Pure braid groups are not residually free

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Abstract. We show that the Artin pure braid group P_n is not residually free for $n \geq 4$. Our results also show that the corank of P_n is equal to 2 for $n \geq 3$.

1 Introduction

A group G is residually free if for every $x \neq 1$ in G, there is a homomorphism f from G to a free group F so that $f(x) \neq 1$ in F. Equivalently, G embeds in a product of free groups (of finite rank). Examples of residually free groups include the fundamental groups of orientable surfaces. In this note, we show that the Artin pure braid group, the kernel $P_n = \ker(B_n \to \Sigma_n)$ of the natural map from the braid group to the symmetric group, is not residually free for $n \geq 4$. (It is easy to see that the pure braid groups P_2 and P_3 are residually free.) We also classify all epimorphisms from the pure braid group to free groups, and determine the corank of the pure braid group. For $n \geq 5$, the fact that P_n is not residually free was established independently by L. Paris (unpublished), see Remark 5.4.

For $n \geq 3$, the braid groups themselves are not residually free. Indeed, the only nontrivial two-generator residually free groups are $\mathbb{Z}$, $\mathbb{Z}^2$, and F_2, the nonabelian free group of rank two, see Wilton [Wil08]. Since B_n can be generated by two elements for $n \geq 3$, it is not residually free. (For $n = 2$, $B_2 = \mathbb{Z}$ is residually free.)

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If G is a group which is not residually free, then any group $\tilde{G}$ with a subgroup isomorphic to G cannot be residually free. Consequently, the (pure) braid groups are "poison" groups for residual freeness. In particular, a group with a subgroup isomorphic to the 4-strand pure braid group P_4 or the 3-strand braid group B_3 is not residually free. Since $P_4 < P_n$ for every $n \geq 4$, our main result follows from the special case $n = 4$. Moreover, the same observation enables us to show that a number of other groups are not residually free. These include (pure) braid groups of orientable surfaces, the (pure) braid groups associated to the full monomial groups, and a number of irreducible (pure) Artin groups of finite type.

This research was motivated by our work in [CFR10, Section 3], which implies the residual freeness of fundamental groups of the complements of certain complex hyperplane arrangements. In particular, the proof of the assertion in the last sentence of [CFR10, Example 3.25] gives the last step in the proof of Theorem 5.2 below.

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2 Automorphisms of the pure braid group

Let B_n be the Artin braid group, with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$, and $\sigma_j\sigma_i = \sigma_i\sigma_j$ for $|j-i| \geq 2$. The Artin pure braid group P_n has generators

$$A_{i,j} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} = \sigma_i^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_i,$$

and relations

$$A_{r,s}^{-1}A_{i,j}A_{r,s} = \begin{cases} A_{i,j} & \text{if } i < r < s < j, \\ A_{i,j} & \text{if } r < s < i < j, \\ A_{r,j}A_{i,j}A_{r,j}^{-1} & \text{if } r < s = i < j, \\ A_{r,j}A_{s,j}A_{i,j}A_{s,j}^{-1}A_{r,j}^{-1} & \text{if } r = i < s < j, \\ [A_{r,j}, A_{s,j}]A_{i,j}[A_{r,j}, A_{s,j}]^{-1} & \text{if } r < i < s < j, \end{cases} \tag{2.1}$$

where $[u, v] = uvu^{-1}v^{-1}$ denotes the commutator. See, for instance, Birman [Bir75] as a general reference on braid groups. It is well known that the pure braid group admits a direct product decomposition

$$P_n = Z \times P_n/Z, \tag{2.2}$$

where $Z = Z(P_n) \cong \mathbb{Z}$ is the center of P_n, generated by

$$Z_n = (A_{1,2})(A_{1,3}A_{2,3}) \cdots (A_{1,n} \cdots A_{n-1,n}). \tag{2.3}$$

Note that $P_3 = Z(P_3) \times P_3/Z(P_3) \cong \mathbb{Z} \times F_2$, where F_2 is the free group on two generators. For any $n \geq 3$, by (2.2), there is a split, short exact sequence

$$1 \to \mathrm{tv}(P_n) \to \mathrm{Aut}(P_n) \leftrightarrows \mathrm{Aut}(P_n/Z) \to 1, \tag{2.4}$$

where the subgroup $\mathrm{tv}(P_n)$ of $\mathrm{Aut}(P_n)$ consists of those automorphisms which become trivial upon passing to the quotient P_n/Z.

For a group G with infinite cyclic center $Z = Z(G) = \langle z \rangle$, a transvection is an endomorphism of G of the form $x \mapsto xz^{t(x)}$, where $t\colon G \to \mathbb{Z}$ is a homomorphism, see Charney and Crisp [CC05]. Such a map is an automorphism if and only if its restriction to Z is surjective, which is the case if and only if $z \mapsto z$ or $z \mapsto z^{-1}$, that is, $t(z) = 0$ or $t(z) = -2$. For the pure braid group P_n, the transvection subgroup $\mathrm{tv}(P_n)$ of $\mathrm{Aut}(P_n)$ consists of automorphisms of the form $A_{i,j} \mapsto A_{i,j}Z_n^{t_{i,j}}$, where $t_{i,j} \in \mathbb{Z}$ and $\sum t_{i,j}$ is either equal to 0 or -2. In the former case, $Z_n \mapsto Z_n$, while $Z_n \mapsto Z_n^{-1}$ in the latter. This yields a surjection $\mathrm{tv}(P_n) \to \mathbb{Z}_2$, with kernel consisting of transvections for which $\sum t_{i,j} = 0$. Since P_n has $\binom{n}{2} = N + 1$ generators, this kernel is free abelian of rank N. The choice $t_{1,2} = -2$ and all other $t_{i,j} = 0$ gives a splitting $\mathbb{Z}_2 \to \mathrm{tv}(P_n)$. Thus, $\mathrm{tv}(P_n) \cong \mathbb{Z}^N \rtimes \mathbb{Z}_2$. Explicit generators of $\mathrm{tv}(P_n)$ are given below.

