic after passage to classifying spaces, and taking Whitney sums. Thus it suffices to assume that each representation has non-trivial first Stiefel-Whitney class.

Thus consider a sum $\Theta = \bigoplus \theta_i$ of surjective representations

$$\theta_i : A_n \rightarrow \mathbb{Z}/2\mathbb{Z}, \quad 1 \leq i \leq q.$$

Fix a basis e_1, \dots, e_n for $H^1(A_n; \mathbb{Z}/2\mathbb{Z})$. Then, for each i ,

$$w_1(\theta_i) = \sum_{1 \leq j \leq n} x_{i,j} e_j$$

with $x_{i,j}$ equal to either 0 or 1, and for each i there is a j with $x_{i,j} \neq 0$. Since each θ_i is a line bundle, the higher Stiefel-Whitney classes vanish, $w_k(\theta_i) = 0$ for $k > 1$.

The Stiefel-Whitney classes of Θ are readily recorded in terms of these data.

Lemma 3.2. *Let $\Theta = \bigoplus \theta_i : A_n \rightarrow O(q)$ be a sum of line bundles as above. Then*

$$w_k(\Theta) = \sum_{1 \leq i_1 < \dots < i_k \leq q} w_1(\theta_{i_1}) \cdots w_1(\theta_{i_k}).$$

In particular,

$$w_1(\Theta) = \sum_{1 \leq i \leq q} w_1(\theta_i) = \sum_{1 \leq i \leq q} \sum_{1 \leq j \leq n} x_{i,j} e_j,$$

and

$$w_2(\Theta) = \sum_{1 \leq i < j \leq q} w_1(\theta_i) \cdot w_1(\theta_j) = \sum_{1 \leq i < j \leq q} \sum_{1 \leq s \neq t \leq n} (x_{i,s}x_{j,t} + x_{i,t}x_{j,s})e_s \cdot e_t.$$

The hypotheses of the vanishing of the total first, and second Stiefel-Whitney classes of these bundles then gives the next lemma.

Lemma 3.3. *Let $\Theta : A_n \rightarrow O(q)$ be a sum of line bundles $\theta_i : A_n \rightarrow \mathbb{Z}/2\mathbb{Z}$, $1 \leq i \leq q$. Assume that*

$$w_1(\Theta) = 0 \quad \text{and} \quad w_2(\Theta) = 0.$$

If e_1, \dots, e_n is a basis for $H^1(A_n; \mathbb{Z}/2\mathbb{Z})$ for which $w_1(\theta_i) = \sum_{1 \leq j \leq n} x_{i,j} e_j$ for each i , $1 \leq i \leq q$, then the following properties hold:

1. *For each fixed j with $1 \leq j \leq n$,*

$$\sum_{1 \leq i \leq q} x_{i,j} = 0.$$

2. *For each pair $\{s, t\}$ with $1 \leq s, t \leq n$ and $s \neq t$,*

$$\sum_{1 \leq i < j \leq q} (x_{i,s}x_{j,t} + x_{i,t}x_{j,s}) = 0.$$

The next result characterizes homomorphisms which both arise from finite dimensional orthogonal representations, and are trivial in real K -theory.

Proposition 3.4. *Let $\Theta : A_n \rightarrow O(q)$ be a sum of line bundles $\theta_i : A_n \rightarrow \mathbb{Z}/2\mathbb{Z}$, $1 \leq i \leq q$. Then $B\Theta$ in $[BA_n, BO]$ is trivial if and only if both $w_1(B\Theta)$ and $w_2(B\Theta)$ vanish. Furthermore, if $B\Theta$ in $[BA_n, BO]$ is trivial, then the element $B\Theta$ in $[BA_n, BO(q)]$ is also trivial. Thus the bundle associated to $B\Theta$ is a trivial n -plane bundle if and only if the representation ρ lifts to $Spin(q)$.*

Proof. Clearly, if $B\Theta$ is trivial, then $w_1(B\Theta) = 0$ and $w_2(B\Theta) = 0$.

For the other implication, the map $\Theta : A_n \rightarrow O(q)$ factors through the inclusion of the maximal $\mathbb{Z}/2\mathbb{Z}$ -torus given by $(\mathbb{Z}/2\mathbb{Z})^q$ by hypothesis. Thus consider the map $\bar{\Theta} : A_n \rightarrow (\mathbb{Z}/2\mathbb{Z})^q$ with image given by an elementary abelian 2-group of rank r for some $r \leq q$. The map $\bar{\Theta}$ is a sum of line bundles, $\bar{\Theta} = \bar{\Theta}_1 \oplus \bar{\Theta}_2 \oplus \dots \oplus \bar{\Theta}_r$.

If $r = 1$ and $w_1(B\bar{\Theta}) = 0$, then $B\bar{\Theta}$ is null-homotopic. If $r > 1$, let e_1, \dots, e_n be a basis for $H^1(A_n; \mathbb{Z}/2\mathbb{Z})$, and choose a basis for $(\mathbb{Z}/2\mathbb{Z})^r$. Write

$$w_1(\bar{\Theta}_i) = \sum_{1 \leq j \leq n} x_{i,j} e_j, \quad 1 \leq i \leq r,$$

as before, and let $X = (x_{i,j})$ be the corresponding matrix of coefficients.

By Lemma 3.3, the vanishing of $w_1(B\bar{\Theta})$ implies that each column of X has an even number of non-zero entries. After an appropriate change of basis in $(\mathbb{Z}/2\mathbb{Z})^r$ if necessary, the matrix of coefficients is of the form

$$X = \begin{pmatrix} 0 & \cdots & 0 & x_{1,s} & x_{1,s+1} & \cdots & x_{1,n} \\ 0 & \cdots & 0 & x_{2,s} & x_{2,s+1} & \cdots & x_{2,n} \\ 0 & \cdots & 0 & 0 & x_{3,s+1} & \cdots & x_{3,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & x_{r,s+1} & \cdots & x_{r,n} \end{pmatrix},$$

with $x_{1,s} = x_{2,s}$ non-zero for some s , $1 \leq s \leq n$. Note that $w_1(B\bar{\Theta}) = 0$ implies that $x_{1,t} + x_{2,t} + x_{3,t} + \cdots + x_{r,t} = 0$ for each t , $s < t \leq n$.

