

## Characteristic varieties of arrangements

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### Abstract

The  $k$ th Fitting ideal of the Alexander invariant  $B$  of an arrangement  $\mathcal{A}$  of  $n$  complex hyperplanes defines a characteristic subvariety,  $V_k(\mathcal{A})$ , of the algebraic torus  $(\mathbb{C}^*)^n$ . In the combinatorially determined case where  $B$  decomposes as a direct sum of local Alexander invariants, we obtain a complete description of  $V_k(\mathcal{A})$ . For any arrangement  $\mathcal{A}$ , we show that the tangent cone at the identity of this variety coincides with  $\mathcal{R}_k^1(A)$ , one of the cohomology support loci of the Orlik–Solomon algebra. Using work of Arapura [1], we conclude that all irreducible components of  $V_k(\mathcal{A})$  which pass through the identity element of  $(\mathbb{C}^*)^n$  are combinatorially determined, and that  $\mathcal{R}_k^1(A)$  is the union of a subspace arrangement in  $\mathbb{C}^n$ , thereby resolving a conjecture of Falk [11]. We use these results to study the reflection arrangements associated to monomial groups.

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### Introduction

A hyperplane arrangement is a finite collection  $\mathcal{A}$  of codimension one subspaces in a finite-dimensional complex vector space  $V$ . Two principal objects associated to  $\mathcal{A}$  are the complement,  $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$ , and the intersection lattice,  $L(\mathcal{A}) = \{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$ . A central problem in the study of arrangements is to elucidate the relationship between these two seemingly disparate objects – one topological, the other combinatorial. The paradigmatic result in this direction is the theorem of Orlik and Solomon [21], which asserts that the cohomology ring of the complement,  $H^*(M(\mathcal{A}); \mathbb{C})$ , is isomorphic to a certain algebra,  $A(\mathcal{A})$ , which is completely determined by the lattice  $L(\mathcal{A})$ .

The above result leads one to investigate the extent to which the lattice  $L = L(\mathcal{A})$  determines the topology of the complement  $M = M(\mathcal{A})$ . Examples of Rybnikov [23]

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show that the fundamental group,  $G = \pi_1(M)$ , is not combinatorially determined in general. However, certain invariants of  $G$ , such as the lower central series quotients (cf. Falk [10]), are determined by the lattice. Thus, it is natural to ask whether a given isomorphism type invariant of  $G$  is determined by the isomorphism type of  $L$ . In this paper, we show that the central characteristic subvarieties of  $G$  are indeed combinatorially determined.

Let  $M'$  be the maximal abelian cover of  $M$  and let  $B(\mathcal{A}) = H_1(M'; \mathbb{Z})$  be the Alexander invariant of  $\mathcal{A}$ , viewed as a module over the Laurent polynomial ring  $\Lambda = \mathbb{Z}\mathbb{Z}^n$ , where  $n = |\mathcal{A}|$ . From the presentation of  $B = B(\mathcal{A})$  found in [6], several invariants of the group  $G = \pi_1(M)$  may be computed. One such invariant is the  $k$ th Fitting ideal,  $F_k(B)$ . This is the ideal of  $\Lambda$  generated by all the codimension  $k - 1$  minors of a presentation matrix for  $B$ , and is well-known to be independent of the presentation. Let  $(\mathbb{C}^*)^n$  be the complex algebraic  $n$ -torus, with coordinate ring  $\Lambda_{\mathbb{C}} = \mathbb{C}\mathbb{Z}^n$ . The  $k$ th characteristic variety of  $\mathcal{A}$  is the subvariety  $V_k(\mathcal{A})$  of  $(\mathbb{C}^*)^n$  defined by the ideal  $F_k(B) \otimes \mathbb{C}$  of  $\Lambda_{\mathbb{C}}$ . These varieties depend only on the group  $G$ , up to a monomial change of basis in  $(\mathbb{C}^*)^n$ . The characteristic varieties first appeared in a more general context in [8, 16, 20] and have been recently studied in [1, 15, 18].

The lattice  $L$  of  $\mathcal{A}$  is a partially ordered set, ordered by reverse inclusion, with rank function given by codimension. (See Orlik and Terao [22] as a general reference for arrangements.) Each flat  $X \in L_2$  of rank two gives rise to a subvariety  $V_X$  of  $V_k(\mathcal{A})$ . In the combinatorially determined instance (identified in [6]) where the module  $B(\mathcal{A})$  decomposes as a direct sum of ‘local’ Alexander invariants, these subvarieties  $V_X$  provide a complete description of the characteristic varieties of  $\mathcal{A}$ .

In general, the characteristic varieties of an arrangement  $\mathcal{A}$  possess ‘non-local’ irreducible components. To analyse such components, we make use of the relation between these varieties and the cohomology of local systems on the complement  $M$  of  $\mathcal{A}$ . Each point  $\mathbf{t} \in (\mathbb{C}^*)^n$  gives rise to a local coefficient system  $\mathbb{C}_{\mathbf{t}}$  on  $M$ . The characteristic variety  $V_k(\mathcal{A})$  may be identified with the cohomology support locus  $W_k^1(M) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \text{rank } H^1(M; \mathbb{C}_{\mathbf{t}}) \geq k\}$ . This variety was studied by Arapura [1], who showed that it is a union of torsion-translated subtori in  $(\mathbb{C}^*)^n$  in more general circumstances. More precise descriptions of characteristic varieties of plane algebraic curves were obtained by Libgober in [17, 18].

A comparable analysis in the Orlik–Solomon algebra has recently been carried out by Falk [11]. Let  $A = A(\mathcal{A})$  be the OS-algebra of  $\mathcal{A}$ , generated by  $a_1, \dots, a_n$ . Each point  $\boldsymbol{\lambda} \in \mathbb{C}^n$  determines an element  $\omega = \sum_{i=1}^n \lambda_i a_i$  of  $A^1$ . Falk’s invariant varieties are the cohomology support loci  $\mathcal{R}_k^1(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^n \mid \text{rank } H^1(A, \mu) \geq k\}$ , where  $A$  is viewed as a complex with differential  $\mu(\eta) = \omega \wedge \eta$ . We establish the relation between these varieties and the characteristic varieties  $V_k(\mathcal{A})$ . We prove:

**THEOREM.** *The tangent cone  $\mathcal{V}_k(\mathcal{A})$  of  $V_k(\mathcal{A})$  at the point  $\mathbf{1}$  coincides with  $\mathcal{R}_k^1(A)$ .*

This result has several significant consequences. First, together with Arapura’s work noted above, it shows that  $\mathcal{R}_k^1(A)$  is the union of a subspace arrangement in  $\mathbb{C}^n$ , resolving a conjecture of Falk [11]. Second, it allows us to conclude that the monomial isomorphism type of the central characteristic subvariety  $\check{V}_k(\mathcal{A})$  (the subvariety of  $V_k(\mathcal{A})$  consisting of those irreducible components passing through  $\mathbf{1}$ ) is determined by the isomorphism type of the lattice  $L(\mathcal{A})$ . Finally, the identification  $\mathcal{R}_k^1(A) = \mathcal{V}_k(\mathcal{A})$  provides a combinatorial means for detecting non-local compo-

nents of  $V_k(\mathcal{A})$ . We illustrate this technique by analysing the first characteristic varieties of braid arrangements and monomial arrangements. As a consequence, we show that the associated generalized pure braid groups are not isomorphic to the corresponding products of free groups, extending a result of [4].

1. Fitting ideals and characteristic varieties

In this section we review the definition of the Alexander invariant, and the associated Fitting ideals and characteristic varieties, for a finite complex.

1.1. Alexander invariant

Let  $M$  be a path-connected space that has the homotopy type of a finite CW-complex. Let  $G = \pi_1(M, *)$  be the fundamental group and  $K = H_1(M)$  its abelianization. Let  $M'$  be the maximal abelian cover of  $M$ , with group of deck-transformations identified with  $K$ . The action of  $K$  on  $M$  induces an action on the homology groups  $H_*(M')$ . This defines on  $H_*(M')$  the structure of a module over the group ring  $\mathbb{Z}K$ . The  $\mathbb{Z}K$ -module  $B = H_1(M')$  is called the (first) *Alexander invariant* of  $M$ . This module depends only on the group  $G$ . Indeed,  $B = G'/G''$ , with the action of  $K = G/G'$  defined by the extension  $1 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G/G' \rightarrow 1$ .

Since  $M$  is up to homotopy a finite complex,  $G$  is a finitely presented group. Now assume  $K$  is free abelian, and choose a system of generators,  $t_1, \dots, t_n$ . This provides an identification of the group ring  $\mathbb{Z}K$  with the ring of finite Laurent series in  $n$  (commuting) variables,  $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ . Thus, we may view the Alexander invariant  $B$  as a module over  $\Lambda$ . As was shown by Crowell,  $B$  is in fact a finitely presented  $\Lambda$ -module. (See [14] for details.)

1.2. Fitting ideals

Recall a standard notion from commutative algebra. Let  $A$  be a module over a commutative ring  $R$ , and assume  $A$  has free presentation  $R^p \xrightarrow{\Omega} R^q \rightarrow A \rightarrow 0$ . The  $k$ th *Fitting ideal*  $F_k(A)$  of the module  $A$  is the ideal of  $R$  generated by the  $(q - k + 1) \times (q - k + 1)$  minors of the matrix  $\Omega$ . We set  $F_k(A) = 0$  if  $k \leq 0$  or  $k \leq q - p$  and  $F_k(A) = R$  if  $k > q$ . The ideal  $F_k(A)$  – also known as the  $(k - 1)$ th determinantal (or elementary) ideal of  $A$  – is independent of the choice of presentation for  $A$  (see e.g. [19]). The Fitting ideals form an ascending chain  $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_q \subseteq F_{q+1} = R$ .

Now let  $M$  be a finite CW-complex, with  $H_1(M; \mathbb{Z}) = \mathbb{Z}^n$ . The Fitting ideals,  $F_k(M)$ , of the Alexander invariant  $H_1(M')$  are homotopy-type invariants of  $M$ . More precisely, if  $f: M \rightarrow N$  is a homotopy equivalence, the extension to group rings of  $f_*: H_1(M; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})$  restricts to an isomorphism  $f_*: F_k(M) \rightarrow F_k(N)$  between the corresponding ideals (see [12, 24]).

1.3. Characteristic varieties

Let  $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$  be the ring of Laurent polynomials with complex coefficients. This is the coordinate ring of the algebraic torus  $(\mathbb{C}^*)^n$ . Consider the complexified Alexander invariant of  $M$ ,  $B_{\mathbb{C}}(M) = H_1(M'; \mathbb{C})$ , viewed as a module over  $\Lambda_{\mathbb{C}}$ , and let  $F_k^{\mathbb{C}}(M)$  be its  $k$ th Fitting ideal. Following Libgober [16], we call the reduced algebraic variety  $V_k(M) = \mathbf{V}(F_k^{\mathbb{C}}(M))$  the  $k$ th *characteristic variety* of  $M$ . The characteristic varieties form a descending tower  $(\mathbb{C}^*)^n = V_0 \supseteq V_1 \supseteq \dots \supseteq V_q \supseteq V_{q+1} = \emptyset$ .

The varieties  $V_k(M)$  are homotopy-type invariants of  $M$ . More precisely, if  $M$  is homotopy equivalent to  $N$ , there is a monomial automorphism  $g: (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ , given by  $g(t_i) = t_1^{a_{i,1}} \cdots t_n^{a_{i,n}}$ , where  $(a_{i,j}) \in \text{GL}(n, \mathbb{Z})$  is such that  $g(V_k(M)) = V_k(N)$ .

The Fitting ideals and characteristic varieties depend only on the fundamental group  $G = \pi_1(M)$ , up to a change of basis as above. Accordingly, we shall denote them, when convenient, by  $F_k(G)$ , resp.  $V_k(G)$ . When  $G$  is the group of an arrangement  $\mathcal{A}$ , we may use the notation  $F_k(\mathcal{A})$ , resp.  $V_k(\mathcal{A})$ .

*Example 1.4.* Let  $G = \mathbb{F}_n$  be the free group of rank  $n$ . Recall the standard free resolution  $C_\bullet$  of  $\mathbb{Z}$  over  $\Lambda = \mathbb{Z}\mathbb{Z}^n$ :

$$0 \rightarrow C_n \xrightarrow{d_n} \cdots \rightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0, \quad (1.1)$$

where  $C_0 = \Lambda$ ,  $C_1 = \Lambda^n$  and  $C_k = \bigwedge^k C_1 = \Lambda^{\binom{n}{k}}$ , and the differentials are given by  $d_k(e_J) = \sum_{r=1}^k (-1)^r (t_{j_r} - 1) \cdot e_{J \setminus \{j_r\}}$ , where  $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$  if  $J = \{j_1, \dots, j_k\}$ .