For $n \geq 4$, Bell and Margalit [BM07] show that the automorphism group of the pure braid group admits a semidirect product decomposition

$$\mathrm{Aut}(P_n) \cong (\mathbb{Z}^N \rtimes \mathbb{Z}_2) \rtimes \mathrm{Mod}(\mathbb{S}_{n+1}). \tag{2.5}$$

Here, $\mathbb{Z}^N \rtimes \mathbb{Z}_2 = \mathrm{tv}(P_n)$ is the transvection subgroup of $\mathrm{Aut}(P_n)$ described above, $\mathbb{S}_{n+1}$ denotes the sphere S^2 with $n+1$ punctures, and $\mathrm{Mod}(\mathbb{S}_{n+1})$ is the extended mapping class group of $\mathbb{S}_{n+1}$, the group of isotopy classes of all self-diffeomorphisms of $\mathbb{S}_{n+1}$. The semidirect product decomposition (2.5) is used in [Coh11] to determine a finite presentation for $\mathrm{Aut}(P_n)$. From this work, it follows that $\mathrm{Aut}(P_n)$ is generated by automorphisms

$$\xi,\ \beta_k\ (1 \leq k \leq n),\ \psi,\ \phi_{p,q}\ (1 \leq p < q \leq n,\ \{p,q\} \neq \{1,2\}), \tag{2.6}$$

given explicitly by

$$
\begin{aligned}
&\xi\colon A_{i,j} \mapsto (A_{i+1,j}\cdots A_{j-1,j})^{-1}A_{i,j}^{-1}(A_{i+1,j}\cdots A_{j-1,j}),\\
&\beta_k\colon A_{i,j} \mapsto \begin{cases} A_{i-1,j} & \text{if } k=i-1,\\ A_{i,i+1}^{-1}A_{i+1,j}A_{i,i+1} & \text{if } k=i<j-1,\\ A_{i,j-1} & \text{if } k=j-1>i,\\ A_{j,j+1}^{-1}A_{i,j+1}A_{j,j+1} & \text{if } k=j,\\ A_{i,j} & \text{otherwise,}\end{cases} \quad \text{for } 1\le k\le n-1,\\
&\beta_n\colon A_{i,j} \mapsto \begin{cases} A_{i,j} & \text{if } j\neq n,\\ (A_{i,n}A_{1,i}\cdots A_{i-1,i}A_{i,i+1}\cdots A_{i,n-1})^{-1} & \text{if } j=n,\end{cases}\\
&\psi\colon A_{i,j} \mapsto \begin{cases} A_{1,2}Z_n^{-2} & \text{if } i=1 \text{ and } j=2,\\ A_{i,j} & \text{otherwise,}\end{cases}\\
&\phi_{p,q}\colon A_{i,j} \mapsto \begin{cases} A_{1,2}Z_n & \text{if } i=1 \text{ and } j=2,\\ A_{p,q}Z_n^{-1} & \text{if } i=p \text{ and } j=q,\\ A_{i,j} & \text{otherwise.}\end{cases}
\end{aligned}
\tag{2.7}
$$

It is readily checked that these are all automorphisms of P_n. The automorphisms ψ and $\phi_{p,q}$ are transvections. For $k \le n-1$, $\beta_k \in \mathrm{Aut}(P_n)$ arises from the conjugation action of B_n on P_n, $\beta_k(A_{i,j}) = \sigma_k^{-1}A_{i,j}\sigma_k$, see Dyer and Grossman [DG81].

Remark 2.1. The presentation of $\mathrm{Aut}(P_n)$ found in [Coh11] is given in terms of the generating set ϵ, ω_k $(1 \le k \le n)$, ψ, $\phi_{p,q}$ $(1 \le p < q \le n,\ \{p,q\} \neq \{1,2\})$, where $\xi = \epsilon \circ \psi$, $\beta_2 = \omega_2 \circ \phi_{1,3}^{-1}$, $\beta_n = \omega_n \circ \psi \circ \phi_{1,n} \circ \phi_{2,n}$, and $\beta_k = \omega_k$ for $k \neq 2, n$. This presentation exhibits the semidirect product structure (2.5) of $\mathrm{Aut}(P_n)$.

3 Epimorphisms to free groups

We study surjective homomorphisms from the pure braid group P_n to the free group F_k on $k \ge 2$ generators. Since $P_2 = \mathbb{Z}$ is infinite cyclic, we assume that $n \ge 3$. We begin by exhibiting a number of specific such homomorphisms.