By the above observation and Lemma 3.2, for $s < t \leq n$, the coefficient of $e_s \cdot e_t$ in $w_2(B\bar{\Theta})$ is given by

$$\begin{aligned} \sum_{1 \leq i < j \leq r} (x_{i,s} x_{j,t} + x_{i,t} x_{j,s}) &= x_{1,s} x_{2,t} + x_{1,t} x_{2,s} + (x_{1,s} + x_{2,s})(x_{3,t} + \cdots + x_{r,t}) \\ &= x_{1,s} x_{2,t} + x_{1,t} x_{2,s} + (x_{1,s} + x_{2,s})(x_{1,t} + x_{2,t}) \\ &= x_{1,s} x_{1,t} + x_{2,s} x_{2,t}. \end{aligned}$$

Since $x_{1,s} = x_{2,s} \neq 0$, the vanishing of $w_2(B\bar{\Theta})$ implies that $x_{1,t} = x_{2,t}$ for each t , $s < t \leq n$. Thus $\bar{\Theta}_1 = \bar{\Theta}_2$. It follows that $B(\bar{\Theta}_1 \oplus \bar{\Theta}_2)$ is null-homotopic, which establishes the result in the case $r = 2$.

For $r > 2$, inductively assume that the result holds for $r - 1$ with $r - 1 \geq 2$. Observe that $B\bar{\Theta}$ is homotopic to $B(\oplus_i \bar{\Theta}_i) = B(\bar{\Theta}_1 \oplus \bar{\Theta}_2 \oplus \Psi)$, where $\Psi = \bar{\Theta}_3 \oplus \cdots \oplus \bar{\Theta}_r$. Since $B(\bar{\Theta}_1 \oplus \bar{\Theta}_2)$ is null-homotopic, $B\bar{\Theta}$ is homotopic to $B\Psi$, and the result follows by induction.

Notice that the above null-homotopy is supported directly on the Whitney sum of the bundles above, and does not require stabilization. Thus, if the first two Stiefel-Whitney classes vanish, the bundle is trivial. □

4. Stiefel-Whitney classes and $KO_{\text{rep}}^0(B\Gamma)$

The first result of this section provides the technical results required to identify the triviality of a bundle arising from a representation and allows passage from “stable” information to “unstable” information.

Lemma 4.1. *Let Γ be a homologically toroidal group. Then there is a bouquet of tori \mathcal{T} and a map $g : \mathcal{T} \rightarrow K(\Gamma, 1)$ which induces a surjection on fundamental groups, as well as a split surjection in integral homology. Let K denote the mapping cone of $g : \mathcal{T} \rightarrow K(\Gamma, 1)$. Then, the following hold.*

1. *There is a cofibre sequence*

$$\mathcal{T} \rightarrow K(\Gamma, 1) \rightarrow K \rightarrow \Sigma(\mathcal{T}) \rightarrow \Sigma(K(\Gamma, 1)) \rightarrow \dots .$$

2. *The mapping cone K of $g : \mathcal{T} \rightarrow K(\Gamma, 1)$ is a retract of $\Sigma(\mathcal{T})$, and is a co- H -space.*

3. *The natural map $K(\Gamma, 1) \rightarrow K$ is null-homotopic,*

4. *For a path-connected CW -complex X , there is a short exact sequence of sets*

$$\dots \rightarrow [K, X] \rightarrow [K(\Gamma, 1), X] \rightarrow [\mathcal{T}, X] \rightarrow \{*\}.$$

Thus the inverse image of the class of the constant map in the set $[\mathcal{T}, X]$ is the class of the constant map in the set $[K(\Gamma, 1), X]$.

Proof. Since Γ is homologically toroidal, there is a bouquet of tori \mathcal{T}' and a map $g' : \mathcal{T}' \rightarrow K(\Gamma, 1)$ which induces a split surjection in integral homology. Let $h : \mathcal{S} \rightarrow K(\Gamma, 1)$ be a map from a bouquet of circles \mathcal{S} to $K(\Gamma, 1)$ which induces a surjection on fundamental groups. Then $\mathcal{T} = \mathcal{T}' \vee \mathcal{S}$ is a bouquet of tori, and the wedge of g' and h is a map $g : \mathcal{T} \rightarrow K(\Gamma, 1)$ which induces both a surjection on fundamental groups and a split surjection in integral homology.

The mapping cone K of g , together with the Barratt-Puppe sequence, gives a cofibre sequence

$$\mathcal{T} \rightarrow K(\Gamma, 1) \rightarrow K \rightarrow \Sigma(\mathcal{T}) \rightarrow \Sigma(K(\Gamma, 1)) \rightarrow \dots .$$

It is shown below that there is a “section” for $\Sigma(\mathcal{T}) \rightarrow \Sigma(K(\Gamma, 1))$. That is, $\Sigma(K(\Gamma, 1))$ is a retract of $\Sigma(\mathcal{T})$ via the retraction $\Sigma(\mathcal{T}) \rightarrow \Sigma(K(\Gamma, 1))$.

Given any cofibre sequence of path-connected CW -complexes $A \rightarrow B \rightarrow C$ for which the map $B \rightarrow C$ is a retraction, there is an induced map $A \vee C \rightarrow B$ (the composite $A \vee C \rightarrow B \vee B \rightarrow B$). This map induces a homology isomorphism. Thus there is a map

$$K \vee \Sigma(K(\Gamma, 1)) \rightarrow \Sigma(\mathcal{T})$$

which is a homology isomorphism.

Since the map $g : \mathcal{T} \rightarrow K(\Gamma, 1)$ induces a surjection on fundamental groups, the group $\pi_1(K)$ is trivial by the Seifert-Van Kampen theorem. Thus the map $K \vee \Sigma(K(\Gamma, 1)) \rightarrow \Sigma(\mathcal{T})$ is a homotopy equivalence, and so the “boundary map” $K \rightarrow \Sigma(\mathcal{T})$ is null-homotopic.