Let  $B = B(\mathbb{F}_n)$  be the Alexander invariant,  $F_k = F_k(\mathbb{F}_n) \subset \Lambda$  the Fitting ideals and  $V_k = V_k(\mathbb{F}_n) \subset (\mathbb{C}^*)^n$  the characteristic varieties. If  $n = 1$ , we have  $B = 0$  and so  $F_0 = 0$ ,  $F_1 = \Lambda$  and  $V_0 = \mathbb{C}^*$ ,  $V_1 = \emptyset$ . If  $n \geq 2$ , a free presentation for  $B$  is given by  $C_3 \xrightarrow{d_3} C_2 \rightarrow B \rightarrow 0$ . A standard computation yields:

$$F_k = \begin{cases} 0 & \text{for } 0 \leq k \leq n-1, \\ I^{\binom{n}{2}-k+1} & \text{for } n \leq k \leq \binom{n}{2}, \\ \Lambda & \text{for } k > \binom{n}{2}, \end{cases} \quad V_k = \begin{cases} (\mathbb{C}^*)^n & \text{for } 0 \leq k \leq n-1, \\ \mathbf{1} & \text{for } n \leq k \leq \binom{n}{2}, \\ \emptyset & \text{for } k > \binom{n}{2}, \end{cases}$$

where  $I = (t_1 - 1, \dots, t_n - 1)$  is the augmentation ideal of  $\Lambda$  and  $\mathbf{1} = (1, \dots, 1)$  is the identity element of  $(\mathbb{C}^*)^n$ .

## 2. Braid monodromy and Alexander invariant of an arrangement

In this section, we review the algorithms for determining the braid monodromy and the Alexander invariant of the complement of a complex hyperplane arrangement, as developed in [5, 6], and record some immediate consequences.

### 2.1. Braid monodromy

The fundamental group of the complement of a complex hyperplane arrangement is isomorphic to that of a generic two-dimensional section. So, for the purpose of studying the Alexander invariant, it is enough to consider affine line arrangements in  $\mathbb{C}^2$ . Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be such an arrangement, with complement  $M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^n H_i$ , and vertices  $\mathcal{V} = \{v_1, \dots, v_s\}$ . If  $v_k = H_{i_1} \cap \cdots \cap H_{i_r}$ , let  $X_k = \{i_1, \dots, i_r\}$  denote the corresponding ‘vertex set’. We identify the set  $L_2 = L_2(\mathcal{A})$  of rank two elements in the lattice of  $\mathcal{A}$  and the collection  $\{X_1, \dots, X_s\}$  of vertex sets of  $\mathcal{A}$ .

The braid monodromy is determined as follows (see [5] for details). Choose coordinates  $(x, z)$  in  $\mathbb{C}^2$  so that the first-coordinate projection map is generic with respect to  $\mathcal{A}$ . Let  $f(x, z) = \prod_{i=1}^n (z - a_i(x))$  be a defining polynomial for  $\mathcal{A}$ . The root map  $a = (a_1, \dots, a_n) : \mathbb{C} \rightarrow \mathbb{C}^n$  restricts to a map from the complement of  $\mathcal{Y} = \text{pr}_1(\mathcal{V})$  to the complement of the braid arrangement  $\mathcal{A}_n = \{\ker(y_i - y_j)\}_{1 \leq i < j \leq n}$ . Identify  $\pi_1(\mathbb{C} \setminus \mathcal{Y})$  with the free group  $\mathbb{F}_s$  and the group of the braid arrangement,  $\pi_1(M(\mathcal{A}_n))$ , with the pure braid group  $P_n$ . Then, the *braid monodromy* of  $\mathcal{A}$  is the homomorphism on fundamental groups,  $\alpha : \mathbb{F}_s \rightarrow P_n$ , induced by the root map.

The generators  $\{\alpha_1, \dots, \alpha_s\}$ , of the image of the braid monodromy can be written

explicitly using a *braided wiring diagram*  $\mathcal{W}$  associated to  $\mathcal{A}$ . Such a diagram, determined by the choices made above, may be specified by a sequence of vertex sets and braids,  $\mathcal{W} = \mathcal{W}_s = \{X_1, \beta_1, X_2, \beta_2, \dots, \beta_{s-1}, X_s\}$ . The braid monodromy generators are given by  $\alpha_k = A_{X_k}^{\delta_k}$ , where  $A_{X_k}$  is the full twist on the strands comprising  $X_k$  and  $\delta_k$  is a pure braid determined by the subdiagram  $\mathcal{W}_k$ .

A presentation for the fundamental group of  $M$  may be obtained from the braid monodromy generators using the Artin representation  $P_n \rightarrow \text{Aut}(\mathbb{F}_n)$ :

$$\pi_1(M) = \langle \gamma_1, \dots, \gamma_n \mid \alpha_k(\gamma_i) = \gamma_i \quad \text{for } i = 1, \dots, n \quad \text{and } k = 1, \dots, s \rangle.$$

Note that all relations are commutators. This presentation can be simplified by Tietze-II moves – eliminating redundant relations. For  $X \in L_2$ , let  $X' = X \setminus \{\min X\}$ . Then, the braid monodromy presentation is

$$\pi_1(M) = \langle \gamma_1, \dots, \gamma_n \mid \alpha_k(\gamma_i) = \gamma_i \quad \text{for } i \in X'_k \quad \text{and } k = 1, \dots, s \rangle. \quad (2.1)$$

The number of relations in this presentation is equal to the second Betti number,  $b = b_2(M) = \sum_{X \in L_2} |X'|$ , of the complement.

## 2.2. Alexander invariant

For an endomorphism  $\alpha$  of the free group  $\mathbb{F}_n$ , let  $\Theta(\alpha): C_1 \rightarrow C_1$  be its abelianized Fox Jacobian. This is a  $\Lambda$ -linear map, whose matrix has rows

$$\Theta(\alpha)(e_i) = \nabla^{\text{ab}}(\alpha(t_i)) = \sum_{i=1}^n \left( \frac{\partial \alpha(t_i)}{\partial t_i} \right)^{\text{ab}} e_i.$$

The restriction of  $\Theta$  to the pure braid group  $P_n < \text{Aut}(\mathbb{F}_n)$  is the Gassner representation,  $\Theta: P_n \rightarrow \text{GL}(n, \Lambda)$ . Applying the Fox Calculus to the presentation (2.1), we obtain the chain complex,  $C_\bullet(M')$ , of the maximal abelian cover of  $M$ :

$$\Lambda^b \xrightarrow{\partial_2} \Lambda^n \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0, \quad (2.2)$$

where  $\partial_1 = d_1 = (t_1 - 1 \cdots t_n - 1)^\top$ , and  $\partial_2$  is the Alexander matrix, with rows  $(\Theta(\alpha_k) - \text{id})(e_i)$ , indexed by  $i \in X'_k$  and  $k \in \{1, \dots, s\}$ . If  $X$  is a non-empty subset of  $[n] = \{1, \dots, n\}$ , let  $C_1[X]$  denote the submodule of  $C_1 = \Lambda^n$  spanned by  $\{e_j \mid j \in X\}$ . Then, via the identification  $C_2(M') \cong \bigoplus_{X \in L_2} C_1[X']$ , the Alexander matrix may be viewed as a map  $\partial_2: \bigoplus_{X \in L_2} C_1[X'] \rightarrow C_1$ .

A presentation for the Alexander invariant,  $B = H_1(M')$ , may be obtained by comparing the complex (2.2) with the resolution (1.1). We paraphrase the construction of [6]. For  $\alpha \in P_n$ , with Artin representation given by  $\alpha(t_i) = z_i t_i z_i^{-1}$ , the homomorphism  $\Phi(\alpha): C_1 \rightarrow C_2$  defined by

$$\Phi(\alpha)(e_i) = e_i \wedge \nabla^{\text{ab}}(z_i)$$

satisfies  $\Theta(\alpha) - \text{id} = d_2 \circ \Phi(\alpha)$ . If  $\alpha = A_X$  is the full twist on  $X = \{i_1, \dots, i_r\}$ , given in terms of the standard generators of  $P_n$  by

$$A_X = (A_{i_1, i_2})(A_{i_1, i_3} A_{i_2, i_3})(A_{i_1, i_4} A_{i_2, i_4} A_{i_3, i_4}) \cdots (A_{i_1, i_r} \cdots A_{i_{r-1}, i_r}),$$

then  $\Phi(A_X): C_1 \rightarrow C_2$  is given by

$$\Phi(A_X)(e_i) = \begin{cases} e_i \wedge \nabla_X & \text{if } i \in X \\ (t_i - 1) \nabla_X \wedge \nabla_{i_X} & \text{if } i_1 \leq i \leq i_r \quad \text{and } i \notin X \\ 0 & \text{otherwise,} \end{cases}$$

where  $X^i = \{j \in X \mid j < i\}$ ,  ${}^iX = \{j \in X \mid i < j\}$ ,  $t_X = \prod_{j \in X} t_j$  and  $\nabla_X = \sum_{j \in X} t_{X^j} e_j$ . In general, if  $\alpha = A_X^\delta$ , then  $\Phi(\alpha) = \Theta_2(\delta) \circ \Phi(A_X) \circ \Theta(\delta^{-1})$ , where  $\Theta_k = \bigwedge^k \Theta: C_k \rightarrow C_k$ .

If  $A_X^\delta$  is the braid monodromy generator corresponding to  $X \in L_2$ , let  $\Phi_X$  denote the restriction of  $\Phi(A_X^\delta)$  to  $C_1[X']$ . Then the map  $\Phi: \bigoplus_{X \in L_2} C_1[X'] \rightarrow C_2$ , defined by  $\Phi|_{C_1[X']} = \Phi_X$ , satisfies

$$\partial_2 = d_2 \circ \Phi. \quad (2.3)$$

Thus, the maps  $\text{id}_{C_0}$ ,  $\text{id}_{C_1}$  and  $\Phi$  constitute a chain map  $\Phi_\bullet$  from the complex (2.2) to the resolution (1.1).

**THEOREM 2.3.** *The Alexander invariant of the arrangement  $\mathcal{A}$  has presentation*

$$K_1 \xrightarrow{\Delta} K_0 \rightarrow B \rightarrow 0,$$

where  $K_0 = C_2$ ,  $K_1 = C_2(M') \oplus C_3 = \bigoplus_{X \in L_2} C_1[X'] \oplus C_3$  and  $\Delta = \begin{pmatrix} \Phi \\ d_3 \end{pmatrix}$ .

This presentation has  $\binom{n}{2}$  generators and  $b + \binom{n}{3}$  relations, where  $b = b_2(M)$ .

*Proof.* Let  $K_\bullet(\Phi)$  denote the mapping cone of the chain map  $\Phi_\bullet$ . We then have an exact sequence of complexes,  $0 \rightarrow C_\bullet \rightarrow K_\bullet(\Phi) \rightarrow C_{\bullet-1}(M') \rightarrow 0$ , explicitly:

$$\begin{array}{ccccccc} \longrightarrow & C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \longrightarrow & C_2(M') \oplus C_3 & \xrightarrow{\begin{pmatrix} \partial_2 & \Phi \\ 0 & d_3 \end{pmatrix}} & C_1 \oplus C_2 & \xrightarrow{\begin{pmatrix} d_1 - \text{id} \\ 0 & d_2 \end{pmatrix}} & C_0 \oplus C_1 & \xrightarrow{\begin{pmatrix} \text{id} \\ d_1 \end{pmatrix}} & C_0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ & C_2(M') & \xrightarrow{\partial_2} & C_1 & \xrightarrow{d_1} & C_0 & & \end{array} \quad (2.4)$$

where the chain maps are the natural inclusion and projection. Since  $(C_\bullet, d_\bullet)$  is a resolution, we have  $B = H_1(M') = H_2(K_\bullet(\Phi))$  and it is immediate from the above diagram that  $H_2(K_\bullet(\Phi)) = \text{coker } \Delta$ .

#### 2.4. Characteristic varieties

We now describe the characteristic varieties of the arrangement  $\mathcal{A}$  in terms of the presentation of the Alexander invariant obtained above. First we establish some notation.

Consider the evaluation map  $\Lambda \times (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ , which takes a Laurent polynomial  $f$  in  $n$  variables and a point  $\mathbf{t} = (t_1, \dots, t_n)$  and yields  $f(\mathbf{t}) = f(t_1, \dots, t_n)$ . For fixed  $f \in \Lambda$ , we get a map  $\mathbf{f}: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ . More generally, we have the map  $\text{Mat}_{p \times q}(\Lambda) \times (\mathbb{C}^*)^n \rightarrow \text{Mat}_{p \times q}(\mathbb{C})$ , which takes a matrix  $F: \Lambda^p \rightarrow \Lambda^q$  and evaluates each entry at  $\mathbf{t}$ , to give  $F(\mathbf{t}): \mathbb{C}^p \rightarrow \mathbb{C}^q$ . For fixed  $F \in \text{Mat}_{p \times q}(\Lambda)$ , we get a map  $\mathbf{F}: (\mathbb{C}^*)^n \rightarrow \text{Mat}_{p \times q}(\mathbb{C})$ .