Let $F_2 = \langle x, y\rangle$ be the free group on two generators, and write $[n] = \{1, 2, \ldots, n\}$. If $I = \{i, j, k\} \subset [n]$ with $i < j < k$, define $f_I\colon P_n \to F_2$

by

$$f_I(A_{r,s}) = \begin{cases} x & \text{if } r = i \text{ and } s = j, \\ y & \text{if } r = i \text{ and } s = k, \\ y^{-1}x^{-1} & \text{if } r = j \text{ and } s = k, \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

If $I = \{i, j, k, l\} \subset [n]$ with $i < j < k < l$, define $f_I \colon P_n \to F_2$ by

$$f_I(A_{r,s}) = \begin{cases} x & \text{if } r = i \text{ and } s = j, \\ y & \text{if } r = i \text{ and } s = k, \\ y^{-1}x^{-1} & \text{if } r = j \text{ and } s = k, \\ y^{-1}x^{-1} & \text{if } r = i \text{ and } s = l, \\ xyx^{-1} & \text{if } r = j \text{ and } s = l, \\ x & \text{if } r = k \text{ and } s = l, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2}$$

In either case (3.1) or (3.2), note that f_I is surjective by construction. It is readily checked that f_I is a homomorphism. We will show that these are, in an appropriate sense, the only epimorphisms from the pure braid group to a nonabelian free group.

Remark 3.1. The epimorphisms $f_I \colon P_n \to F_2$ are induced by maps of topological spaces. Let

$$F(\mathbb{C}, n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$$

be the configuration space of n distinct ordered points in $\mathbb{C}$. It is well known that $P_n = \pi_1(F(\mathbb{C}, n))$ and that $F(\mathbb{C}, n)$ is a $K(P_n, 1)$-space.

For a subset I of $[n]$ of cardinality k, let $p_I \colon F(\mathbb{C}, n) \to F(\mathbb{C}, k)$ denote the projection which forgets all coordinates not indexed by I. The induced map on pure braid groups forgets the corresponding strands. Additionally, let $q_n \colon F(\mathbb{C}, n) \to F(\mathbb{C}, n)/\mathbb{C}^*$ denote the natural projection, where $\mathbb{C}^*$ acts by scalar multiplication. In particular, $q_3 \colon F(\mathbb{C}, 3) \to F(\mathbb{C}, 3)/\mathbb{C}^* \cong \mathbb{C} \times (\mathbb{C} \setminus \{\text{two points}\})$. Finally, define $g \colon F(\mathbb{C}, 4) \to F(\mathbb{C}, 3)$ by

$$g(z_1,z_2,z_3,z_4) = \big((z_1+z_2-z_3-z_4)^2, (z_1+z_3-z_2-z_4)^2, (z_1+z_4-z_2-z_3)^2\big).$$

One can check that if $|I| = 3$, then $f_I = (q_3 \circ p_I)_*$, while if $|I| = 4$, $f_I = (q_3 \circ g \circ p_I)_*$.

In the case $n = 4$, $F(\mathbb{C}, n)$ is diffeomorphic to $\mathbb{C} \times M$, where M is the complement of the Coxeter arrangement $\mathcal{A}$ of type D_3. With this identification, the mappings p_I and g correspond to the components of the mapping constructed in [CFR10, Example 3.25]. The map g is the pencil associated with the non-local component of the first resonance variety of $H^*(M;\mathbb{C})$ as in [LY00, FY07], see below.

One can also check that the homomorphism $g_*\colon P_4 \to P_3$ is the restriction to pure braid groups of the famous homomorphism $B_4 \to B_3$ of full braid groups, given by $\sigma_1 \mapsto \sigma_1, \sigma_2 \mapsto \sigma_2, \sigma_3 \mapsto \sigma_1$.

Let G and H be groups, and let f and g be (surjective) homomorphisms from G to H. Call f and g equivalent if there are automorphisms $\phi \in \mathrm{Aut}(G)$ and $\psi \in \mathrm{Aut}(H)$ so that $g \circ \phi = \psi \circ f$. If f and g are equivalent, we write $f \sim g$.

Proposition 3.2. *If I and J are subsets of $[n]$ of cardinalities 3 or 4, then the epimorphisms f_I and f_J from P_n to F_2 are equivalent.*

Proof. If $n = 3$, there is nothing to prove. So assume that $n \geq 4$.

Let $I = \{i_1, \dots, i_q\} \subset [n]$ with $q \geq 2$ and $i_1 < i_2 < \cdots < i_q$. Define $\alpha_I \in B_n$ by

$$\alpha_I = (\sigma_{i_1-1}\cdots\sigma_1)(\sigma_{i_2-1}\cdots\sigma_2)\cdots(\sigma_{i_q-1}\cdots\sigma_q),$$

where $\sigma_{i_k-1}\cdots\sigma_k = 1$ if $i_k = k$. Then $\alpha_I^{-1}A_{i_r,i_s}\alpha_I = A_{r,s}$ for $1 \leq r < s \leq q$. This can be seen by checking that, for instance, the geometric braids $\alpha_I A_{r,s}\alpha_I^{-1}$ and A_{i_r,i_s} are equivalent. Denote the automorphism $A_{i,j} \mapsto \alpha_I^{-1}A_{i,j}\alpha_I$ of P_n by the same symbol,

$$\alpha_I = (\beta_{i_1-1}\cdots\beta_1)(\beta_{i_2-1}\cdots\beta_2)\cdots(\beta_{i_q-1}\cdots\beta_q) \in \mathrm{Aut}(P_n).$$

Then, $\alpha_I(A_{i_r,i_s}) = A_{r,s}$ for $1 \leq r < s \leq q$.