For a path-connected CW-complex X , the Barratt-Puppe sequence induces an exact sequence of sets (but not necessarily groups)

$$\cdots \rightarrow [K, X] \rightarrow [K(\Gamma, 1), X] \rightarrow [\mathcal{T}, X] \rightarrow \{*\}.$$

Consequently, the inverse image of the class of the constant map $\mathcal{T} \rightarrow X$ is in the image of the map $[K, X] \rightarrow [K(\Gamma, 1), X]$. This last map is constant as the map $K(\Gamma, 1) \rightarrow K$ is null-homotopic by the above remarks.

The lemma follows. □

The main results of this section are the following theorem, and a number of related results, including Theorems 1.3–1.5.

Theorem 4.2. *Let Γ be a homologically toroidal group, and let $\rho : \Gamma \rightarrow O(n)$ be a representation with vanishing first two Stiefel-Whitney classes. Then $B\rho : B\Gamma \rightarrow BO(n)$ and the composition $B\Gamma \xrightarrow{B\rho} BO(n) \rightarrow BO$ are both null-homotopic, and the associated vector bundles are trivial.*

Proof. Let $\rho : \Gamma \rightarrow O(n)$ be a representation. Since Γ is homologically toroidal, there is a group $\mathcal{F} = \coprod_{1 \leq i \leq m} G_i$ with each group G_i free abelian group of finite rank, and a map $w : \mathcal{F} \rightarrow \Gamma$ which is surjective by Lemma 4.1, and induces a split epimorphism on homology (with any trivial coefficients). Consequently, the composite

$$G_i \rightarrow \Gamma \rightarrow O(n) \rightarrow O$$

has vanishing first two Stiefel-Whitney classes for every i . Hence the induced map $\alpha : G_i \rightarrow O$ gives the trivial element in $[BG_i, BO]$ for every i . It follows that the element $B\rho$ is trivial in $[B\Gamma, BO]$, and that the associated vector bundle is trivial.

The proof that the map $B\rho : B\Gamma \rightarrow BO(n)$ is null-homotopic, and that the associated bundle is trivial, is given next.

Let $g : \mathcal{T} \rightarrow K(\Gamma, 1)$ be a map from a bouquet of tori inducing $w : \mathcal{F} \rightarrow \Gamma$ on fundamental groups, and a split surjection in integral homology as in Lemma 4.1. If $f \in [K(\Gamma, 1), BO(n)]$ restricts to a trivial element in $[\mathcal{T}, BO(n)]$, then f “comes from” an element in $[K, BO(n)]$, where K is the mapping cone of $g : \mathcal{T} \rightarrow K(\Gamma, 1)$. But the map $[K, BO(n)] \rightarrow [K(\Gamma, 1), BO(n)]$ is trivial by Lemma 4.1. Thus the element f in $[K(\Gamma, 1), BO(n)]$ was trivial to start.

Consequently, bundles (not just stable bundles) which restrict trivially to \mathcal{T} are trivial bundles. □

Proposition 4.3. *Let Γ be a homologically toroidal group.*

1. *Any group homomorphism $\Gamma \rightarrow U(n)$ induces a null-homotopic map $B\Gamma \rightarrow BU$, and hence a trivial map in complex K -theory.*

2. None of the non-trivial elements in $[B\Gamma, BU]$ are induced by representations of Γ .
3. Every element in $[B\Gamma, BO]$ which arises as a representation has order 2.

Proof. If \mathcal{F} is a finite free product of free abelian groups of finite rank as in the definition of a homologically toroidal group, then by Proposition 2.8, any homomorphism $\mathcal{F} \rightarrow U(n)$ induces a trivial map in K -theory. Given any epimorphism $\mathcal{F} \rightarrow \Gamma$ which induces a split surjection in homology, together with any homomorphism $\Gamma \rightarrow U(n)$, the induced map $B\Gamma \rightarrow BU$ is null-homotopic as observed earlier since $B\Gamma$ is a split summand in $B\mathcal{F}$ after suspension.

By Lemma 3.1, twice any orthogonal representation of A_n gives a null-homotopic map after passage to classifying spaces as these are sums of $SO(2)$ representations. Thus twice any orthogonal representation of Γ gives a null-homotopic map after passage to classifying spaces. \square

Observe that Theorem 1.3 follows from Theorem 4.2 and Proposition 4.3.

Now recall that Theorem 1.4 asserts that any bundle over the complement $M(\mathcal{A})$ of a complex hyperplane arrangement \mathcal{A} which arises from a Spin representation of the fundamental group $\Gamma = \pi_1 M(\mathcal{A})$ is a trivial bundle (even though the complement might not be a $K(\Gamma, 1)$ space).

Proof of Theorem 1.4. By Proposition 2.6, the complement $M(\mathcal{A})$ of an arrangement \mathcal{A} is a homologically toroidal space. So there is a bouquet of tori \mathcal{T} , and a map $\mathcal{T} \rightarrow M(\mathcal{A})$ which induces a split epimorphism in integral homology. Let A_n denote the fundamental group of one of these tori. The associated bundle over any one of these tori is induced by a representation as the natural map from a torus to $K(\Gamma, 1)$ is homotopic to $B\rho$ for some choice of homomorphism $A_n \rightarrow \Gamma$. Thus if the original bundle is induced by a representation of the fundamental group, then the associated bundle obtained over the torus arises as a Spin representation of the fundamental group of the torus.

By Proposition 4.3, a Spin representation of the fundamental group of a wedge of tori induces a trivial stable bundle over the wedge of tori. By Lemma 4.1 the bundle over $M(\mathcal{A})$ is trivial. The theorem follows. \square

Next, recall that Theorem 1.5 asserts that any two classes in the first and second cohomology of the fundamental group of the complement of any $K(\Gamma, 1)$ complex hyperplane arrangement may be realized as the first and second Stiefel-Whitney classes of a representation. This is a consequence of the following.