The above considerations, when applied to the presentation matrix  $\Delta$  for the Alexander invariant from Theorem 2.3, yield the following description of the characteristic varieties of  $\mathcal{A}$ :

$$V_k(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \text{rank } \Delta(\mathbf{t}) \leq \binom{n}{2} - k\}. \quad (2.5)$$

For each point  $\mathbf{t} \neq \mathbf{1}$  in  $(\mathbb{C}^*)^n$ , the complex  $(C_\bullet \otimes \mathbb{C}, \mathbf{d}_\bullet(\mathbf{t}))$  is acyclic. Consequently,

for each such  $\mathbf{t}$ , we have  $\text{rank } \mathbf{d}_k(\mathbf{t}) = \binom{n-1}{k-1}$ . In particular,  $\text{rank } \mathbf{d}_3(\mathbf{t}) = \binom{n-1}{2}$  and it follows that  $\text{rank } \mathbf{\Delta}(\mathbf{t}) \geq \binom{n-1}{2}$  for  $\mathbf{t} \neq \mathbf{1}$ . For  $\mathbf{t} = \mathbf{1}$ , since  $\mathbf{d}_3(\mathbf{1})$  is the zero matrix, we have  $\text{rank } \mathbf{\Delta}(\mathbf{1}) = \text{rank } \mathbf{\Phi}(\mathbf{1}) = b$  (the latter equality is a consequence of the proof of theorem 6.5 of [6] and is established by other means in the proof of Theorem 5.2 below). Thus we have

$$V_k(\mathcal{A}) = \begin{cases} \mathbf{1} & \text{for } n \leq k \leq \binom{n}{2} - b, \\ \emptyset & \text{for } k > \binom{n}{2} - b. \end{cases} \quad (2.6)$$

### 3. Characteristic varieties of decomposable arrangements

In this section we study the case where the Alexander invariant of an arrangement decomposes as a direct sum of ‘local’ invariants. We begin with a general formula for the characteristic varieties of a product of spaces.

#### 3.1. Products

Let  $M_1$  and  $M_2$  be two path-connected finite CW-complexes, with  $K_i = H_1(M_i)$  free abelian. Let  $\mathbf{T}_i$  be the complex torus whose coordinate ring is  $\mathbb{C}K_i$ . Let  $M = M_1 \times M_2$  be the product CW-complex, with first homology group  $K = K_1 \oplus K_2$ . Finally, let  $\mathbf{T}$  be the complex torus with coordinate ring  $\mathbb{C}K$ .

**THEOREM 3.2.** *With respect to the canonical decomposition  $\mathbf{T} = \mathbf{T}_1 \times \mathbf{T}_2$ , the characteristic varieties of the product  $M = M_1 \times M_2$  are given by*

$$V_k(M_1 \times M_2) = (V_k(M_1) \times \mathbf{1}) \cup (\mathbf{1} \times V_k(M_2)).$$

*Proof.* As noted in [6], the Alexander invariant of  $M$  decomposes as  $B = B_1 \oplus B_2$ , where  $B_1 = (H_1(M_1) \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K_2} \mathbb{Z}$  and  $B_2 = (H_1(M_2) \otimes_{\mathbb{Z}K_2} \mathbb{Z}K) \otimes_{\mathbb{Z}K_1} \mathbb{Z}$ . A standard property of Fitting ideals (see e.g. [19]) gives

$$F_k(B) = \sum_{i=1}^k F_i(B_1) \cdot F_{k-i+1}(B_2).$$

Thus:

$$\begin{aligned} V_k(M) &= \bigcap_{i=1}^k (\mathbf{V}(F_i(B_1)) \cup \mathbf{V}(F_{k-i+1}(B_2))) \\ &= \mathbf{V}(F_k(B_1)) \cup \mathbf{V}(F_k(B_2)), \end{aligned}$$

and the conclusion readily follows.

*Example 3.3.* Recall two standard constructions in arrangement theory (see [22] for details). The *cone* of an affine arrangement  $\mathcal{A}$  of  $n$  hyperplanes in  $\mathbb{C}^\ell$  is a central arrangement  $\mathbf{c}\mathcal{A}$  of  $n+1$  hyperplanes in  $\mathbb{C}^{\ell+1}$ . The *decone* of a central arrangement  $\mathcal{A}$  of  $n$  hyperplanes in  $\mathbb{C}^\ell$  is an affine arrangement  $\mathbf{d}\mathcal{A}$  of  $n-1$  hyperplanes in  $\mathbb{C}^{\ell-1}$ . The complements are related by  $M(\mathcal{A}) = M(\mathbf{d}\mathcal{A}) \times \mathbb{C}^*$ . Thus,  $\pi_1(M(\mathcal{A})) \cong \pi_1(M(\mathbf{d}\mathcal{A})) \times \mathbb{Z}$ , where the generator of  $\mathbb{Z}$  can be taken to be  $\gamma_1 \cdots \gamma_n$ , the product of the meridians about the hyperplanes of  $\mathcal{A}$ . Choosing  $H_n \in \mathcal{A}$  as the hyperplane ‘at infinity’ in the decone  $\mathbf{d}\mathcal{A}$ , we derive the following from Theorem 3.2.

$$V_k(\mathcal{A}) = \{ \mathbf{t} \in (\mathbb{C}^*)^n \mid (t_1, \dots, t_{n-1}) \in V_k(\mathbf{d}\mathcal{A}) \text{ and } t_1 \cdots t_n = 1 \}.$$

*Example 3.4.* Let  $\mathcal{A}$  be a central arrangement of  $n$  lines in  $\mathbb{C}^2$ . Then the group of  $\mathcal{A}$  is  $G \cong \mathbb{F}_{n-1} \times \mathbb{Z}$ , with  $\mathbb{Z} = \langle \gamma_1 \cdots \gamma_n \rangle$ . Thus, by Example 1.4 and Theorem 3.2,  $V_k(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$  for  $1 \leq k \leq n-2$ ,  $V_k(\mathcal{A}) = \mathbf{1}$  for  $n-1 \leq k \leq \binom{n-1}{2}$  and  $V_k(\mathcal{A}) = \emptyset$  for  $k > \binom{n-1}{2}$ .

*Example 3.5.* More generally, the characteristic varieties of a finite direct product of finitely generated free groups are given by  $V_k(\mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_\ell}) = \bigcup_{i=1}^{\ell} \tilde{V}_k(\mathbb{F}_{n_i})$ , where  $\tilde{V}_k(\mathbb{F}_{n_i}) = \mathbf{1} \times \cdots \times V_k(\mathbb{F}_{n_i}) \times \cdots \times \mathbf{1}$  (see Example 1.4 and compare [15, lemma 3.3.1]).

### 3.6. Local Alexander invariants

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement of  $n$  lines in  $\mathbb{C}^2$  that is transverse to infinity (that is, no two lines of  $\mathcal{A}$  are parallel). For each  $X \in L_2 = L_2(\mathcal{A})$ , let  $\mathcal{A}_X = \{H_{X,1}, \dots, H_{X,n}\}$  denote the arrangement obtained from  $\mathcal{A}$  by perturbing the lines so that all lines except those passing through the vertex corresponding to  $X$  are in general position.

The group of the arrangement  $\mathcal{A}_X$  has presentation

$$G_X = \langle \gamma_1, \dots, \gamma_n \mid [\gamma_X, \gamma_i] = 1 \text{ for } i \in X \text{ and } [\gamma_j, \gamma_i] = 1 \text{ for } j \notin X, i \in [n] \rangle,$$

where  $\gamma_X = \prod_{i \in X} \gamma_i$ . If  $|X| = m$ , then clearly,  $G_X \cong \mathbb{F}_{m-1} \times \mathbb{Z}^{n-m+1}$ . Let  $B_X = B(G_X)$  be the ‘local’ Alexander invariant associated to  $X \in L_2$ . Note that  $B_X$  is trivial if  $m = 2$ . For  $|X| = m \geq 3$ , set

$$V_X = \{(t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid t_X - 1 = 0 \text{ and } t_j - 1 = 0 \text{ for } j \notin X\}.$$

As in the above examples, we have  $V_k(\mathcal{A}_X) = V_X$  for  $1 \leq k \leq m-2$ ,  $V_k(\mathcal{A}_X) = \mathbf{1}$  for  $m-1 \leq k \leq \binom{m-1}{2}$ , and  $V_k(\mathcal{A}_X) = \emptyset$  for  $k > \binom{m-1}{2}$ .

### 3.7. Decomposable Alexander invariants

Given an arrangement  $\mathcal{A}$  in  $\mathbb{C}^2$  that is transverse to infinity, let  $\mathcal{A}^{\text{cc}} = \prod_{X \in L_2} \mathcal{A}_X$  denote the product (see [22]) of the arrangements  $\mathcal{A}_X$  constructed above. Define the *coarse (combinatorial) Alexander invariant* of  $\mathcal{A}$  to be the module  $B^{\text{cc}}(\mathcal{A}) = B(\mathcal{A}^{\text{cc}}) \otimes_{\mathbb{Z}\mathbb{Z}^{ns}} \mathbb{Z}\mathbb{Z}^n$  induced from the Alexander invariant of  $\mathcal{A}^{\text{cc}}$  by the projection  $t_{X,j} \mapsto t_j$ . It is readily seen that

$$B^{\text{cc}}(\mathcal{A}) = \bigoplus_{X \in L_2} B_X.$$

In [6], we defined a homomorphism  $\Pi: B \rightarrow B^{\text{cc}}$ . This map is always surjective, but is not in general a bijection. The failure of injectivity is measured by the cokernel of a certain linear map  $\bar{\Psi}_3: \bar{C}_3 \rightarrow \bigoplus_X \bar{C}_2[X'] \wedge \bar{C}_1$ , where  $\bar{C}$  denotes the image of a (free)  $\Lambda$ -module  $C$  under the augmentation map  $\epsilon: \Lambda \rightarrow \mathbb{Z}$ . The map  $\bar{\Psi}_3$  is determined solely by the intersection lattice (see [6, section 7.6] for details). If  $\text{coker } \bar{\Psi}_3 = 0$ , and thus  $B \cong B^{\text{cc}}$ , we say that  $\mathcal{A}$  is *decomposable*.

*Remark 3.8.* In [6], the implications of the surjectivity of the map  $\bar{\Psi}_3$  are stated in terms of the  $I$ -adic completions of the modules  $B$  and  $B^{\text{cc}}$ . That these implications apply to the modules themselves follows from the ‘reflection of isomorphism from the completion’ discussed in [9, exercise 7.5].

Now let  $V_k^{\text{cc}}(\mathcal{A}) = V_k(B^{\text{cc}})$  be the  $k$ th coarse characteristic variety of  $\mathcal{A}$ . Since the map  $\Pi: B \rightarrow B^{\text{cc}}$  is surjective, we have  $V_k^{\text{cc}}(\mathcal{A}) \subseteq V_k(\mathcal{A})$ . We shall refer to the irreducible components of  $V_k^{\text{cc}}(\mathcal{A})$  as *local* components of  $V_k(\mathcal{A})$  and to the other irreducible components of  $V_k(\mathcal{A})$  as *non-local* components. If  $\mathcal{A}$  is decomposable, then  $V_k^{\text{cc}}(\mathcal{A}) = V_k(\mathcal{A})$ . An inductive argument, analogous to the proof of Theorem 3.2, yields:

**THEOREM 3.9.** *The coarse characteristic varieties of an arrangement  $\mathcal{A}$  are determined by the lattice of  $\mathcal{A}$ , as follows:*

$$V_k^{\text{cc}}(\mathcal{A}) = \bigcup_{X \in L_2} V_k(G_X) = \bigcup_{|X| \geq k+2} V_X.$$

*In particular, if  $\mathcal{A}$  is decomposable, then  $V_k(\mathcal{A}) = \bigcup_{|X| \geq k+2} V_X$ .*

*Remark 3.10.* The module  $B_X = B(G_X)$  depends only on the cardinality  $|X|$  of the vertex set  $X$ . Consequently, the module  $B^{\text{cc}}$  depends only on the number and multiplicities of the elements of  $L_2(\mathcal{A})$ . On the other hand, the monomial isomorphism type of the coarse characteristic variety  $V_k^{\text{cc}}(\mathcal{A})$  depends on more information from the lattice, as the following example of Falk demonstrates.

*Example 3.11.* Consider the arrangements  $\mathcal{A}_1$  and  $\mathcal{A}_2$  from example 4.10 in [11]; these are decomposable:  $B(\mathcal{A}_1) = B_{\{1,2,3\}} \oplus B_{\{1,4,5\}} \oplus B_{\{3,5,6\}} \oplus B_{\{4,6,7\}}$  and  $B(\mathcal{A}_2) = B_{\{1,2,3\}} \oplus B_{\{1,4,5\}} \oplus B_{\{3,5,6\}} \oplus B_{\{1,6,7\}}$ . Thus,  $B(\mathcal{A}_1) \cong B(\mathcal{A}_2)$ . From Theorem 3.9, we get:

$$V_1(\mathcal{A}_1) = V_{\{1,2,3\}} \oplus V_{\{1,4,5\}} \oplus V_{\{3,5,6\}} \oplus V_{\{4,6,7\}},$$

$$V_1(\mathcal{A}_2) = V_{\{1,2,3\}} \oplus V_{\{1,4,5\}} \oplus V_{\{3,5,6\}} \oplus V_{\{1,6,7\}}.$$

Thus, the varieties  $V_1(\mathcal{A}_1)$  and  $V_1(\mathcal{A}_2)$  are (abstractly) isomorphic. Nevertheless, there is no monomial isomorphism  $(\mathbb{C}^*)^7 \rightarrow (\mathbb{C}^*)^7$  taking  $V_1(\mathcal{A}_1)$  to  $V_1(\mathcal{A}_2)$ . This is proved exactly as in [11], using the fact that the ‘polymatroids’ associated to  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are distinct.