If I has cardinality 3, then, by the above, we have $f_I = f_{[3]} \circ \alpha_I$, so $f_I \sim f_{[3]}$. Similarly, if $|I| = 4$, then $f_I = f_{[4]} \circ \alpha_I$ and $f_I \sim f_{[4]}$. Thus it suffices to show that $f_I \sim f_{[3]}$ for some I with $|I| = 4$. This can be established by checking that $f_I = f_{[3]} \circ \beta_n$ for $I = \{1, 2, 3, n\}$. $\square$

Remark 3.3. The homomorphisms f_I also have a natural interpretation in terms of the moduli space $\mathfrak{M}_{0,n}$ of genus-zero curves with n marked points. By definition, $\mathfrak{M}_{0,n}$ is the quotient of the configuration space $F(S^2,n)$ of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ by the action of $\mathrm{PSL}(2,\mathbb{C})$. The map $h_n\colon F(\mathbb{C}, n) \to \mathfrak{M}_{0,n+1}$ given by $h_n(z_1, \dots, z_n) = [(z_1, \dots, z_n, \infty)]$ induces a homeomorphism

$$F(\mathbb{C}, n)/\mathrm{Aff}(\mathbb{C}) \to \mathfrak{M}_{0,n+1},$$

where $\mathrm{Aff}(\mathbb{C}) \cong \mathbb{C} \rtimes \mathbb{C}^*$ is the affine group, and a homotopy equivalence

$$\overline{h}_n \colon F(\mathbb{C}, n)/\mathbb{C}^* \to \mathfrak{M}_{0,n+1}.$$

For $1 \le i \le 5$, let $\delta_i \colon \mathfrak{M}_{0,5} \to \mathfrak{M}_{0,4}$ be defined by forgetting the i-th point. Then, in the notation of Remark 3.1, for $1 \le i \le 4$, $\delta_i \circ h_4 = \overline{h}_3 \circ q_3 \circ p_I$, where $I = [4] \setminus \{i\}$. Up to a linear change of coordinates in $F(\mathbb{C}, 3)$, $\delta_5 \circ h_4 = \overline{h}_3 \circ q_3 \circ g$. (See also Pereira [Per10, Example 3.1].) The maps $\delta_i \colon \mathfrak{M}_{0,5} \to \mathfrak{M}_{0,4}$, $1 \le i \le 5$, are clearly equivalent up to diffeomorphism of the source. Applying Remark 3.1, this gives an alternate proof of Proposition 3.2.

For our next result, we require some properties of the cohomology ring of the pure braid group. Let $A = \bigoplus_{k \ge 0} A^k$ be a graded algebra over a field $\Bbbk$ that is connected ($A^0 \cong \Bbbk$), graded-commutative ($b \cdot a = (-1)^{pq} a \cdot b$ for $a \in A^p$ and $b \in A^q$), and satisfies $\dim A^1 < \infty$. Since $a \cdot a = 0$ for each $a \in A^1$, multiplication by a defines a cochain complex (A, δ_a):

$$A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} A^2 \xrightarrow{\delta_a} \cdots\cdots \xrightarrow{\delta_a} A^\ell,$$

where $\delta_a(x) = ax$. The resonance varieties $\mathcal{R}^d(A)$ of A are defined by

$$\mathcal{R}^d(A) = \{a \in A^1 \mid H^d(A, \delta_a) \neq 0\}.$$

If $\dim A^d < \infty$, then $\mathcal{R}^d(A)$ is an algebraic set in A^1.

In the case where $A = H^*(M(\mathcal{A}); \Bbbk)$ is the cohomology ring of the complement of a complex hyperplane arrangement, and $\Bbbk$ has characteristic zero, work of Libgober and Yuzvinsky [LY00] (see also [FY07]) shows that $\mathcal{R}^1(A)$ is the union of the maximal isotropic subspaces of A^1 for the quadratic form

$$\mu \colon A^1 \otimes A^1 \to A^2, \;\; \mu(a \otimes b) = ab \tag{3.3}$$

having dimension at least two. Note that, for any field $\Bbbk$, any isotropic subspace of A^1 of dimension at least two is contained in $\mathcal{R}^1(A)$.