Proposition 4.4. *Let \mathcal{A} be a complex hyperplane arrangement with complement $M(\mathcal{A})$. Given cohomology classes $\zeta_i \in H^i(M(\mathcal{A}); \mathbb{Z}/2\mathbb{Z})$, $i = 1, 2$, there is a vector bundle ξ over $M(\mathcal{A})$ for which the first and second Stiefel-Whitney classes are $w_1(\xi) = \zeta_1$ and $w_2(\xi) = \zeta_2$.*

Proof. Assume that \mathcal{A} is an arrangement of n hyperplanes in \mathbb{C}^ℓ . The cohomology of $M(\mathcal{A})$ is isomorphic to the Orlik-Solomon algebra of \mathcal{A} , which is a quotient of an exterior algebra on n generators, see [23]. This may be realized topologically as follows: The complement $M(\mathcal{A})$ is homeomorphic to a slice $W \cap (\mathbb{C}^*)^n$ of the complex n -torus, where W is an appropriate ℓ -dimensional affine subspace of \mathbb{C}^n . It is readily checked that the inclusion $i : W \cap (\mathbb{C}^*)^n \hookrightarrow (\mathbb{C}^*)^n$ induces a surjection in cohomology.

Now if ξ is a vector bundle over $(\mathbb{C}^*)^n$, then the Stiefel-Whitney classes of the induced bundle $i^*(\xi)$ over $M(\mathcal{A}) = W \cap (\mathbb{C}^*)^n$ satisfy $w_j(i^*(\xi)) = i^*(w_j(\xi))$. Thus, since i^* is surjective, it suffices to prove the proposition in the case where \mathcal{A} is the arrangement of coordinate hyperplanes in \mathbb{C}^n and $M(\mathcal{A}) = (\mathbb{C}^*)^n$. This is a straightforward exercise using the calculations of Section 3. \square

Observe that the above vector bundles over $(\mathbb{C}^*)^n$ arise from representations of the fundamental group $\pi_1((\mathbb{C}^*)^n) = A_n$. For any arrangement \mathcal{A} of n hyperplanes, the abelianization of the fundamental group, $\Gamma = \pi_1 M(\mathcal{A})$, of the complement is isomorphic to A_n . So if \mathcal{A} is a $K(\Gamma, 1)$ arrangement, the above result shows that any two classes in the first and second cohomology of Γ may be realized as the first and second Stiefel-Whitney classes of an orthogonal representation which factors through the abelianization of Γ .

Example 4.5. Taking \mathcal{A} to be the braid arrangement, the above results show that the only possible non-trivial elements in the real K -theory of the pure braid group that arise from representations are torsion elements. The relevant representations, and associated bundles, will be constructed explicitly below.

Recall that the abelianization of the pure braid group P_n is free abelian of rank $m = \binom{n}{2}$. The abelianization map $P_n \rightarrow \mathbb{Z}^m$, followed by mod-2 reduction, yields a homomorphism $P_n \rightarrow (\mathbb{Z}/2\mathbb{Z})^m$. Projection to products of the form $(\mathbb{Z}/2\mathbb{Z})^q$ followed by inclusions as 2-tori in orthogonal groups or special orthogonal groups give families of useful representations. Stiefel-Whitney classes of some of these will now be discussed.

For $1 \leq j < i \leq n$, let $A_{i,j} \in H^1(P_n; \mathbb{Z}/2\mathbb{Z})$ be the cohomology class dual to the standard generator $\gamma_{i,j}$ which links strands i and j . Recall that a basis for $H^t(P_n)$ is given by products $A_{I,J} = A_{i_1,j_1} A_{i_2,j_2} \cdots A_{i_t,j_t}$ where $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $j_m < i_m$ for each m .

Certain choices of maps are listed next. These maps may be used to give an explicit proof for the pure braid groups of the existence of representations which have any fixed choice of first, and second Stiefel-Whitney class in Theorem 1.5.

Associated to $A_{I,J} \in H^t(P_n; \mathbb{Z}/2\mathbb{Z})$, define representations

1. $\alpha(A_{I,J}) : P_n \rightarrow O(t)$ to be the composite of the projection specified by $A_{I,J} : P_n \rightarrow (\mathbb{Z})^t \rightarrow (\mathbb{Z}/2\mathbb{Z})^t$ followed by the inclusion of $(\mathbb{Z}/2\mathbb{Z})^t$ in $O(t)$; and

2. $\beta(A_{I,J}) : P_n \rightarrow SO(t + 1)$ to be the composite of the projection specified by $A_{I,J} : P_n \rightarrow (\mathbb{Z})^t \rightarrow (\mathbb{Z}/2\mathbb{Z})^t$ followed by the inclusion of $(\mathbb{Z}/2\mathbb{Z})^t$ in $SO(t + 1)$.

For $A_{I,J} = A_{i_1,j_1} A_{i_2,j_2} \cdots A_{i_t,j_t} \in H^t(P_n; \mathbb{Z}/2\mathbb{Z})$, the (first two) Stiefel-Whitney classes of the representations $\alpha(A_{I,J})$ and $\beta(A_{I,J})$ are given by

$$\begin{aligned} \alpha(A_{I,J})^*(w_1) &= \sum_{m=1}^t A_{i_m,j_m}, & \alpha(A_{I,J})^*(w_2) &= \sum_{1 \leq m < n \leq t} A_{i_m,j_m} A_{i_n,j_n}, \\ \beta(A_{I,J})^*(w_1) &= 0, & \beta(A_{I,J})^*(w_2) &= \sum_{1 \leq m < n \leq t} A_{i_m,j_m} A_{i_n,j_n}. \end{aligned}$$

By choosing appropriate basis elements $A_{I,J}$, and forming Whitney sums, one can use the representations $\alpha(A_{I,J})$ and $\beta(A_{I,J})$ to produce a representation for which the first two Stiefel-Whitney classes are any two classes in $H^1(P_n; \mathbb{Z}/2\mathbb{Z})$ and $H^2(P_n; \mathbb{Z}/2\mathbb{Z})$.

A similar explicit construction can be carried out for an arbitrary $K(\Gamma, 1)$ arrangement.

5. Pure braid group representations in generalized Heisenberg groups

The point of this section is to show how the relations in the cohomology algebra for the pure braid groups are equivalent to the existence of certain choices of homomorphisms to groups which themselves are generalizations of the classical Heisenberg group, and which arise from Riemann surfaces. The main point is that the standard quadratic relations in the cohomology of the pure braid groups correspond to liftings. These points are described below, and are variations of the representations used in the previous section.