#### 4. Cohomology support loci

In this section, we identify the characteristic varieties of  $\mathcal{A}$  with certain cohomology support loci of the complement. A recent result of Arapura then enables us to show that each irreducible component of the variety  $V_k(\mathcal{A})$  is a subtorus of the complex algebraic torus. Furthermore, we identify the tangent cone of  $V_k(\mathcal{A})$  at the identity of the torus.

##### 4.1. Rank one local systems

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a hyperplane arrangement in  $\mathbb{C}^\ell$ . Let  $M$  be the complement of  $\mathcal{A}$ , and let  $G = \langle \gamma_1, \dots, \gamma_n | r_1, \dots, r_b \rangle$  be the braid presentation for its fundamental group, as in (2.1). The generators  $\gamma_i$  are represented by meridional loops around  $H_i$ , with orientations given by the complex structure. Since all relations  $r_k$  are commutators,  $H_1(M; \mathbb{Z})$  is isomorphic to  $\mathbb{Z}^n = \langle t_1, \dots, t_n | [t_i, t_j] = 1 \rangle$ , with identification given by  $\gamma_i \mapsto t_i$ . Furthermore,  $H^1(M; \mathbb{C}^*)$  is isomorphic to the algebraic torus  $(\mathbb{C}^*)^n$ , with coordinates  $\mathbf{t} = (t_1, \dots, t_n)$ .

Each point  $\mathbf{t}$  in  $(\mathbb{C}^*)^n$  determines a representation  $G \rightarrow \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$ ,  $\gamma_i \mapsto t_i$  and an associated rank one local system, which we denote by  $\mathbb{C}_{\mathbf{t}}$ . For generic  $\mathbf{t}$ , the (co)homology of  $M$  with coefficients in  $\mathbb{C}_{\mathbf{t}}$  vanishes. Those  $\mathbf{t}$  for which  $H^r(M; \mathbb{C}_{\mathbf{t}})$  does not vanish comprise the *cohomology support loci*

$$W_k^r(M) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim H^r(M; \mathbb{C}_{\mathbf{t}}) \geq k\}.$$

These loci are algebraic subvarieties of  $(\mathbb{C}^*)^n$ , which depend only on the homotopy type of  $M$  and a generating set for  $G = \pi_1(M)$ . We now relate  $W_k^1(M)$  to the characteristic variety  $V_k(\mathcal{A})$ .

If  $\mathcal{A}$  is an arrangement in general position (through rank 2), then the group of  $\mathcal{A}$  is free abelian and the Alexander invariant is trivial. Thus,  $V_k(\mathcal{A}) = \emptyset$  in this instance. On the other hand,  $W_k^1(M) = \{\mathbf{1}\}$  for  $1 \leq k \leq n$  and  $W_k^1(M) = \emptyset$  for  $k > n$  (see e.g. [13]).

If  $\mathcal{A}$  is not in general position, we have the following. As shown by Hironaka [15] (see also [18]), an analogous result holds for an arbitrary CW-complex  $M$  with torsion-free first homology (and non-trivial Alexander invariant).

**THEOREM 4.2.** *Let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes, with complement  $M$ . Set  $N = \min\{n, \binom{n}{2} - b_2(M)\}$ . If  $H_1(M') \neq 0$ , then, for  $1 \leq k \leq N$ , the characteristic variety  $V_k(\mathcal{A})$  coincides with the cohomology support locus  $W_k^1(M)$ .*

*Proof.* Let  $\mathcal{A}$  be an arrangement in  $\mathbb{C}^\ell$  with complement  $M$ , and let  $S \subset \mathbb{C}^\ell$  be a two-dimensional affine subspace that is transverse to  $\mathcal{A}$ . Then, by a well-known Lefschetz-type theorem, the first homology of  $M$  with coefficients in the local system  $\mathbb{C}_{\mathbf{t}}$  is isomorphic to that of the section  $M \cap S$ . Furthermore, since the arrangements  $\mathcal{A} \subset \mathbb{C}^\ell$  and  $\mathcal{A} \cap S \subset S$  are combinatorially identical through rank two ( $L_k(\mathcal{A}) = L_k(\mathcal{A} \cap S)$  for  $k \leq 2$ ), the second Betti number of  $M$  is equal to that of  $M \cap S$ .

In light of the above, we may assume without loss of generality that  $\mathcal{A}$  is an (affine) arrangement in  $\mathbb{C}^2$ . Then the homology,  $H_*(M; \mathbb{C}_{\mathbf{t}})$ , of the complement with coefficients in the local system  $\mathbb{C}_{\mathbf{t}}$  is given by the homology of the chain complex  $C_\bullet(M; \mathbb{C}_{\mathbf{t}}) := C_\bullet(M') \otimes_{\Lambda} \mathbb{C}_{\mathbf{t}}$ , where  $\Lambda = \mathbb{Z}\mathbb{Z}^n$  and  $(C_\bullet(M'), \partial_\bullet)$  is the chain complex of the maximal abelian cover of  $M$  specified in (2.2). The terms of  $C_\bullet(M; \mathbb{C}_{\mathbf{t}})$  are finite-dimensional complex vector spaces and the boundary maps,  $\partial_k(\mathbf{t})$ , are the evaluations of  $\partial_k$  at  $\mathbf{t}$ .

Let  $W_1^k(M) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim H_1(M; \mathbb{C}_{\mathbf{t}}) \geq k\}$  denote the homology support loci of  $M$ . Evidently, these varieties coincide with the cohomology support loci defined above. From the above description of the complex  $C_\bullet(M; \mathbb{C}_{\mathbf{t}})$ , it is clear that  $W_1^n(M) = \{\mathbf{1}\}$  and  $W_1^k(M) = \emptyset$  for  $k > n$ , while as noted in (2.6),  $V_k(\mathcal{A}) = \{\mathbf{1}\}$  for  $n \leq k \leq \binom{n}{2} - b$ , where  $b = b_2(M)$ . Thus if  $n \leq \binom{n}{2} - b$ , we have  $V_n(\mathcal{A}) = W_1^n(M)$ .

For  $k < n$ , since  $\partial_1(\mathbf{t}) = \mathbf{d}_1(\mathbf{t})$  is of rank at most one, the homology support loci are given by  $W_1^k(M) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \mathrm{rank} \partial_2(\mathbf{t}) \leq n - k - 1\}$ . Recall that the Alexander invariant  $B = B(\mathcal{A})$  is presented by the matrix  $\Delta$  of Theorem 2.3. Tensoring the terms of the diagram (2.4) with  $\mathbb{C}$ , and evaluating the maps at  $\mathbf{t}$ , it follows that

$$W_1^k(M) = \{\mathbf{t} \mid \mathrm{rank} \partial_2(\mathbf{t}) \leq n - k - 1\} = \{\mathbf{t} \mid \mathrm{rank} \Delta(\mathbf{t}) \leq \binom{n}{2} - k\} = V_k(\mathcal{A})$$

for  $1 \leq k \leq N$ .

We now pursue a more precise description of the varieties  $V_k(\mathcal{A})$ . First, note that  $M = \mathbb{C}\mathbb{P}^\ell \setminus \bigcup_{H \in \mathcal{A}^*} H$ , where  $\mathcal{A}^*$  is the projectivization of the cone  $\mathbf{c}\mathcal{A}$ . By blowing up the singularities, we see that  $M$  is biholomorphically equivalent to the complement of a normal-crossing divisor in a smooth, simply-connected projective variety. (This holds more generally for complex subspace arrangements, as shown by De Concini and Procesi [7].) The following result of Arapura [1] describes the support loci of such quasiprojective varieties.

**THEOREM 4.3** (Arapura [1]). *Let  $M$  be the complement of a normal-crossing divisor in a compact Kähler manifold with vanishing first homology. The cohomology support locus  $W_k^1(M)$  is a finite union of torsion-translated subtori of the algebraic torus  $(\mathbb{C}^*)^n$ , where  $n = b_1(M)$ .*

In other words, each irreducible component of  $W_k^1(M)$  is of the form  $\mathbf{q} \cdot (\mathbb{C}^*)^s$ , for some integer  $s \geq 0$  and some torsion element  $\mathbf{q} = (q_1, \dots, q_n) \in (\mathbb{C}^*)^n$ . In [17] and [18], Libgober studies in detail the case where  $M$  is the complement of a plane algebraic curve, obtaining more information about the possible translations that can occur in this case. However, a complete understanding of what torsion-translated subtori can occur is still lacking.

In view of this, let us define the  $k$ th central characteristic subvariety of the arrangement  $\mathcal{A}$  to be the subvariety  $\check{V}_k(\mathcal{A})$  of  $(\mathbb{C}^*)^n$  consisting of all irreducible components of  $V_k(\mathcal{A})$  passing through  $\mathbf{1}$ . In all examples we have examined,  $\check{V}_1(\mathcal{A}) = V_1(\mathcal{A})$ . In general however,  $\check{V}_k(\mathcal{A}) \subsetneq V_k(\mathcal{A})$ , as the following example illustrates.

*Example 4.4.* Consider the central 3-arrangement  $\mathcal{A} = \{H_1, \dots, H_7\}$ , with defining polynomial  $Q = x(x+y+z)(x+y-z)y(x-y-z)(x-y+z)z$ . The lattice of  $\mathcal{A}$  has six rank two flats of multiplicity three. The first characteristic variety,  $V_1(\mathcal{A})$ , is the union of nine 2-dimensional subtori of  $(\mathbb{C}^*)^7$ , six of which are local (see Theorem 3.9). The remaining three 2-tori,

$$\begin{aligned} &\{\mathbf{t} \in (\mathbb{C}^*)^7 \mid t_1 = t_4, \quad t_2 = t_3, \quad t_5 = t_7, \quad t_6 = 1, \quad t_1 t_2 t_5 = 1\}, \\ &\{\mathbf{t} \in (\mathbb{C}^*)^7 \mid t_1 = t_5, \quad t_2 = t_6, \quad t_4 = t_7, \quad t_3 = 1, \quad t_1 t_2 t_4 = 1\}, \\ &\{\mathbf{t} \in (\mathbb{C}^*)^7 \mid t_1 = t_7, \quad t_3 = t_6, \quad t_4 = t_5, \quad t_2 = 1, \quad t_1 t_3 t_4 = 1\}, \end{aligned}$$

arise from subarrangements that are isomorphic to the braid arrangement  $\mathcal{A}_4$  (see Sections 5.6 and 6.8 below). The second characteristic variety,  $V_2(\mathcal{A})$ , consists of the two points of intersection of the three 2-tori above,  $\mathbf{1}$  and  $(-1, 1, 1, -1, -1, 1, -1)$ , whereas  $\check{V}_2(\mathcal{A}) = \mathbf{1}$ .

#### 4.5. Tangent cones

We now study the tangent cone at the origin  $\mathbf{1} \in (\mathbb{C}^*)^n$  of the variety  $V_k(\mathcal{A}) = W_k^1(M)$ . The tangent space of  $H^1(M, \mathbb{C}^*) = (\mathbb{C}^*)^n$  at  $\mathbf{1}$  is  $H^1(M, \mathbb{C}) = \mathbb{C}^n$ , with coordinates  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ . The exponential map  $T_{\mathbf{1}}((\mathbb{C}^*)^n) \rightarrow (\mathbb{C}^*)^n$  is just the coefficient map  $H^1(M, \mathbb{C}) \rightarrow H^1(M, \mathbb{C}^*)$  induced by  $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$ ,  $\lambda_i \mapsto e^{\lambda_i} = t_i$ .

Let  $\mathcal{V}_k(\mathcal{A})$  denote the tangent cone at  $\mathbf{1}$  of the variety  $V_k(\mathcal{A})$ . Clearly, this coincides with the tangent cone at  $\mathbf{1}$  of  $\check{V}_k(\mathcal{A})$ . From Theorem 4.3, we know that  $\check{V}_k(\mathcal{A})$  is a (central) arrangement of subtori in  $(\mathbb{C}^*)^n$ . Thus,  $\mathcal{V}_k(\mathcal{A})$  is a central arrangement of subspaces in  $\mathbb{C}^n$ . We shall find defining equations for this variety and relate them to the presentation of the Alexander invariant of  $M$  from Theorem 2.3.