For our purposes, it will suffice to take $\Bbbk = \mathbb{Q}$. Let $A = H^*(P_n; \mathbb{Q})$ be the rational cohomology ring of the pure braid group, that is, the cohomology of the complement of the braid arrangement in $\mathbb{C}^n$. By work of Arnold [Arn69] and F. Cohen [Coh76], A is generated by degree one elements $a_{i,j}$, $1 \le i < j \le n$, which satisfy (only) the relations

$$a_{i,j}a_{i,k} - a_{i,j}a_{j,k} + a_{i,k}a_{j,k} \text{ for } 1 \le i < j < k \le n,$$

and their consequences. The irreducible components of the first resonance variety $\mathcal{R}^1(A)$ may be obtained from work of Cohen and Suciu [CS00] (see also [Per10]), and may be described as follows. The rational vector space $A^1 = H^1(P_n; \mathbb{Q})$ is of dimension $\binom{n}{2}$, and has basis $\{a_{i,j} \mid 1 \leq i < j \leq n\}$. If $I = \{i, j, k\} \subset [n]$ with $i < j < k$, let V_I be the subspace of A^1 defined by

$$V_I = \operatorname{span}\{a_{i,j} - a_{j,k}, a_{i,k} - a_{j,k}\}.$$

If $I = \{i, j, k, l\} \subset [n]$ with $i < j < k < l$, let V_I be the subspace of A^1 defined by

$$V_I = \operatorname{span}\{a_{i,j} + a_{k,l} - a_{j,k} - a_{i,l}, a_{i,k} + a_{j,l} - a_{j,k} - a_{i,l}\}.$$

The 2-dimensional subspaces V_I of A^1, where $|I| = 3$ or $|I| = 4$, are the irreducible components of $\mathcal{R}^1(A) = \bigcup_{k=3}^{4} \bigcup_{|I|=k} V_I$.

Remark 3.4. One can check that $V_I = f_I^*(H^1(F_2; \mathbb{Q}))$, where $f_I\colon P_n \to F_2$ is defined by (3.1) if $|I| = 3$ and by (3.2) if $|I| = 4$. Work of Schenck and Suciu [SS06, Lemma 5.3] implies that for any two components V_I and V_J of $\mathcal{R}^1(A)$, there is an isomorphism of $A^1 = H^1(P_n; \mathbb{Q})$ taking V_I to V_J. This provides an analog, on the level of (degree-one) cohomology, of Proposition 3.2.

Theorem 3.5. *Let $n \geq 3$ and $k \geq 2$, and consider the pure braid group P_n and the free group F_k.*

1. *If $k \geq 3$, there are no epimorphisms from P_n to F_k.*
2. *If $k = 2$, there is a single equivalence class of epimorphisms from P_n to F_2.*

Proof. For part (3.5), if $f\colon P_n \to F_k$ is an epimorphism, then f splits, so

$$f^*\colon H^1(F_k; \mathbb{Q}) \to H^1(P_n; \mathbb{Q})$$

is injective. Consequently, $f^*(H^1(F_k; \mathbb{Q}))$ is a k-dimensional isotropic subspace of $A^1 = H^1(P_n; \mathbb{Q})$ for the form (3.3). Since this subspace is isotropic, it must be contained in an irreducible component of $\mathcal{R}^1(A)$. Since these components are all of dimension 2, we cannot have $k \geq 3$.

For part (3.5), by Proposition 3.2, it suffices to show that an arbitrary epimorphism $f\colon P_n \to F_2$ is equivalent to f_I for some I of cardinality 3 or 4. We will extensively use that fact that if $[a, b] = 1$ in F_2, then $\langle a, b\rangle < F_2$ is free and abelian, so $a = z^m$ and $b = z^n$ for some $z \in F_2$ and $m, n \in \mathbb{Z}$. Additionally, if the homology class $[a]$ of a in $H_1(F_2; \mathbb{Z})$

is part of a basis and $[a, b] = 1$, then b is a power of a. Indeed, suppose $\{[a], [a']\}$ is a basis for $H_1(F_2; \mathbb{Z})$. Write $[z] = c_1[a] + c_2[a']$ with $c_1, c_2 \in \mathbb{Z}$. Then we have $[a] = m[z] = mc_1[a]+mc_2[a']$, which implies $c_2 = 0$ and $m = c_1 = \pm 1$, yielding the assertion.

Let $f\colon P_n \to F_2$ be an epimorphism. Then $f^*(H^1(F_2; \mathbb{Q}))$ is a 2-dimensional isotropic subspace of $H^1(P_n; \mathbb{Q})$ for the form (3.3). Since the irreducible components of $\mathcal{R}^1(A)$ are all of dimension 2, we must have $f^*(H^1(F_2; \mathbb{Q})) = V_I$ for some $I \subset [n]$ of cardinality 3 or 4. Since the cohomology rings of P_n and F_2 are torsion-free, passing to integer coefficients, we have $f^*(H^1(F_2; \mathbb{Z})) = V_I \cap \mathbb{Z}^{\binom{n}{2}} \subset H^1(P_n; \mathbb{Z})$. Consequently, there is an automorphism φ^* of $H^1(F_2; \mathbb{Z})$ so that $f^* = f_I^* \circ \varphi^*$. Passing to homology (again using torsion-freeness), we have $f_* = \varphi_* \circ (f_I)_*$, where $\varphi_* \in \operatorname{Aut}(H_1(F_2; \mathbb{Z}))$ is dual to φ^*. Let $\varphi \in \operatorname{Aut}(F_2)$ be an automorphism which induces φ_*.