First define the generalized Heisenberg group π_g as the central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_g \rightarrow \mathbb{Z}^{2g} \rightarrow 1$$

with characteristic class χ_g given by the cup product form for the cohomology ring of a closed orientable surface of genus g . This characteristic class is given explicitly by $\chi_g = \sum_{1 \leq i \leq g} x_i y_i$, where $x_i, y_i, 1 \leq i \leq g$, generate the integral cohomology ring of \mathbb{Z}^{2g} , which is an exterior algebra on these generators. The classifying space $B\pi_g$ is a 2-stage Moore-Postnikov tower. Thus π_g is a torsion free nilpotent group of nilpotence class 2.

Notice that there is a “mod- n ” reduction of π_g , say $\bar{\pi}_g$, obtained by both replacing \mathbb{Z} by $\mathbb{Z}/n\mathbb{Z}$, and the characteristic class χ_g by the naturally associated mod- n reduction. Note that there is a morphism of group extensions:

$$\begin{array}{ccccc}
 \mathbb{Z} & \longrightarrow & \pi_g & \longrightarrow & \mathbb{Z}^{2g} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \bar{\pi}_g & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})^{2g}
 \end{array}$$

The group $\bar{\pi}_g$ is an example of the tensor product $G \otimes_{\mathbb{Z}} R$ for a discrete group G , and R the commutative ring $\mathbb{Z}/n\mathbb{Z}$. Informally, this construction is obtained by (i) replacing an abelian group by the tensor product with R ; (ii) replacing the collection of abelian groups given by the filtration quotients of G obtained from the descending central series for G by their tensor product with R ; and (iii) reassembling this data into a single group $G \otimes_{\mathbb{Z}} R$ via characteristic classes of central extensions. Details are given below.

Let G denote a discrete group with Γ^n the n -th stage of the descending central series for G , defined inductively by $\Gamma^1 = G$, and $\Gamma^{n+1} = [\Gamma^n, G]$ for $n \geq 1$. Since the descending central series quotient Γ^n / Γ^{n+1} is an abelian group, it is a module over the integers, and $\Gamma^n / \Gamma^{n+1} \otimes_{\mathbb{Z}} R$ is defined for any commutative ring R . Observe that there is central extension

$$1 \rightarrow \Gamma^n / \Gamma^{n+1} \rightarrow G / \Gamma^{n+1} \rightarrow G / \Gamma^n \rightarrow 1.$$

Now define $G / \Gamma^2 \otimes_{\mathbb{Z}} R = \Gamma^1 / \Gamma^2 \otimes_{\mathbb{Z}} R$, and assume that $G / \Gamma^n \rightarrow G / \Gamma^n \otimes_{\mathbb{Z}} R$ has been defined along with a commutative diagram

$$\begin{array}{ccc}
 K(G / \Gamma^n, 1) & \longrightarrow & K(G / \Gamma^n \otimes_{\mathbb{Z}} R, 1) \\
 \downarrow & & \downarrow \\
 K(\Gamma^n / \Gamma^{n+1}, 2) & \longrightarrow & K(\Gamma^n / \Gamma^{n+1} \otimes_{\mathbb{Z}} R, 2)
 \end{array}$$

Notice that the homotopy theoretic fibre of

$$K(G / \Gamma^n \otimes_{\mathbb{Z}} R, 1) \rightarrow K(\Gamma^n / \Gamma^{n+1} \otimes_{\mathbb{Z}} R, 2)$$

is a $K(H, 1)$ space, where $H = G / \Gamma^{n+1} \otimes_{\mathbb{Z}} R$. Denote this fibre by $K(G / \Gamma^{n+1} \otimes_{\mathbb{Z}} R, 1)$. With some mild hypotheses, there is a homotopy commutative diagram

$$\begin{array}{ccc}
 K(G / \Gamma^{n+1}, 1) & \longrightarrow & K(G / \Gamma^{n+1} \otimes_{\mathbb{Z}} R, 1) \\
 \downarrow & & \downarrow \\
 K(\Gamma^{n+1} / \Gamma^{n+2}, 2) & \longrightarrow & K(\Gamma^{n+1} / \Gamma^{n+2} \otimes_{\mathbb{Z}} R, 2)
 \end{array}$$

Then define $K(G \otimes_{\mathbb{Z}} R, 1)$ as the inverse limit of the tower of the $K(G / \Gamma^n \otimes_{\mathbb{Z}} R, 1)$. The reader is cautioned that this completion is different (on the level of fundamental groups) from the completion usually used in the theory of discrete groups. This is the tensor product as defined in Bousfield and Kan [5] for nilpotent groups.

Returning to the pure braid group, consider the abelianization map $P_n \rightarrow \mathbb{Z}^m$, where $m = \binom{n}{2}$ as above. There are maps $P_n \rightarrow \mathbb{Z}^3$ constructed from the cohomology algebra of P_n as follows. Recall that the cohomology of P_n is the quotient of the exterior algebra generated by one-dimensional classes $A_{i,j}$ for $1 \leq j < i \leq n$ by the ideal generated by $A_{i,j}A_{i,t} - A_{t,j} \cdot A_{i,t} + A_{t,j} \cdot A_{i,j}$ for $1 \leq j < t < i \leq n$.

Write (i, t, j) for any triple of integers $1 \leq j < t < i \leq n$. Define $p_{(i,t,j)} : P_n \rightarrow \mathbb{Z}^3$ by the associated cohomology class. That is, the cohomology of \mathbb{Z}^3 has three fundamental cycles ι_s for $s = 1, 2, 3$ which are of degree 1. Define $p_{(i,t,j)}$ by the formula

$$p_{(i,t,j)}^*(\iota_s) = \begin{cases} A_{i,j} & \text{if } s = 1, \\ A_{i,t} & \text{if } s = 2, \\ A_{t,j} & \text{if } s = 3. \end{cases}$$

An associated homomorphism $\rho_{(i,t,j)} : P_n \rightarrow \pi_3$ is constructed as follows. Define $\pi_{(i,t,j)} : P_n \rightarrow \mathbb{Z}^6$ to be the composite $\pi_{(i,t,j)} = \sigma \circ \Delta \circ p_{(i,t,j)}$, where $\sigma : \mathbb{Z}^6 \rightarrow \mathbb{Z}^6$ is the permutation given by the formula $\sigma(n_1, n_2, n_3, n_4, n_5, n_6) = (n_1, n_3, n_2, n_5, n_4, n_6)$, and $\Delta : \mathbb{Z}^3 \rightarrow \mathbb{Z}^6$ is defined by the formula $\Delta(n_1, n_2, n_3) = (n_1, -n_1, n_2, n_2, n_3, n_3)$.