Let  $f \in \Lambda$  be a Laurent polynomial, and  $\mathbf{f}: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$  the corresponding map, defined in Section 2.4. The derivative of this map at the identity,  $\mathbf{f}_*: \mathbf{T}_1((\mathbb{C}^*)^n) \rightarrow \mathbb{C}$ , is given by  $\mathbf{f}_*(\boldsymbol{\lambda}) = d/dx|_{x=0} f(e^{x\lambda_1}, \dots, e^{x\lambda_n})$ . If  $f$  and  $g$  are two Laurent polynomials, the Product Rule yields  $(\mathbf{fg})_*(\boldsymbol{\lambda}) = \mathbf{f}_*(\boldsymbol{\lambda})\mathbf{g}(\mathbf{1}) + \mathbf{f}(\mathbf{1})\mathbf{g}_*(\boldsymbol{\lambda})$ . More generally, for  $F \in \text{Mat}_{p \times q}(\Lambda)$  and  $G \in \text{Mat}_{q \times r}(\Lambda)$ , matrix multiplication and the differentiation rules yield

$$(\mathbf{F} \cdot \mathbf{G})_*(\boldsymbol{\lambda}) = \mathbf{F}_*(\boldsymbol{\lambda}) \cdot \mathbf{G}(\mathbf{1}) + \mathbf{F}(\mathbf{1}) \cdot \mathbf{G}_*(\boldsymbol{\lambda}). \quad (4.1)$$

Recall the boundary map  $d_k: C_k \rightarrow C_{k-1}$  from the standard resolution (1.1). Let  $\mathbf{d}_k$  be the corresponding map, and  $\boldsymbol{\delta}_k = (\mathbf{d}_k)_*$  its derivative at  $\mathbf{1}$ . Note that  $\mathbf{d}_k(\mathbf{1}) = \mathbf{0}$  is the zero matrix, and that  $\boldsymbol{\delta}_k(\lambda_1, \dots, \lambda_n) = \mathbf{d}_k(1 - \lambda_1, \dots, 1 - \lambda_n)$ .

Recall also the presentation matrix,  $\Delta = \begin{pmatrix} \Phi \\ d_3 \end{pmatrix}: \Lambda^p \rightarrow \Lambda^q$ , for the Alexander invariant of  $M$  from Theorem 2.3, with  $p = b_2(M) + \binom{n}{3}$  and  $q = \binom{n}{2}$ . For  $\mathbf{t} \in (\mathbb{C}^*)^n$ , let  $\mathbf{\Delta}(\mathbf{t}): \mathbb{C}^p \rightarrow \mathbb{C}^q$  be the corresponding map, given by

$$\mathbf{\Delta}(\mathbf{t}) = \begin{pmatrix} \mathbf{\Phi}(\mathbf{t}) \\ \mathbf{d}_3(\mathbf{t}) \end{pmatrix}. \quad (4.2)$$

As noted above, we have  $V_k(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \text{rank } \mathbf{\Delta}(\mathbf{t}) \leq \binom{n}{2} - k\}$ . We now obtain an analogous description of the tangent cone  $\mathcal{V}_k(\mathcal{A})$ .

**THEOREM 4.6.** *Let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes, with complement  $M$ , and non-trivial Alexander invariant  $B$ , presented by  $\Delta = \begin{pmatrix} \Phi \\ d_3 \end{pmatrix}$ . Then the tangent cone,  $\mathcal{V}_k(\mathcal{A})$ , of the characteristic variety  $V_k(\mathcal{A})$  at the point  $\mathbf{1}$  is a subspace arrangement in  $\mathbb{C}^n$ , defined by the equations*

$$\mathcal{V}_k(\mathcal{A}) = \left\{ \boldsymbol{\lambda} \in \mathbb{C}^n \mid \text{rank} \begin{pmatrix} \mathbf{\Phi}(\mathbf{1}) \\ \boldsymbol{\delta}_3(\boldsymbol{\lambda}) \end{pmatrix} \leq \binom{n}{2} - k \right\}.$$

*Proof.* The identification in Theorem 4.2 of the characteristic variety  $V_k(\mathcal{A})$  and the (co)homology support locus shows that  $V_k(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \text{rank } \boldsymbol{\partial}_2(\mathbf{t}) \leq n - k - 1\}$ , where  $\boldsymbol{\partial}_2(\mathbf{t})$  is obtained from the Alexander matrix  $\partial_2$  of (2.2) by evaluation at  $\mathbf{t}$ . Thus the tangent cone at  $\mathbf{1}$  is given by  $\mathcal{V}_k(\mathcal{A}) = \{\boldsymbol{\lambda} \in \mathbb{C}^n \mid \text{rank } (\boldsymbol{\partial}_2)_*(\boldsymbol{\lambda}) \leq n - k - 1\}$ .

Recall that the image of a free  $\Lambda$ -module  $C$  under the augmentation map is denoted by  $\overline{C}$ . Also, recall from (2.3) that  $\partial_2 = d_2 \circ \Phi$ . Consequently, for each  $\mathbf{t}$ , we have  $\boldsymbol{\partial}_2(\mathbf{t}) = \mathbf{\Phi}(\mathbf{t}) \cdot \mathbf{d}_2(\mathbf{t})$ . Using (4.1) and the fact that  $\mathbf{d}_2(\mathbf{1}) = \mathbf{0}$ , we obtain  $(\boldsymbol{\partial}_2)_*(\boldsymbol{\lambda}) = \mathbf{\Phi}(\mathbf{1}) \cdot \boldsymbol{\delta}_2(\boldsymbol{\lambda})$ . Thus, we have a commuting diagram

$$\begin{array}{ccccc} \bigoplus_{X \in L_2} \overline{C}_1[X'] & \xrightarrow{(\boldsymbol{\partial}_2)_*(\boldsymbol{\lambda})} & \overline{C}_1 & \xrightarrow{\boldsymbol{\delta}_1(\boldsymbol{\lambda})} & \overline{C}_0 \\ & & \downarrow \mathbf{\Phi}(\mathbf{1}) & & \downarrow \text{id} \\ \overline{C}_3 & \xrightarrow{\boldsymbol{\delta}_3(\boldsymbol{\lambda})} & \overline{C}_2 & \xrightarrow{\boldsymbol{\delta}_2(\boldsymbol{\lambda})} & \overline{C}_1 & \xrightarrow{\boldsymbol{\delta}_1(\boldsymbol{\lambda})} & \overline{C}_0 \end{array}$$

and an exercise in homological algebra, using a mapping cone construction as in the proof of Theorem 2.3, completes the proof.

*Remark 4.7.* Recall from 2.2 that the map  $\Phi: \bigoplus_{X \in L_2} C_1[X'] \rightarrow C_2$  is defined by  $\Phi|_{C_1[X']} = \Theta_2(\delta) \circ \Phi(A_X) \circ \Theta_1(\delta^{-1})|_{C_1[X']}$ , where  $A_X^\delta$  is the braid monodromy generator corresponding to  $X \in L_2$ . Since pure braids are IA-automorphisms of the free group, we have  $\Theta_k(\delta)(\mathbf{1}) = \text{id}$  for all  $\delta \in P_n$ . Thus,  $\mathbf{\Phi}(\mathbf{1})|_{\overline{C}_1[X']} = \mathbf{\Phi}_X(\mathbf{1})$ , and it

is readily checked that the latter is given by  $\Phi_X(\mathbf{1})(e_i) = e_i \wedge \bar{\nabla}_X$  for each  $i \in X'$ , where  $\bar{\nabla}_X = \sum_{j \in X} e_j$ . It follows that the map  $\Phi(\mathbf{1})$  is determined by the vertex sets  $X \in L_2$ . However, it is not a priori clear that this map is combinatorial, as the vertex sets and their ordering are dictated by the braid monodromy. By comparison with the Orlik–Solomon algebra, the combinatorial nature of the map  $\Phi(\mathbf{1})$  will be established in the next section.

### 5. Relation to the Orlik–Solomon algebra

In this section, we compare Falk’s invariant of the Orlik–Solomon algebra of an arrangement of hyperplanes  $\mathcal{A}$  with the characteristic varieties of the group of  $\mathcal{A}$ .

#### 5.1. Orlik–Solomon algebra

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be a (central) arrangement and  $E$  the graded exterior algebra over  $\mathbb{C}$  generated by  $1$  and  $e_1, \dots, e_n$ . Note that  $E^r = \bar{C}_r$ . Define  $\bar{d}_r: E^r \rightarrow E^{r-1}$  by  $\bar{d}_r(e_J) = \sum_{q=1}^r (-1)^r e_{J \setminus \{j_q\}}$ . Let  $I = I(\mathcal{A})$  be the homogeneous ideal in  $E$  generated by  $\{\bar{d}(e_J) \mid \dim \bigcap_{j \in J} H_j < |J|\}$ . The Orlik–Solomon algebra of  $\mathcal{A}$  is the graded algebra  $A = E/I$ .

Let  $p: E \rightarrow A$  denote the natural projection, and write  $a_i = p(e_i)$ . Note that  $I^0 = I^1 = 0$  and so the maps  $p: E^r \rightarrow A^r$  are isomorphisms for  $r \leq 1$ . An **nbc** (no broken circuit) basis for  $A^2$ , corresponding to the ordering  $H_1, \dots, H_n$  of the hyperplanes, consists of all generators  $a_j \wedge a_k$ , except those for which there exists  $i < \min\{j, k\}$  such that  $H_i \cap H_j \cap H_k \in L_2(\mathcal{A})$  (see [22]). With this choice of basis, for  $H_i \cap H_j \cap H_k \in L_2(\mathcal{A})$ , the projection  $p: E^2 \rightarrow A^2$  is given by  $p(e_i \wedge e_j) = a_i \wedge a_j$ ,  $p(e_i \wedge e_k) = a_i \wedge a_k$  and  $p(e_j \wedge e_k) = a_i \wedge a_k - a_i \wedge a_j$ .

Associated to each  $\lambda \in \mathbb{C}^n$ , we have an element  $\omega = \sum_{i=1}^n \lambda_i a_i$  of  $A^1$ . Left-multiplication by  $\omega$  defines a map  $\mu: A^r \rightarrow A^{r+1}$ . Clearly,  $\mu \circ \mu = 0$ , so  $(A, \mu)$  is a complex. The cohomology support loci of the OS-algebra are Falk’s invariant varieties  $\mathcal{R}^r(A) = \{\lambda \in \mathbb{C}^n \mid H^r(A, \mu) \neq 0\}$  (denoted in [11] by  $\mathcal{R}_r(A)$ ). We shall consider here only the first such locus, filtered by the family of subvarieties  $\mathcal{R}_k^1(A) = \{\lambda \in \mathbb{C}^n \mid \text{rank } H^1(A, \mu) \geq k\}$ . (See [11] and [22] for detailed discussions of the OS-algebra and the variety  $\mathcal{R}^1(A)$ .)

If the arrangement  $\mathcal{A}$  is in general position (through rank 2), it is well-known that  $\mathcal{R}^1(A) = \{0\}$  (see e.g. [25]). If  $\mathcal{A}$  is not in general position, we have the following.

**THEOREM 5.2.** *Let  $\mathcal{A}$  be a central arrangement of complex hyperplanes with non-trivial Alexander invariant. Then the cohomology support locus  $\mathcal{R}_k^1(A)$  of the Orlik–Solomon algebra of  $\mathcal{A}$  coincides with the tangent cone  $\mathcal{V}_k(\mathcal{A})$  of the characteristic variety  $V_k(\mathcal{A})$  at the point  $\mathbf{1}$ .*

*Proof.* Given  $\lambda \in \mathbb{C}^n$ , let  $\hat{\omega} = \sum \lambda_i e_i \in E^1$ . Left-multiplication by  $\hat{\omega}$  defines a map  $\hat{\mu}: E^r \rightarrow E^{r+1}$  and, as above,  $(E, \hat{\mu})$  is a complex. The natural projection  $p: E \rightarrow A$  is a chain map with kernel  $I$ . If  $\lambda \neq 0$ , the complex  $(E, \hat{\mu})$  is acyclic and we have  $H^r(A, \mu) = H^{r+1}(I, \hat{\mu})$ . Thus to study  $H^*(A, \mu)$ ,  $\mathcal{R}_k^1(A)$ , etc., one can pass from the OS-algebra to the OS-ideal (see [11], [25]). However, to facilitate comparison with the Alexander invariant, we opt for a slightly different approach.

Consider the mapping cone  $P$  of  $p$ . This is a complex with terms  $P^r = A^{r-1} \oplus E^r$ , and differentials  $\xi: P^r \rightarrow P^{r+1}$  given by  $\xi(x, y) = (p(y) - \omega \wedge x, \hat{\omega} \wedge y)$ . The natural inclusion and projection yield an exact sequence  $0 \rightarrow A^{\bullet-1} \rightarrow P^\bullet \rightarrow E^\bullet \rightarrow 0$  of

complexes. As above, we have  $H^r(A, \mu) = H^{r+1}(P, \xi)$  if  $\lambda \neq \mathbf{0}$ . Thus we can write  $\widehat{\mathcal{R}}_k^1(A) = \{\lambda \in \mathbb{C}^n \setminus \{\mathbf{0}\} \mid \text{rank } H^2(P, \xi) \geq k\}$ , where  $\widehat{\mathcal{R}}_k^1(A) = \mathcal{R}_k^1(A) \setminus \{\mathbf{0}\}$ .