From the definitions (3.1) and (3.2) of the epimorphisms f_I, there exists $\{i, j, k\}$ with $1 \le i < j < k \le n$, $f_I(A_{i,j}) = x$, and $f_I(A_{i,k}) = y$, where $F_2 = \langle x, y\rangle$. Let $u = f(A_{i,j})$ and $v = f(A_{i,k})$. Using the equation $f_* = \varphi_* \circ (f_I)_*$, we have

$$[u] = [f(A_{i,j})] = f_*([A_{i,j}]) = \varphi_*([f_I(A_{i,j})] = \varphi_*([x]) = [\varphi(x)],$$

and similarly $[v] = [\varphi(y)]$. Thus $\{[u], [v]\}$ is a basis for $H_1(F_2; \mathbb{Z})$.

Let $w = f(A_{j,k})$. Using the pure braid relations (2.1), we have $A_{i,j}A_{i,k}A_{j,k} = A_{i,k}A_{j,k}A_{i,j} = A_{j,k}A_{i,j}A_{i,k}$. Applying f, these imply that $[uv, w] = 1$ and $[u, vw] = 1$ in F_2. Since $\{[u], [u]+[v]\}$ is a basis for $H_1(F_2; \mathbb{Z})$, these imply that $w = (uv)^m$ and $vw = u^n$ for some $m, n \in \mathbb{Z}$. A calculation with homology classes reveals that $m = n = -1$. Hence $w = v^{-1}u^{-1}$, i.e., $uvw = 1$.

Suppose that $f(A_{r,s}) = 1$ for all $\{r, s\} \not\subset \{i, j, k\}$. Then the image of f is contained in the subgroup $\langle u, v\rangle$ of F_2. Since f is by hypothesis an epimorphism, we have $\langle u, v\rangle = F_2$. Letting λ be the automorphism of F_2 taking u to x and v to y, we have $\lambda \circ f = f_{\{i,j,k\}}$.

Now suppose that $f(A_{r,s}) \neq 1$ for some $\{r, s\} \not\subset \{i, j, k\}$. First assume that $\{r, s\} \cap \{i, j, k\} = \emptyset$. We claim that $f(A_{r,s}) = 1$. There are various cases depending on the relative positions of $r < s$ and $i < j < k$. We consider the case $i < r < j < k < s$ and leave the remaining analogous cases to the reader. In this instance, we have relations $[A_{j,k}, A_{r,s}] = 1$ and $A_{i,j}^{-1}A_{r,s}A_{i,j} = [A_{i,s}, A_{j,s}]A_{r,s}[A_{i,s}, A_{j,s}]^{-1}$. The second of these, together with the pure braid relations (2.1) may be used to show that $[A_{i,j}, A_{j,s}^{-1}A_{r,s}A_{j,s}] = 1$. Applying f, we have $[w, f(A_{r,s})] = 1$ and $[u, z^{-1}f(A_{r,s})z] = 1$ in F_2, where $z = f(A_{j,s})$. Since any two element subset of $\{[u],[v],[w]\}$ forms a basis for $H_1(F_2; \mathbb{Z})$, these relations imply that

$f(A_{r,s}) = w^m$ and $z^{-1} f(A_{r,s}) z = u^n$ for some $m, n \in \mathbb{Z}$. Consequently, $m[w]=n[u]$ in $H_1(F_2; \mathbb{Z})$, which forces $m = n = 0$ and $f(A_{r,s}) = 1$.

Thus, we must have $f(A_{r,s}) \neq 1$ for some $\{r, s\}$ with $|\{r, s\} \cap \{i, j, k\}| = 1$. As above, there are several cases, and we consider a representative one, leaving the other, similar, cases to the reader.

Assume that $r = k$, so that $i < j < k < s$, and that $f(A_{k,s}) \neq 1$. Applying f to the pure braid relations $[A_{i,j}, A_{k,s}] = 1$, $[A_{j,k}, A_{i,s}] = 1$, and $[A_{i,k}, A_{k,s}^{-1} A_{j,s} A_{k,s}] = 1$ yields $[u, f(A_{k,s})] = 1$, $[w, f(A_{i,s})] = 1$, and $[v, f(A_{k,s}^{-1} A_{j,s} A_{k,s})] = 1$ in F_2. It follows that $f(A_{k,s}) = u^m$, $f(A_{i,s}) = w^n$, and $f(A_{j,s}) = u^m v^l u^{-m}$ for some $m, n, l \in \mathbb{Z}$. Since $f(A_{k,s}) \neq 1$, we have $m \neq 0$. Then, applying f to the pure braid relations $[A_{i,k}, A_{i,s} A_{k,s}] = 1$ and $[A_{j,k}, A_{j,s} A_{k,s}] = 1$, we obtain $[v, w^n u^m] = 1$ and $[w, u^m v^l] = 1$ in F_2. Thus, $w^n u^m = v^p$ and $u^m v^l = w^q$ for some $p, q \in \mathbb{Z}$. Passing to homology, using the fact that $[u] + [v] + [w] = 0$ in $H_1(F_2; \mathbb{Z})$ since $uvw = 1$ in F_2, reveals that $m = n = l$.