Lemma 5.1. *The characteristic class of the extension for the generalized Heisenberg group π_3 pulls back to zero along $\pi_{(i,t,j)}^*$, that is $\pi_{(i,t,j)}^*(\chi_3) = 0$. Thus there is a lift to $\rho_{(i,t,j)} : P_n \rightarrow \pi_3$. The product*

$$\rho = \prod_{1 \leq j < t < i \leq n} \rho_{(i,t,j)} : P_n \rightarrow (\pi_3)^{\binom{n}{3}}$$

of the $\binom{n}{3}$ maps $\rho_{(i,t,j)} : P_n \rightarrow \pi_3$ induces a surjection in cohomology.

Proof. Notice that $\pi_{(i,t,j)}^*(\chi_3) = A_{i,j} \cdot A_{i,t} - A_{t,j} \cdot A_{i,t} + A_{t,j} \cdot A_{i,j}$ for $j < t < i$. This is precisely the three-term relation for the cohomology algebra of P_n . In addition, the elements $A_{i,j}$ are in the image of ρ^* . The lemma follows. \square

There are natural Spin representations obtained from the maps ρ as follows. Consider the mod 2 reductions of the maps $\pi_{(i,t,j)}$ to obtain $\bar{\pi}_{(i,t,j)} : P_n \rightarrow (\mathbb{Z}/2\mathbb{Z})^6$. The group $(\mathbb{Z}/2\mathbb{Z})^6$ is a maximal elementary abelian 2-group in $SO(7)$ while the characteristic class defining $Spin(7)$ pulls back to that defining π_3 . Thus, there are embeddings of $\pi_3 \otimes \mathbb{Z}/2\mathbb{Z} = \bar{\pi}_3$ in the group $Spin(7)$, where the tensor product is as defined above.

There are maps given by taking products of the above composites:

$$\psi_{(i,t,j)} : P_n \rightarrow Spin(7) \quad \text{and} \quad \Psi : P_n \rightarrow (Spin(7))^{\binom{n}{3}} \rightarrow Spin(7) \cdot \binom{n}{3}.$$

Notice that the maps $\psi_{(i,t,j)}$ are non-trivial. Furthermore, these maps with targets given by the Spinor groups do not factor through the abelianization of P_n .

Remark. Taking products, there are homomorphisms to products of Heisenberg groups that factor through $\Gamma^1(P_n)/\Gamma^3(P_n)$ associated to the descending central series for P_n . This homomorphism with source $\Gamma^1(P_n)/\Gamma^3(P_n) \otimes \mathbb{Z}/2\mathbb{Z}$ is a monomorphism by inspection in case $n = 3$. For $n > 3$, verification that this map is a monomorphism is left to the reader.

The above discussion is summarized in the following.

Theorem 5.2.

1. For any triple of integers $1 \leq j < t < i \leq n$ there are non-trivial Spin representations of P_n given by

$$\psi_{(i,t,j)} : P_n \rightarrow Spin(7) \quad \text{and} \quad \Psi : P_n \rightarrow Spin(7)^{\binom{n}{3}} \rightarrow Spin(7 \cdot \binom{n}{3}).$$

Furthermore, there is a map, which is an embedding for $n = 3$,

$$\Gamma^1(P_n)/\Gamma^3(P_n) \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow Spin(7)^{\binom{n}{3}}.$$

2. Every Spin representation of Γ where Γ is homologically toroidal induces the trivial map in real K -theory.

Notice that this last theorem gives many choices of non-trivial Spin representations of the pure braid group for which the corresponding fibre bundles are trivial.

6. The Burau representation, K -theory, and the stable braid group

Recall that Proposition 1.7 concerns the Burau representation of the full braid group.

Let $\sigma_k, 1 \leq k \leq n - 1$, denote the standard generators of the full braid group B_n . The Burau representation is the homomorphism

$$b : B_n \rightarrow GL(n, \mathbb{Z}[t, t^{-1}])$$

given on generators by

$$b(\sigma_k) = \begin{pmatrix} \mathbb{I}_{k-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{n-k-1} \end{pmatrix},$$

where \mathbb{I}_m denotes the $m \times m$ identity matrix. Among other interpretations, this representation may be realized as the action of B_n on the one-chains of an infinite cyclic cover of a bouquet of circles induced by the Artin representation. See Birman [4] for a detailed account of the Burau representation.

To prove Proposition 1.7, first consider the specialization at $t = 1$ of the Burau representation. Notice that this specialization is given by the natural map of the braid group to the symmetric group followed by the natural inclusion of the symmetric group in $GL(n, \mathbb{Z})$.

Complexification of this representation gives the trivial bundle via the Vandermonde matrix. A complicated proof of this fact was simplified in [13], and is included here for convenience of the reader.

This bundle is given by the complex n -plane bundle

$$F(\mathbb{C}, n) \times_{\Sigma_n} \mathbb{C}^n \rightarrow F(\mathbb{C}, n)/\Sigma_n.$$

A trivialization is as follows. Let $z = (z_1, \dots, z_n)$ be a point in configuration space and $x = (x_1, x_2, \dots, x_n)$ in \mathbb{C}^n . Define $\lambda : F(\mathbb{C}, n) \times \mathbb{C}^n \rightarrow F(\mathbb{C}, n) \times \mathbb{C}^n$ by the formula $\lambda(z, x) = (z, y)$ where $y = (y_1, y_2, \dots, y_n)$ is given by $y_i = \sum_{j=1}^n (z_j)^{j-1} x_j$. Thus there is an induced bundle isomorphism

$$F(\mathbb{C}, n) \times_{\Sigma_n} \mathbb{C}^n \rightarrow F(\mathbb{C}, n)/\Sigma_n \times \mathbb{C}^n.$$

The bundle on the right-hand side is trivial. Thus the bundle on the left-hand side is also trivial, as is the tensor product with any other bundle.