Define an automorphism  $\psi: P^2 \rightarrow P^2$  by  $\psi(x, y) = (x, y - \hat{\omega} \wedge x)$ . Then, using the fact that  $A^r = E^r$  for  $r \leq 1$ , we see that  $\psi \circ \xi: P^1 \rightarrow P^2$  is given by  $(x, y) \mapsto (y - \omega \wedge x, \mathbf{0})$  and  $\xi \circ \psi^{-1}: P^2 \rightarrow P^3$  by  $(x, y) \mapsto (p(y), \hat{\omega} \wedge y)$ . It follows that  $H^2(P, \xi) = \ker \phi$ , where  $\phi = (p, \hat{\mu}): E^2 \rightarrow A^2 \oplus E^3$  is the restriction of  $\xi \circ \psi^{-1}$  to  $E^2 \subseteq P^2$ . Thus  $\widehat{\mathcal{R}}_k^1(A)$  consists of all  $\lambda \neq \mathbf{0}$  for which  $\dim \ker \phi \geq k$ , i.e. for which  $\text{rank } \phi(\lambda) \leq \binom{n}{2} - k$ . For an arrangement  $\mathcal{A}$  with non-trivial Alexander invariant, it is readily checked that  $\text{rank } \phi(\mathbf{0}) < \binom{n}{2}$ . For such  $\mathcal{A}$ , we have  $\mathcal{R}_k^1(A) = \{\lambda \in \mathbb{C}^n \mid \text{rank } \phi(\lambda) \leq \binom{n}{2} - k\}$ .

The choice of **nb**c basis for  $A^2$  above provides a natural identification  $A^2 \leftrightarrow \bigoplus_{X \in L_2} \overline{C}_1[X']$ . Using this identification, the fact that  $E^k = \overline{C}_k$  and the description of the map  $\Phi(\mathbf{1})$  from Remark 4.7, it is readily checked that  $p$  is dual to  $\Phi(\mathbf{1})$  and  $\hat{\mu}(\lambda)$  is dual to  $\delta_3(\lambda)$ . It follows that  $\phi(\lambda) = (p, \hat{\mu}(\lambda))$  is the transpose of the matrix  $\begin{pmatrix} \Phi(\mathbf{1}) \\ \delta_3(\lambda) \end{pmatrix}$  from Theorem 4.6. Thus,  $\mathcal{R}_k^1(A) = \mathcal{V}_k(\mathcal{A})$ .  $\square$

Recall from Theorem 4.3 that all irreducible components of the variety  $V_k(\mathcal{A})$  passing through  $\mathbf{1}$  are subtori of  $(\mathbb{C}^*)^n$ . Thus, all irreducible components of the tangent cone  $\mathcal{V}_k(\mathcal{A})$  are (linear) subspaces of  $\mathbb{C}^n$ . As a consequence of Theorem 5.2, we have the following, which resolves a conjecture made by Falk ([11, conjecture 4.7]).

**COROLLARY 5.3.** *The variety  $\mathcal{R}_k^1(A)$  is the union of a subspace arrangement.*

Since the subspace arrangement  $\mathcal{V}_k(\mathcal{A}) = \mathcal{R}_k^1(A)$  is determined, up to a linear change of basis in  $\mathbb{C}^n$ , by the intersection lattice  $L(\mathcal{A})$ , the central arrangement of tori  $\check{V}_k(\mathcal{A})$  is determined, up to a monomial change of basis in  $(\mathbb{C}^*)^n$ , by  $L(\mathcal{A})$ . Thus, as another consequence of Theorem 5.2, we have the following.

**COROLLARY 5.4.** *The monomial isomorphism type of the central characteristic subvariety  $\check{V}_k(\mathcal{A})$  is a combinatorial invariant of the arrangement  $\mathcal{A}$ .*

*Remark 5.5.* Libgober proves Corollary 5.3 by other means in [18]. However, the assertion there that the (monomial isomorphism type of the) entire characteristic variety  $V_k(\mathcal{A})$  is combinatorial does not follow immediately from this result. As Example 4.4 illustrates, the variety  $V_k(\mathcal{A})$  may contain irreducible components, which do not pass through  $\mathbf{1}$  and which are not a priori combinatorially determined. Thus, the determination of the precise relationship between the intersection lattice and the characteristic varieties of an arrangement remains an open problem.

### 5.6. Essential tori and non-local components

We use the identification of the variety  $\mathcal{R}^1(A)$  and the tangent cone of the first characteristic variety provided by Theorem 5.2 to obtain a combinatorial means for detecting components of  $\check{V}_1(\mathcal{A})$ .

Positive-dimensional, irreducible components of the characteristic variety  $V_k(\mathcal{A})$  which are not contained in any coordinate torus  $t_i = 1$  of  $(\mathbb{C}^*)^n$  are called *essential tori* in [18]. For an arrangement  $\mathcal{A}$  of cardinality  $n$ , with rank two lattice elements of multiplicities only  $m$  and 2, Libgober gives an algorithm for detecting essential tori which pass through  $\mathbf{1}$  (see [18, section 3]). Ingredients of this algorithm include polytopes of quasi-adjunction and the calculation of the cohomology of a certain ideal sheaf.

Using Theorem 5.2 above, we obtain a combinatorial alternative to this approach. While this alternative method applies only to the first characteristic variety  $V_1(\mathcal{A})$ , it has the advantage of being applicable to an arbitrary arrangement. To describe this method, we must first recall Falk's description of the variety  $\mathcal{R}^1(A)$  from [11].

Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an arrangement with lattice  $L = L(\mathcal{A})$  and consider  $\mathbb{C}^n$  with basis  $e_i$  and coordinates  $\lambda_i$  corresponding to the elements of  $\mathcal{A}$  (resp. of  $L_1$ ). A partition  $\Pi$  of  $[n]$  (or of  $L_1$ ) is *neighbourly* if, for all flats  $X \in L_2$  and all blocks  $\pi$  of  $\Pi$ ,  $|\pi \cap X| \geq |X| - 1$  implies that  $X \subseteq \pi$ . Partitions with more than one block will be called non-trivial. A flat  $X$  that is contained in a single block of  $\Pi$  is said to be *monochrome* and is called *polychrome* otherwise. Note that all rank two flats of multiplicity two are necessarily monochrome.

For each flat  $X \in L_2$ , let  $H_X$  be the hyperplane in  $\mathbb{C}^n$  defined by  $\sum_{i \in X} \lambda_i = 0$  and let  $H_{[n]} = \{\sum_{i=1}^n \lambda_i = 0\}$ . If  $\Pi$  is a neighbourly partition, let  $S_\Pi$  denote the subspace of  $\mathbb{C}^n$  defined by  $S_\Pi = H_{[n]} \cap \bigcap H_X$ , where  $X$  ranges over all polychrome flats in  $L_2$ . Associated to  $\Pi$ , we also have a vector-valued skew-symmetric bilinear form  $\langle \lambda, \lambda' \rangle_\Pi$  on  $\mathbb{C}^n$ , the components of which are the  $2 \times 2$  determinants  $\begin{vmatrix} \lambda_i & \lambda_j \\ \lambda'_i & \lambda'_j \end{vmatrix}$ , for  $i, j \in \pi \subset \Pi$ . Falk's characterization of the variety  $\mathcal{R}^1(A)$  is the following.

**THEOREM 5.7** (Falk [11, theorem 3.10]).  *$\lambda \in \mathcal{R}^1(A)$  if and only if there exists a subarrangement  $\mathcal{A}'$  of  $\mathcal{A}$  and a neighbourly partition  $\Pi$  of  $L_1(\mathcal{A}')$  such that*

- (i)  $\lambda \in S_\Pi$ ; and
- (ii) *there exists  $\lambda' \in S_\Pi$ , not proportional to  $\lambda$ , such that  $\langle \lambda, \lambda' \rangle_\Pi = 0$ .*

If  $\Pi$  is a neighbourly partition, let  $\mathcal{V}_\Pi$  denote the subvariety of the subspace  $S_\Pi$  on which the form  $\langle \cdot, \cdot \rangle_\Pi$  is degenerate. In all known examples,  $\mathcal{V}_\Pi = S_\Pi$ , see [11] and below. By Theorem 5.2, the variety  $\mathcal{V}_\Pi$  is linear in general. This does not, however, rule out the possibility that the containment  $\mathcal{V}_\Pi \subset S_\Pi$  is strict.

*Example 5.8.* Let  $\mathcal{A}$  be the Hessian configuration, with defining polynomial  $Q = x_1x_2x_3 \prod_{i,j=0,1,2} (x_1 + \omega^i x_2 + \omega^j x_3)$ , where  $\omega = \exp(2\pi i/3)$ . The characteristic varieties  $V_k(\mathcal{A})$  were studied by Libgober in [18, example 5]. In particular, he finds an essential torus  $V \subset V_1(\mathcal{A})$  of dimension three. Consequently, the variety  $\mathcal{R}^1(A)$  also has a three-dimensional (linear) component  $S$ . This component is readily recovered using Theorem 5.7. Note that  $L_2(\mathcal{A})$  consists of 9 flats of multiplicity 4 and 12 flats of multiplicity 2. Let  $H_i = \ker x_i$  and let  $H_{i,j} = \ker(x_1 + \omega^i x_2 + \omega^j x_3)$ . The partition  $\Pi = (H_1, H_2, H_3 \mid H_{0,0}, H_{1,2}, H_{2,1} \mid H_{0,1}, H_{1,0}, H_{2,2} \mid H_{0,2}, H_{1,1}, H_{2,0})$  is neighbourly, and all flats of multiplicity 4 are polychrome. The subspace  $S_\Pi$  is three-dimensional. Checking that the form  $\langle \cdot, \cdot \rangle_\Pi$  is trivial on  $S_\Pi$ , we have  $S = S_\Pi$ .

*Remark 5.9.* As illustrated by the above example, for an arbitrary arrangement  $\mathcal{A}$ , the variety  $\mathcal{R}^1(A)$  may have non-local components of dimension greater than two. This leaves open the possibility that the Tutte polynomial of (the matroid of) an arrangement is not determined by the Orlik–Solomon algebra, (see [11, section 3]).

In practice (see below), we use Falk's characterization of  $\mathcal{R}^1(A)$  to detect essential tori and non-local components of  $V_1(\mathcal{A})$ . Theorems 5.2 and 5.7 yield:

**PROPOSITION 5.10.** *Let  $\mathcal{A}$  be an arrangement of  $n$  complex hyperplanes. If  $\Pi$  is an associated neighbourly partition of  $[n]$  and  $\mathcal{V}_\Pi$  is the subspace of  $\mathbb{C}^n$  on which the form*

$\langle \cdot, \cdot \rangle_{\Pi}$  is degenerate, then  $V_{\Pi} = \exp(\mathcal{V}_{\Pi})$  is an essential torus of the first characteristic variety  $V_1(\mathcal{A})$ .

### 6. Monomial arrangements

In this section, we use the results on characteristic varieties obtained above to study the arrangements associated with the full monomial groups  $G(r, 1, \ell)$  and the irreducible subgroups  $G(r, r, \ell)$ ,  $r \geq 2$ . Defining polynomials for the reflection arrangements  $\mathcal{A}_{r,1,\ell}$  and  $\mathcal{A}_{r,r,\ell}$  corresponding to these groups are given by

$$Q(\mathcal{A}_{r,1,\ell}) = x_1 \cdots x_{\ell} \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r) \quad \text{and} \quad Q(\mathcal{A}_{r,r,\ell}) = \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

Note that the arrangements  $\mathcal{A}_{2,1,\ell}$  and  $\mathcal{A}_{2,2,\ell}$  are the Coxeter arrangements of type B and D respectively. The fundamental group,  $P(r, k, \ell) = \pi_1(M(\mathcal{A}_{r,k,\ell}))$ , is the generalized pure braid group associated to the complex reflection group  $G(r, k, \ell)$ ,  $k = 1, r$ . Presentations for the (full) braid groups associated to these reflection groups were recently found in [3].

The arrangement  $\mathcal{A}_{r,1,\ell}$  is fibre-type, with exponents  $\{n_1, n_2, \dots, n_{\ell}\}$ , where  $n_i = (i-1)r + 1$ . Hence, the group  $P(r, 1, \ell)$  admits the structure of an iterated semidirect product of free groups:  $P(r, 1, \ell) \cong \mathbb{F}_{n_{\ell}} \rtimes \cdots \rtimes \mathbb{F}_{n_2} \rtimes \mathbb{F}_{n_1}$ .