In P_n, we also have the relation $[A_{j,s}, A_{k,s} A_{j,k}] = 1$. Applying f we obtain the relation $[u^m v^m u^{-m}, u^m w] = 1$ in F_2. Hence, $u^m v^m u^{-m} = z^p$ and $u^m w = z^q$ for some $z \in F_2$ and $p, q \in \mathbb{Z}$. Writing $[z] = c_1[u] + c_2[v]$, we have

$$m[v] = pc_1[u] + pc_2[v] \text{ and } (m-1)[u] - [v] = qc_1[u] + qc_2[v]$$

in $H_1(F_2; \mathbb{Z})$. It follows that $m = n = l = 1$, and therefore $f(A_{i,s}) = w = v^{-1}u^{-1}$, $f(A_{j,s}) = uvu^{-1}$, and $f(A_{k,s}) = u$. Thus the image of f is contained in the subgroup $\langle u, v \rangle$ of F_2. As before, $\langle u, v \rangle = F_2$ since f is an epimorphism. Recalling that λ is the automorphism of F_2 taking u to x and v to y, we have $\lambda \circ f = f_{\{i,j,k,s\}}$. $\square$

The proof of Theorem 3.5(a) actually yields the following general result.

Theorem 3.6. *Let G be a finitely generated group, and $\Bbbk$ an algebraically closed field. Then there are no epimorphisms from G to F_k for $k > \dim \mathcal{R}^1(H^*(G, \Bbbk))$.*

Recall that the corank of a group G is the largest natural number k for which the free group F_k is an epimorphic image of G. The corank of $P_2 \cong \mathbb{Z}$ is 1. For larger n, as an immediate consequence of Theorem 3.5, we obtain the following.

Corollary 3.7. *For $n \geq 3$, the corank of the pure braid group P_n is equal to* 2.

Theorem 3.6 yields a more general result.

Corollary 3.8. *Let G be a finitely generated group and $\Bbbk$ an algebraically closed field. Then the corank of G is bounded above by* $\dim \mathcal{R}^1(H^*(G, \Bbbk))$.

In fact, results of Dimca, Papadima, and Suciu [DPS09] imply that the corank of G is equal to $\dim \mathcal{R}^1(H^*(G, \mathbb{C}))$ for a wide class of quasi-Kahler groups, including fundamental groups of complex projective hypersurface complements.

4 Epimorphisms of the lower central series Lie algebra

An analogue of Theorem 3.5 holds on the level of lower central series Lie algebras. For a group G, let G_k be the k-th lower central series subgroup, defined inductively by $G_1 = G$ and $G_{k+1} = [G_k, G]$ for $k \geq 1$. Let $\mathfrak{g}(G) = \bigoplus_{k\geq 1} G_k/G_{k+1}$. The map $G \times G \to G$ given by the commutator, $(x, y) \mapsto [x, y]$, induces a bilinear map $\mathfrak{g}(G) \times \mathfrak{g}(G) \to \mathfrak{g}(G)$ which defines a Lie algebra structure on $\mathfrak{g}(G)$.

The structure of the lower central series Lie algebra $\mathfrak{g}(P_n)$ of the pure braid group was first determined rationally by Kohno [Koh85]. The following description holds over the integers as well, see Papadima [Pap02]. For each $j \geq 1$, let $L[j]$ be the free Lie algebra generated by elements $a_{1,j+1}, \ldots, a_{j,j+1}$. Then $\mathfrak{g}(P_n)$ is additively isomorphic to $\bigoplus_{j=1}^{n-1} L[j]$, and the Lie bracket relations in $\mathfrak{g}(P_n)$ are the infinitesimal pure braid relations, given by

$$\begin{aligned} &[a_{i,j} + a_{i,k} + a_{j,k}, a_{m,k}] = 0, \quad \text{for } m \in \{i, j\}, \\ &[a_{i,j}, a_{k,l}] = 0, \quad \text{for } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned} \tag{4.1}$$

Let $\mathfrak{f}_k$ be the free Lie algebra (over $\mathbb{Z}$) generated by $x_1, \ldots, x_k$. The homomorphisms $f_I \colon P_n \to F_2$, $|I| = 3, 4$, induce surjective Lie algebra homomorphisms $\mathfrak{g}(P_n) \to \mathfrak{f}_2$. Calculations with (3.1) and (3.2) reveal that these are given by

$$(f_{\{i,j,k\}})_*(a_{r,s}) = \begin{cases} x_1 & \text{if } \{r, s\} = \{i, j\}, \\ x_2 & \text{if } \{r, s\} = \{i, k\}, \\ -x_1 - x_2 & \text{if } \{r, s\} = \{j, k\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(f_{\{i,j,k,l\}})_*(a_{r,s}) = \begin{cases} x_1 & \text{if } \{r, s\} = \{i, j\} \text{ or } \{r, s\} = \{k, l\}, \\ x_2 & \text{if } \{r, s\} = \{i, k\} \text{ or } \{r, s\} = \{j, l\}, \\ -x_1 - x_2 & \text{if } \{r, s\} = \{j, k\} \text{ or } \{r, s\} = \{i, l\}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Up to isomorphism, the maps* $(f_I)_*\colon \mathfrak{g}(P_n) \to \mathfrak{f}_2$, $|I| = 3, 4$, *are the only epimorphisms from the lower central series Lie algebra* $\mathfrak{g}(P_n)$ *to a free Lie algebra of rank at least* 2.