Since \mathbb{C}^* is path-connected, any specialization of the Burau representation induces a map in K -theory analogous to that induced by setting $t = 1$. Thus parts (1) and (3) of Proposition 1.7 follow.

To prove part (2), recall that Formanek [17] shows that any irreducible representation in $\text{Irr}(B_n, GL(m, \mathbb{C}))$ for $m < n$, and $n > 6$, is given by a specialization of a tensor product of a one-dimensional representation with the reduced Burau representation. The proposition follows.

Next consider a finite dimensional unitary representation of the stable braid group

$$\rho : B_\infty \rightarrow U(n).$$

Notice that any invariant subspace of \mathbb{C}^n admits an orthogonal complement as the representation is unitary. Thus a unitary representation splits as a sum of irreducible representations. Assume that $m \gg n$. In [17, Lemma 9], Formanek shows that an irreducible complex representation of B_m of dimension $m - 1$ for $m > 2$ does not extend to B_{m+2} . Thus any finite dimensional irreducible unitary representation of B_∞ of dimension at least two is trivial. Consequently, these representations give maps which are trivial after passage to classifying spaces, and they induce trivial maps on K -theory.

The remaining case is $n = 1$. Since $U(1)$ is abelian, any one-dimensional unitary representation $\rho : B_\infty \rightarrow U(1)$ factors through \mathbb{Z} , the abelianization of B_∞ , and thus gives a trivial element in complex K -theory. The next proposition follows at once.

Proposition 6.1. *Let $\rho : B_\infty \rightarrow U(n)$ be a homomorphism. The induced element in complex K -theory $B\rho : B_\infty \rightarrow BU$ is trivial.*

The next lemma is included for future use as it provides a method for considering general maps from BB_∞ to BU .

Lemma 6.2. *The natural map from the direct limit $\lim BB_n \rightarrow BB_\infty$ induces a map*

$$[BB_\infty, X] \rightarrow \varprojlim_n [BB_n, X]$$

which is an isomorphism of groups whenever X is an infinite loop space.

Proof. Since X is an infinite loop space, there is a homotopy equivalence $\Omega^q X_q \rightarrow X$ for some choice of space X_q , and any strictly positive integer q . Thus the induced map $[BB_\infty, X] \rightarrow [\Sigma^q BB_\infty, \Omega X_{q+1}]$ is an isomorphism of groups. Recall [11] that $\Sigma^{2L} BB_\infty$ is homotopy equivalent to $\Sigma^{2L} BB_L \vee \Sigma^{2L} Y_L$ where Y_L is the cofibre of the natural map $BB_L \rightarrow BB_\infty$, and that Y_L is $\lfloor L/2 \rfloor$ -connected.

Recall Milnor’s lim^1 exact sequence from [22], as described in Bousfield-Kan [5] on the level of function spaces: let $E = colim_n E_n$ be a filtered space, where (E_n, E_{n-1}) is an NDR pair in the terminology of Steenrod; then there is an exact sequence of groups

$$1 \rightarrow lim^1[E_n, X] \rightarrow [E, X] \rightarrow \varprojlim_n [E_n, X] \rightarrow 1.$$

Next let E_n denote BB_n . By the previous remarks concerning the braid groups, the restriction maps $[E_n, X] \rightarrow [E_{n-1}, X]$ are split epimorphisms of groups (as X is assumed to be an infinite loop space). The Mittag-Leffler condition is satisfied, and thus Milnor’s lim^1 term is zero (see [5, Prop. 2.4, Cor. 3.3]). The lemma follows. □

7. An analogue of a classifying space for KO_{rep}^0

Let Γ be any group, with $g : \Gamma \rightarrow O(n)$ a group homomorphism. Composing g with the natural map $O(n) \rightarrow O$ yields a composite homomorphism $G : \Gamma \rightarrow O$. There is a function

$$\Phi : [B\Gamma, BO] \rightarrow H^1(B\Gamma; \mathbb{Z}/2\mathbb{Z}) \oplus H^2(B\Gamma; \mathbb{Z}/2\mathbb{Z})$$

obtained by evaluating a map on w_1 and w_2 , the first two Steifel-Whitney classes.

The purpose of this section is to give a natural description of the above function in terms of spaces. Recall from real Bott periodicity that there is a homotopy equivalence $BO \rightarrow \Omega(SU/SO)$, and that $BSpin$ is the 2-connected cover of BO .

Consider the map $Sq^2 : K(\mathbb{Z}/2\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}/2\mathbb{Z}, 4)$ given by sending the fundamental cycle in the mod-2 cohomology of $K(\mathbb{Z}/2\mathbb{Z}, 4)$ to the cup square of the fundamental cycle for the mod-2 cohomology of $K(\mathbb{Z}/2\mathbb{Z}, 2)$. Let E denote the homotopy theoretic fibre of this map. The space E is the first two stages of the Postnikov tower for the single delooping of BO .

Notice that the loop space of E is a product given by $K(\mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{Z}/2\mathbb{Z}, 2)$, but this decomposition does not preserve the loop structure. Thus there is a bijection between the sets $[X, \Omega(E)]$, and $[X, K(\mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{Z}/2\mathbb{Z}, 2)]$, but they may have different group structures.

A standard and classical exercise carried out below gives a loop map $ev : SU/SO \rightarrow E$ which induces an isomorphism on the first five homotopy groups. Identifying BO with $\Omega(SU/SO)$, there is a double loop map $\Omega(ev) : BO \rightarrow \Omega(E)$. The reader is referred to Cartan’s computations in [7] for the details of the proof.

There are isomorphisms

$$H^*(U/O; \mathbb{F}_2) \cong E[x_1, x_2, \dots, x_n, \dots] \quad \text{and}$$

$$H_*(U/O; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_3, \dots, y_{2n+1}, \dots].$$

Thus the unique non-trivial map $SU/O \rightarrow K(\mathbb{Z}/2\mathbb{Z}, 2)$ lifts to E , and induces an isomorphism on the first non-vanishing homotopy group. Furthermore, all maps are loop maps as Sq^2 is stable. Consequently, there is a lift which is a loop-map, and induces an isomorphism up to H_3 . This map induces an isomorphism on the first three homotopy groups. This suffices by Bott periodicity and the definition, as the next possibly non-vanishing group is in dimension five.