The groups  $P(r, r, \ell)$  also admit such structure. Define  $\pi: \mathbb{C}^{\ell} \rightarrow \mathbb{C}^{\ell-1}$  by  $\pi(x_1, \dots, x_{\ell-1}, x_{\ell}) = (x_1^r - x_{\ell}^r, \dots, x_{\ell-1}^r - x_{\ell}^r)$ . The restriction of  $\pi$  to the complement  $M(\mathcal{A}_{r,r,\ell})$  is a generalized Brieskorn bundle (see [2] for the case  $r = 2$ ). The base space of this bundle is homotopy equivalent to the complement of the braid arrangement  $\mathcal{A}_{\ell}$  and the fibre is a surface of genus  $(r^{\ell-1}(\ell-3) - (r-2)(\ell-1) + 2)/2$  with  $r^{\ell-1}$  punctures. It follows that  $P(r, r, \ell) \cong \mathbb{F}_{m_{\ell}} \rtimes \cdots \rtimes \mathbb{F}_{m_2} \rtimes \mathbb{F}_{m_1}$ , where  $m_i = i$  for  $1 \leq i \leq \ell-1$  and  $m_{\ell} = r^{\ell-1}(\ell-2) + (r-2)(\ell-1) + 1$ .

Let  $\Pi(r, 1, \ell) = \mathbb{F}_{n_{\ell}} \times \cdots \times \mathbb{F}_{n_2} \times \mathbb{F}_{n_1}$  and  $\Pi(r, r, \ell) = \mathbb{F}_{m_{\ell}} \times \cdots \times \mathbb{F}_{m_2} \times \mathbb{F}_{m_1}$  denote the corresponding direct products of free groups.

**THEOREM 6.1.** *For  $\ell \geq 3$ , the groups  $P(r, k, \ell)$  and  $\Pi(r, k, \ell)$  are not isomorphic.*

We establish this result by distinguishing the characteristic varieties  $V_1(P(r, k, \ell))$  and  $V_1(\Pi(r, k, \ell))$ . The latter may be determined using Examples 3.3 and 3.5. In particular, this variety has  $\ell-1$  irreducible components. As we shall see in Proposition 6.11 and Remark 6.12, the variety  $V_1(P(r, k, \ell))$  has many more components.

An analogous result for Artin's pure braid group  $P_{\ell+1}$  and the direct product  $\Pi_{\ell+1} = \mathbb{F}_{\ell} \times \cdots \times \mathbb{F}_2 \times \mathbb{F}_1$  was proved in [4] by other means. Proposition 6.9 below provides another proof of this fact.

#### 6.2. Monomial arrangements in $\mathbb{C}^3$

We now determine the structure of the characteristic variety  $V_1(\mathcal{A})$  of every monomial arrangement  $\mathcal{A} = \mathcal{A}_{r,r,3}$  in  $\mathbb{C}^3$ . We first show that the variety  $V_1(\mathcal{A})$  contains an essential torus of dimension two. The cases  $r = 2$  and  $r = 3$  were considered previously by Falk [11] and Libgober [18].

A defining polynomial for  $\mathcal{A}$  is given by  $Q(\mathcal{A}) = (x_1^r - x_2^r)(x_1^r - x_3^r)(x_2^r - x_3^r)$ . Denote the hyperplanes of  $\mathcal{A}$  by  $H_{i,j}^{(k)} = \ker(x_i - \zeta^k x_j)$ , where  $\zeta = \exp(2\pi i/r)$ ,  $1 \leq i < j \leq 3$

and  $1 \leq k \leq r$ . In this notation, the rank two elements of the lattice of  $\mathcal{A}$  are given by

$$L_2 = \left\{ \bigcap_{k=1}^r H_{i,j}^{(k)}, 1 \leq i < j \leq 3, \quad H_{1,2}^{(p)} \cap H_{2,3}^{(q)} \cap H_{1,3}^{(s)}, \text{ where } s \equiv p + q \pmod{r} \right\}.$$

Consider the partition  $\Pi = (H_{1,2}^{(1)}, \dots, H_{1,2}^{(r)} \mid H_{1,3}^{(1)}, \dots, H_{1,3}^{(r)} \mid H_{2,3}^{(1)}, \dots, H_{2,3}^{(r)})$  of the rank one elements of  $L(\mathcal{A})$ . It is readily checked that this partition is neighbourly, with polychrome flats all of the triple points  $H_{1,2}^{(p)} \cap H_{2,3}^{(q)} \cap H_{1,3}^{(s)}$  noted above. Let  $n = 3r = |\mathcal{A}|$  and consider  $\mathbb{C}^n$  with basis  $e_{i,j}^{(k)}$  and coordinates  $\lambda_{i,j}^{(k)}$ ,  $1 \leq i < j \leq 3$  and  $1 \leq k \leq r$ . Let  $H_{[n]}$  denote the hyperplane defined by  $\sum \lambda_{i,j}^{(k)} = 0$  and let  $L' \subset L_2$  denote the set of polychrome flats. In this notation, for  $X \in L_2$ , we have  $H_X = \{\sum \lambda_{i,j}^{(k)} = 0 \mid \lambda_{i,j}^{(k)} \in X\}$ .

The subspace  $S_\Pi = H_{[n]} \cap \bigcap_{X \in L'} H_X$  arising from the neighbourly partition  $\Pi$  may be realized as the nullspace of the  $r^2 + 1$  by  $3r$  matrix  $K$  whose rows are determined by  $H_{[n]}$  and the  $H_X$  above. It is readily checked that the vectors

$$v_1 = \sum_{k=1}^r (e_{1,2}^{(k)} - e_{2,3}^{(k)}) \quad \text{and} \quad v_2 = \sum_{k=1}^r (e_{1,3}^{(k)} - e_{2,3}^{(k)})$$

are in  $S_\Pi$ . Furthermore, it is also easy to find  $3r - 2$  linearly independent rows of the matrix  $K$ . Thus  $\{v_1, v_2\}$  is a basis for  $S_\Pi$ . Finally, using this basis, we see that the form  $\langle \cdot, \cdot \rangle_\Pi$  is trivial on  $S_\Pi$ . Thus  $S_\Pi$  is an irreducible component of  $\mathcal{R}^1(A)$ .

Since  $\mathcal{R}^1(A) = \mathcal{V}_1(\mathcal{A})$  is the tangent cone of the characteristic variety  $V_1(\mathcal{A})$  at the point  $\mathbf{1}$ , we obtain a two-dimensional irreducible component,  $V_\Pi$ , of the latter by exponentiating. Using coordinates  $t_{i,j}^{(k)}$  for the complex torus  $(\mathbb{C}^*)^n$ , we have

$$V_\Pi = \left\{ \mathbf{t} \in (\mathbb{C}^*)^n \mid t_{i,j}^{(1)} = \dots = t_{i,j}^{(r)}, 1 \leq i < j \leq 3 \quad \text{and} \quad \prod_{\substack{1 \leq i < j \leq 3 \\ 1 \leq k \leq r}} t_{i,j}^{(k)} = 1 \right\}.$$

If  $r = 3$ , the set of rank one elements of the lattice of  $\mathcal{A} = \mathcal{A}_{3,3,3}$  admits three neighbourly partitions in addition to  $\Pi = \Pi_1$  above which give rise to essential 2-tori. These partitions are

$$\begin{aligned} \Pi_2 &= (H_{1,2}^{(1)}, H_{1,3}^{(2)}, H_{2,3}^{(1)} \mid H_{1,2}^{(2)}, H_{1,3}^{(1)}, H_{2,3}^{(2)} \mid H_{1,2}^{(3)}, H_{1,3}^{(3)}, H_{2,3}^{(3)}), \\ \Pi_3 &= (H_{1,2}^{(1)}, H_{1,3}^{(1)}, H_{2,3}^{(3)} \mid H_{1,2}^{(2)}, H_{1,3}^{(2)}, H_{2,3}^{(1)} \mid H_{1,2}^{(3)}, H_{1,3}^{(2)}, H_{2,3}^{(2)}), \\ \Pi_4 &= (H_{1,2}^{(1)}, H_{1,3}^{(3)}, H_{2,3}^{(2)} \mid H_{1,2}^{(2)}, H_{1,3}^{(2)}, H_{2,3}^{(3)} \mid H_{1,2}^{(3)}, H_{1,3}^{(1)}, H_{2,3}^{(1)}). \end{aligned}$$

Let  $V_{\Pi_k}$  denote the essential torus arising from  $\Pi_k$  and write  $V(3) := \bigcup_{k=1}^4 V_{\Pi_k}$ .

**LEMMA 6.3.** *If  $r \geq 4$ , the variety  $V(r) := V_\Pi$  is the only essential torus of the monomial arrangement  $\mathcal{A} = \mathcal{A}_{r,r,3}$ .*

*Proof.* It suffices to show that the partition  $\Pi$  above is the only neighbourly partition of  $L_1(\mathcal{A})$  which gives rise to an essential torus. Recall that  $L_2(\mathcal{A})$  consists of flats  $X_{i,j} = \bigcap_{k=1}^r H_{i,j}^{(k)}$ ,  $1 \leq i < j \leq 3$ , with  $|X_{i,j}| = r$  and  $Y_{p,q,s} = H_{1,2}^{(p)} \cap H_{2,3}^{(q)} \cap H_{1,3}^{(s)}$ , where  $s \equiv p + q \pmod{r}$ , with  $|Y_{p,q,s}| = 3$ .

Let  $\Gamma$  be a neighbourly partition of  $L_1(\mathcal{A})$ . If all three flats  $X_{i,j}$  are monochrome, then either  $\Gamma$  is trivial and gives rise to no essential torus, or  $\Gamma = \Pi$  is the partition considered above, giving rise to the torus  $V(r)$ . So assume that one of the flats  $X_{i,j}$

is polychrome. If all elements of  $L_2(\mathcal{A})$  are polychrome, it is readily checked that the subspace  $S_\Gamma$  arising from  $\Gamma$  consists only of the origin,  $S_\Gamma = \{\mathbf{0}\}$ , and yields no essential torus. Thus we may assume further that there is a monochrome flat.

Without loss of generality, suppose that the flat  $X_{1,2}$  is polychrome. If, say  $X_{1,3}$  is monochrome, then since  $\Gamma$  is neighbourly by assumption, all other elements of  $L_2(\mathcal{A})$  must be polychrome. Checking that  $S_\Gamma = \{\mathbf{0}\}$ , we see that  $\Gamma$  does not yield an essential torus in this instance.

Suppose now that  $Y_{r,r,r}$  is monochrome (and that  $X_{1,2}$  is polychrome). If  $Y_{r,r,r}$  is the only monochrome flat, one can check that  $S_\Gamma = \{\mathbf{0}\}$ . So suppose there is another monochrome flat. By the above considerations, we can assume that this flat is of multiplicity three, say  $Y_{r-1,r-2,r-1}$ . With these assumptions, the fact that  $\Gamma$  is neighbourly yields  $3r - 9$  polychrome flats:  $X_{i,j}$ ,  $1 \leq i < j \leq 3$ ,  $Y_{p,r,p}$ ,  $Y_{p,r-p,r}$ ,  $Y_{r,p,p}$ ,  $1 \leq p \leq r - 1$  and  $Y_{r-1,p+1,p}$ ,  $Y_{p+1,r-1,p}$ ,  $Y_{p,r-p-2,r-2}$ ,  $1 \leq p \leq r - 3$ . An exercise in linear algebra then reveals that  $S_\Gamma = \{\mathbf{0}\}$  in this case as well.

*Remark 6.4.* A comparable analysis reveals that the characteristic variety  $V_1(\mathcal{A})$  of every full monomial arrangement  $\mathcal{A} = \mathcal{A}_{r,1,3}$  also contains a two-dimensional essential torus. Briefly, we have  $\mathcal{A} = \mathcal{A}_{r,r,3} \cup \{H_1, H_2, H_3\}$ , where  $H_i = \ker x_i$ , so  $|\mathcal{A}| = n + 3$ . The partition  $\Pi = (H_3, H_{1,2}^{(i)} \mid H_2, H_{1,3}^{(j)} \mid H_1, H_{2,3}^{(k)})$  of the rank one elements of  $L(\mathcal{A})$  is neighbourly. All elements of  $L_2(\mathcal{A})$  of multiplicity greater than 2 are polychrome. The subspace  $S_\Pi \subset \mathbb{C}^{n+3}$  has basis  $\{v_1 + r(e_3 - e_1), v_2 + r(e_2 - e_1)\}$ , is an irreducible component of  $\mathcal{R}^1(A)$  and gives rise to the essential torus

$$\{\mathbf{t} \in (\mathbb{C}^*)^{n+3} \mid (t_{i,j}^{(1)})^r = \dots = (t_{i,j}^{(r)})^r = t_q, 1 \leq i < j \leq 3, q \neq i, j \text{ and } T = 1\}$$

of  $V_1(\mathcal{A})$ , where  $T = t_1 t_2 t_3 \prod t_{i,j}^{(k)}$ , the product over  $1 \leq i < j \leq 3$  and  $1 \leq k \leq r$ .

Returning to the arrangement  $\mathcal{A} = \mathcal{A}_{r,r,3}$ , we now find all irreducible components of the variety  $V_1(\mathcal{A})$  of positive dimension. We first require some preliminary results.

**LEMMA 6.5.** *If  $r = pq$ , then the monomial arrangement  $\mathcal{A}_{r,r,3}$  has  $p^2$  subarrangements lattice-isomorphic to the arrangement  $\mathcal{A}_{q,q,3}$ .*

*Proof.* Let  $K\zeta^a$  and  $K\zeta^b$  be two cosets of  $K = \langle \zeta^p \rangle = \mathbb{Z}_q$  in  $\mathbb{Z}_r$ . Note that there are  $p = r/q$  such cosets. It is then readily checked that the subarrangement

$$\{H_{1,2}^{(a+kp)}, H_{2,3}^{(b+kp)}, H_{1,3}^{(a+b+kp)} \mid k = 0, \dots, q-1\},$$

where the indices  $a + b + kp$  are taken mod  $r$ , is lattice-isomorphic to  $\mathcal{A}_{q,q,3}$ .  $\square$

By the discussion in Section 6.2, each of the subarrangements  $\mathcal{A}_{q,q,3}$  above gives rise to a two-dimensional component of  $V_1(\mathcal{A})$  (four such components if  $q = 3$ ). Denote these components by  $V(q; a, b) \cong V(q)$ . Note that the local component arising from  $H_{1,2}^{(a)} \cap H_{2,3}^{(b)} \cap H_{1,3}^{(a+b)} \in L_2(\mathcal{A})$  may be expressed as  $V(1; a, b)$ , and that essential tori of  $V_1(\mathcal{A})$  may be expressed as  $V(r; 1, 1)$ . We now show that these components and the local components  $V_{X_{i,j}}$  arising from  $X_{i,j} = \bigcap_{k=1}^r H_{i,j}^{(k)} \in L_2(\mathcal{A})$  are the only positive-dimensional components of  $V_1(\mathcal{A})$ .

**LEMMA 6.6.** *If  $\mathcal{B}$  is a subarrangement of  $\mathcal{A} = \mathcal{A}_{r,r,3}$  which for each  $q$  is not lattice-isomorphic to  $\mathcal{A}_{q,q,3}$ , then  $\mathcal{B}$  does not give rise to a non-local component of  $V_1(\mathcal{A})$ .*

*Proof.* A subarrangement  $\mathcal{B}$  of  $\mathcal{A} = \mathcal{A}_{r,r,3}$  may be specified by choosing three subsets  $I, J, K \subset [r]$ :  $\mathcal{B} = \{H_{1,2}^{(i)}, H_{1,3}^{(j)}, H_{2,3}^{(k)} \mid i \in I, j \in J, k \in K\}$ . It suffices to check that a subarrangement satisfying the hypothesis admits no non-trivial neighbourly partition. Since  $\mathcal{B}$  is, by assumption, not isomorphic to a monomial arrangement, we can find say  $i \in I$  and  $j \in J$  so that  $H_{2,3}^{(i-j)} \notin \mathcal{B}$ . Thus  $H_{1,2}^{(i)} \cap H_{1,3}^{(j)} \in L_2(\mathcal{B})$  is of multiplicity two. If  $\Pi$  is a neighbourly partition of  $\mathcal{B}$ , then  $H_{1,2}^{(i)}$  and  $H_{1,3}^{(j)}$  must lie in the same block of  $\Pi$  for this  $i$  and this  $j$ . It follows that  $H_{1,2}^{(i)}$  and  $H_{1,3}^{(j)}$  must lie in the same block of  $\Pi$  for all  $i$  and  $j$ . Consequently,  $\Pi$  is trivial.  $\square$

We summarize the above discussion with the following.

**PROPOSITION 6.7.** *The first central characteristic subvariety of the monomial arrangement  $\mathcal{A}_{r,r,3}$  is given by*

$$\check{V}_1(\mathcal{A}_{r,r,3}) = \bigcup_{1 \leq i < j \leq 3} V_{X_{i,j}} \cup \bigcup_{q \mid r} V(q: a, b), \quad (6.1)$$

and thus consists of 3 tori of dimension  $r - 1$  and  $\sum_{q \mid r} c_q(r^2/q^2)$  tori of dimension two, where  $c_3 = 4$  and  $c_q = 1$  for  $q \neq 3$ .

We now use the above results to identify (non-local) components of the characteristic variety  $V_1(\mathcal{A})$  for the monomial arrangements  $\mathcal{A}_{r,k,\ell}$ , for general  $\ell$ . Though it is not technically a monomial arrangement, we first consider the braid arrangement.

### 6.8. The braid arrangement

Let  $\mathcal{A}_\ell = \{H_{i,j} \mid 1 \leq i < j \leq \ell\}$  denote the braid arrangement in  $\mathbb{C}^\ell$ , with group  $P_\ell$ , Artin's pure braid group. Let  $n = \binom{\ell}{2} = |\mathcal{A}_\ell|$  and choose coordinates  $t_{i,j}$  for  $(\mathbb{C}^*)^n$ . The rank two elements of  $L(\mathcal{A}_\ell)$  are

$$L_2 = \{H_{i,j} \cap H_{p,q}, \{i, j\} \cap \{p, q\} = \emptyset \text{ and } H_{i,j} \cap H_{i,k} \cap H_{j,k}, 1 \leq i < j < k \leq \ell\}.$$

Thus the coarse combinatorial Alexander invariant is  $B^{\text{cc}}(\mathcal{A}_\ell) = \bigoplus_{1 \leq i < j < k \leq \ell} B_{i,j,k}$ , where  $B_{i,j,k}$  is the local Alexander invariant associated to  $H_{i,j} \cap H_{i,k} \cap H_{j,k} \in L_2$ . As noted in Theorem 3.9, this yields local components  $V_1^{\text{cc}}(\mathcal{A}_\ell) = \bigcup V_{i,j,k}$  of the characteristic variety  $V_1(\mathcal{A}_\ell)$ , where

$$V_{i,j,k} = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid t_{i,j}t_{i,k}t_{j,k} = 1 \text{ and } t_{p,q} = 1 \text{ if } |\{p, q\} \cap \{i, j, k\}| \leq 1\}.$$

Non-local components of the variety  $V_1(\mathcal{A}_\ell)$  may be detected as follows. Since the Coxeter groups  $D_3$  and  $A_3$  are isomorphic, the pure braid group  $P_4$  coincides with the generalized pure braid group  $P(2, 2, 3)$ . Thus, the calculation in Section 6.2 above yields a non-local component  $V \cong V(2)$  of  $V_1(\mathcal{A}_4)$ . In the current notation, this essential torus is given by

$$V = \{\mathbf{t} \in (\mathbb{C}^*)^6 \mid t_{1,2} = t_{3,4}, \quad t_{1,3} = t_{2,4}, \quad t_{2,3} = t_{1,4}, \quad \prod_{i < j} t_{i,j} = 1\}.$$

In general, recall that  $n = \binom{\ell}{2}$ , write  $T = \prod_{1 \leq i < j \leq \ell} t_{i,j}$  and for each 4-element subset  $I$  of  $[\ell]$ , let

$$V_I = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid t_{i,j} = t_{p,q} \text{ if } \{i, j\} \cup \{p, q\} = I, \quad t_{p,q} = 1 \text{ if } \{p, q\} \notin I, T = 1\}.$$

Each such subset  $I$  gives rise to a subarrangement  $\mathcal{A}_I \subset \mathcal{A}_\ell$  that is lattice-isomorphic to the arrangement  $\mathcal{A}_4 \cong \mathcal{A}_{2,2,3}$ . Thus, each torus  $V_I$  is a component of  $V_1(\mathcal{A}_\ell)$ .

PROPOSITION 6-9. *The first central characteristic subvariety of the braid arrangement  $\mathcal{A}_\ell$  is given by*

$$\check{V}_1(\mathcal{A}_\ell) = V_1^{\text{cc}}(\mathcal{A}_\ell) \cup \bigcup_{\substack{I \subset [\ell] \\ |I|=4}} V_I,$$

and thus consists of  $\binom{\ell}{3} + \binom{\ell}{4} = \binom{\ell+1}{4}$  tori of dimension two.

*Proof.* The inclusion  $V_1^{\text{cc}}(\mathcal{A}_\ell) \cup \bigcup_{|I|=4} V_I \subseteq \check{V}_1(\mathcal{A}_\ell)$  follows from the above discussion. For the reverse inclusion, note that every subarrangement of the braid arrangement is a direct product of braid arrangements of smaller rank and that for  $\ell \geq 5$ , the arrangement  $\mathcal{A}_\ell$  admits no non-trivial neighbourly partition.

#### 6-10. The monomial arrangements $\mathcal{A}_{r,r,\ell}$

We now obtain a similar description of the characteristic variety  $V_1(\mathcal{A})$  for the monomial arrangement  $\mathcal{A} = \mathcal{A}_{r,r,\ell}$ . Recall that the hyperplanes of  $\mathcal{A}$  are denoted by  $H_{i,j}^{(k)} = \ker(x_i - \zeta^k x_j)$ , where  $\zeta = \exp(2\pi i/r)$ . Each subset  $I = \{i_1, i_2, i_3\}$  of  $[\ell]$  gives rise to a subarrangement  $\mathcal{A}_I = \{H_{i_p, i_q}^{(k)} \mid 1 \leq k \leq r, 1 \leq p < q \leq 3\}$  of  $\mathcal{A}$  that is lattice-isomorphic to  $\mathcal{A}_{r,r,3}$ . Let  $V_I$  denote the variety specified by Section 6-1 above, in the appropriate coordinates.

PROPOSITION 6-11. *The first central characteristic subvariety of the monomial arrangement  $\mathcal{A}_{r,r,\ell}$  is given by*

$$\check{V}_1(\mathcal{A}_{r,r,\ell}) = V_1^{\text{cc}}(\mathcal{A}_{r,r,\ell}) \cup \bigcup_{\substack{I \subset [\ell] \\ |I|=3}} V_I.$$

*Proof.* Write  $\mathcal{A} = \mathcal{A}_{r,r,\ell}$ . The inclusion  $V_1^{\text{cc}}(\mathcal{A}) \cup \bigcup_{|I|=3} V_I \subseteq \check{V}_1(\mathcal{A})$  follows from the above discussion. For the reverse inclusion, it suffices to show that  $\check{V}_1(\mathcal{A})$  has only those non-local components specified above. For this, first note that if  $\ell = 4$ , then  $\mathcal{A}$  has a non-trivial neighbourly partition  $\Pi = (H_{1,2}^{(*)}, H_{3,4}^{(*)} \mid H_{1,3}^{(*)}, H_{2,4}^{(*)} \mid H_{2,3}^{(*)}, H_{1,4}^{(*)})$ . However, an exercise in linear algebra reveals that the subspace  $S_\Pi$  associated to this partition consists only of the origin,  $S_\Pi = \{\mathbf{0}\}$ . If  $\ell \geq 5$ , the arrangement  $\mathcal{A}_{r,r,\ell}$  admits no non-trivial neighbourly partition. Thus, the monomial arrangements  $\mathcal{A}_{r,r,\ell}$  of rank greater than three have no essential tori.

Finally, if  $\mathcal{B}$  is a subarrangement of  $\mathcal{A} = \mathcal{A}_{r,r,\ell}$  that is not lattice-isomorphic to a monomial arrangement  $\mathcal{A}_{q,q,k}$ , then an argument similar to the proof of Lemma 6-6 shows that  $\mathcal{B}$  does not give rise to a non-local component of  $V_1(\mathcal{A})$ .

*Remark 6-12.* The characteristic variety  $V_1(\mathcal{A}_{r,1,\ell})$  of the full monomial arrangement may be analysed in an analogous fashion and the above calculations may be used to identify non-local components of this variety. For instance, let  $\mathcal{B}_\ell = \mathcal{A}_{2,1,\ell}$  be the Coxeter arrangement of type B. Associated to each subset  $I = \{i_1, i_2, i_3\}$  of  $[\ell]$  is a subarrangement  $\mathcal{B}_I$  of  $\mathcal{B}_\ell$  that is lattice-isomorphic to  $\mathcal{B}_3 = \mathcal{A}_{2,1,3}$ . Each such subarrangement gives rise to 12 two-dimensional non-local components of  $V_1(\mathcal{B}_\ell)$ , one corresponding to the subarrangement itself (see Remark 6-4) and the remaining 11 corresponding to subarrangements of  $\mathcal{B}_I$  lattice-isomorphic to  $\mathcal{A}_4$ . In this way, we obtain  $12\binom{\ell}{3}$  two-dimensional non-local components of  $V_1(\mathcal{B}_\ell)$ . Similarly, for general  $r$ , subsets  $I$  of  $[\ell]$  as above yield subarrangements  $\mathcal{A}_I$  of  $\mathcal{A}_{r,1,\ell}$ , and associated

non-local components of  $V_1(\mathcal{A}_{r,1,\ell})$ . Further analysis of the characteristic varieties of the full monomial arrangements is left to the interested reader.

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