Proposition 7.1.

1. *The homotopy theoretic fibre of $BO \rightarrow \Omega(E)$ is $BSpin$. Thus for any pointed space X , there is a long exact sequence of abelian groups*

$$\begin{aligned} \dots \rightarrow [X, \Omega^2(E)] \rightarrow [X, BSpin] \rightarrow [X, BO] \\ \rightarrow [X, \Omega(E)] \rightarrow [X, B^2Spin] \rightarrow \dots \end{aligned}$$

2. *For any path-connected CW-complex X there is a short exact sequence of abelian groups*

$$0 \rightarrow H^2(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow [X, \Omega(E)] \rightarrow H^1(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow 0$$

which is split as sets. Thus $H^2(X; \mathbb{Z}/2\mathbb{Z})$ acts on $[X, \Omega(E)]$ which is isomorphic to $H^1(X; \mathbb{Z}/2\mathbb{Z}) \times H^2(X; \mathbb{Z}/2\mathbb{Z})$ as an $H^2(X; \mathbb{Z}/2\mathbb{Z})$ -set.

3. *In case $X = B\Gamma$ for a homologically toroidal group Γ , the composite*

$$KO_{\text{rep}}^0(B\Gamma) \rightarrow KO^0(B\Gamma) \rightarrow [B\Gamma, \Omega(E)]$$

is an isomorphism of groups, and $[B\Gamma, \Omega(E)]$ is isomorphic to $H^1(B\Gamma; \mathbb{Z}/2\mathbb{Z}) \oplus H^2(B\Gamma; \mathbb{Z}/2\mathbb{Z})$ with this choice of isomorphism given by evaluating a map on w_1 and w_2 , the first two Stiefel-Whitney classes.

Proof. Most of the proof has been outlined above. Notice that the natural map $\Omega(\text{ev}) : BO \rightarrow \Omega(E)$ induces an isomorphism on the first three homotopy groups. Since E has two non-trivial homotopy groups, and the map $\Omega(\text{ev})$ is an isomorphism of these two homotopy groups, the map induces an isomorphism on the first 3 homotopy groups by Bott periodicity. Thus the fibre of $\Omega(\text{ev}) : BO \rightarrow \Omega(E)$ is $BSpin$, and the long exact sequence of part (1) follows at once.

To prove part (2), notice that there is a fibration $K(\mathbb{Z}/2\mathbb{Z}, 3) \rightarrow E \rightarrow K(\mathbb{Z}/2\mathbb{Z}, 2)$ which admits a section after looping. Thus there is a short exact sequence

$$0 \rightarrow H^2(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow [X, \Omega(E)] \rightarrow H^1(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow 0.$$

Notice that $KO_{\text{rep}}^0(B\Gamma)$ is the subgroup of $KO^0(B\Gamma)$ generated by elements for which at least one of the first two Stiefel-Whitney classes is non-zero. This is precisely the group $[X, \Omega(E)]$. Since every element in $KO_{\text{rep}}^0(B\Gamma)$ has order 2 as shown in Proposition 4.3, this sequence is split as groups, part (3) follows. \square

8. Final remarks: representations into $GL(n, R)$

Rather than restricting to either orthogonal or unitary representations of a group, it is natural to also consider more general representations with values in $GL(n, R)$ for R either the real, or complex numbers, and to pose the same questions about K -theory. Some remarks concerning more general representations are included in this final section. With this change, the situation regarding K -theory, and representations might be more complicated than that given by orthogonal or unitary representations depending on the choice of group. This will be illustrated in three ways below.

As a first example, consider representations of a surface group π in $SL(2, \mathbb{R})$ so that the action of π on the homogeneous space $SL(2, \mathbb{R})/SO(2)$, regarded as the upper half-plane, is properly discontinuous. The existence of such representations dates back to the nineteenth century. Notice that such a representation cannot compress to $SO(2)$ as the associated action of a purported compression is not properly discontinuous.

A second type of situation occurs from the fact that an arbitrary representation in $GL(n, \mathbb{C})$ may not be a sum of irreducible representations if the group π is not finite. In the arguments in Section 3 above, an elementary but essential use is made of the fact that unitary representations split as sums of irreducible representations.

Next, consider the 6-stranded braid group for the 2-sphere denoted by Γ^6 . This group is a quotient of the mapping class group Γ_2 for a genus 2 surface. The natural homomorphism $\Gamma_2 \rightarrow Sp(4, \mathbb{Z})$ descends to a homomorphism

$$\phi : \Gamma^6 \rightarrow PGL(2, \mathbb{C}),$$

obtained by regarding $Sp(4, \mathbb{Z})$ as a subgroup of $Sp(4, \mathbb{R})$ which, in turn, is a subgroup of $GL(2, \mathbb{C})$. A number of properties of this representation are listed below. Of these, (1) is easily verified, while (2) may be established using (1) and the results of [3].

1. The map ϕ induces a monomorphism in mod-2 cohomology on the level of classifying spaces $B\phi : B\Gamma^6 \rightarrow BPGL(2, \mathbb{C})$ by restricting to an elementary abelian 2-group of rank 2 in Γ^6 .
2. The map $B\phi$ is not homotopic to a map $B\rho$ arising from a representation

$$\rho : \Gamma^6 \rightarrow SO(3).$$

3. The group $SO(3)$ is a maximal compact subgroup of $PGL(2, \mathbb{C})$.
4. The representation ϕ is thus not orthogonal, but is both natural and interesting.
5. The pure braid group associated to Γ^6 is homologically toroidal, and the results proven here for orthogonal, and unitary representations applies to this group.

The point of these remarks is that there are representations of the braid groups which are not orthogonal and yet contain relevant topological information. While some of the methods of this paper apply to these more general representations in the case of pure braid groups, a comprehensive analysis of the K -theoretic contributions of these representations remains to be carried out.

Acknowledgments. Some of this work was carried out while the second two authors visited the Department of Mathematics at the University of Wisconsin-Madison in the Spring of 1999. These authors thank the Department and the University for their hospitality, and for providing an exciting and productive mathematical environment.

We are also grateful to the referee for valuable comments.

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