Sketch of Proof. If $\varphi\colon \mathfrak{g}(P_n) \to \mathfrak{f}_k$ is an epimorphism, there is an induced epimorphism $\mathfrak{g}(P_n)\otimes\mathbb{Q} \to \mathfrak{f}_k\otimes\mathbb{Q}$ which we denote by the same symbol. Let $b_{i,j} = \varphi(a_{i,j})$. Then the relations (4.1) imply that

$$\begin{aligned} &[b_{r,s}, b_{i,j} + b_{i,k} + b_{j,k}] = 0 \quad \text{if } \{r, s\} \subset \{i, j, k\}, \text{ and} \\ &[b_{i,j}, b_{k,l}] = 0 \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned} \tag{4.2}$$

Since $\mathfrak{f}_k \otimes \mathbb{Q}$ is free, we conclude that $b_{r,s}$ is a scalar multiple of $b_{i,j} + b_{i,k} + b_{j,k}$ for each $\{r, s\} \subset \{i, j, k\}$ and that $b_{k,l}$ is a scalar multiple of $b_{i,j}$ if $\{i, j\} \cap \{k, l\} = \emptyset$. Write $b_{r,s} = \sum_{j=1}^{k} c_{r,s;j}x_j$ as a linear combination of the generators $x_1, \ldots, x_k$ of $\mathfrak{f}_k \otimes \mathbb{Q}$. For $1 \le p \le k$, let $\eta_p = \sum_{1\le r<s\le n} c_{r,s;p}a^*_{r,s} \in H^1(P_n; \mathbb{Q})$, where $a^*_{r,s}$ is dual to $a_{r,s} \in H_1(P_n; \mathbb{Q})$.

If $1 \le p < q \le k$, it follows from (4.2) that the 2×2 determinants

$$\begin{vmatrix} c_{r,s;p} & c_{r,s;q} \\ c_{i,j;p} + c_{i,k;p} + c_{j,k;p} & c_{i,j;q} + c_{i,k;q} + c_{j,k;q} \end{vmatrix}$$

and

$$\begin{vmatrix} c_{u,v;p} & c_{u,v;q} \\ c_{i,j;p} + c_{l,k;p} & c_{i,j;q} + c_{k,l;q} \end{vmatrix}$$

vanish for $\{r, s\} \subset \{i, j, k\}$, and $\{u, v\} = \{i, j\}$ or $\{k, l\}$ where $\{i, j\} \cap \{k, l\} = \emptyset$. By [Fa97,LY00], this implies that $\eta_p \wedge \eta_q = 0$. Consequently, $\{\eta_1, \ldots, \eta_k\}$ spans an isotropic subspace of A^1, where $A = H^*(P_n; \mathbb{Q})$. Then $k \le \dim \mathcal{R}^1(A, \mathbb{Q})$ as in the proof of Theorem 3.5. One concludes that if $\varphi\colon \mathfrak{g}(P_n) \twoheadrightarrow \mathfrak{f}_k$, then k is at most 2.

Finally, one shows that every surjection $\mathfrak{g}(P_n) \to \mathfrak{f}_2$ is induced by f_I for some I with $|I| = 3$ or 4, using a linear version of the argument in the proof of Theorem 3.5 and the infinitesimal pure braid relations (4.1). □

5 Pure braid groups are not residually free

Recall that a group G is residually free if for every $x \neq 1$ in G, there is a homomorphism f from G to a free group F so that $f(x) \neq 1$ in F. In this section, we show that P_n is not residually free for $n \ge 4$, and derive some consequences. Since $P_2 \cong \mathbb{Z}$ and $P_3 \cong \mathbb{Z} \times F_2$, these groups are residually free.

For each $I \subset [n]$ of cardinality 3 or 4, let $K_I = \ker(f_I \colon P_n \to F_2)$. Define

$$K_n = \bigcap_{k=3}^{4} \bigcap_{|I|=k} K_I.$$

Proposition 5.1. *The subgroup K_n of P_n is characteristic.*

Proof. It suffices to show that if β is one of the generators of $\mathrm{Aut}(P_n)$ listed in (2.6), then $\beta(K_n) = K_n$.

If $\beta = \psi$ or $\beta = \phi_{p,q}$ is a transvection, then for each I, $f_I \circ \beta = f_I$ since $f_I(Z_n) = 1$, where Z_n is the generator of the center $Z(P_n)$ of P_n recorded in (2.3). It follows that $\beta(K_I) = K_I$ for each I, which implies that $\beta(K_n) = K_n$.

If $\beta = \xi$, then for each I, it is readily checked that $f_I \circ \xi = \lambda \circ f_I$, where $\lambda \in \mathrm{Aut}(F_2)$ is defined by $\lambda(x) = x^{-1}$ and $\lambda(y) = xy^{-1}x^{-1}$. Thus, $\xi(K_I) = K_I$, and $\xi(K_n) = K_n$.

If $\beta = \beta_k$ for $1 \le k \le n-1$, let τ_k denote the permutation induced by β_k. Let $I = \{i_1, \dots, i_l\}$ where $l = 3$ or 4, and let $\tau_k(I)$ denote the set $\{\tau_k(i_1), \dots, \tau_k(i_l)\}$ with the elements in increasing order. Define automorphisms $\lambda_1, \lambda_2 \in \mathrm{Aut}(F_2)$ by

$$\lambda_1 \colon \begin{cases} x \mapsto x, \\ y \mapsto x^{-1}y^{-1}, \end{cases} \qquad \text{and} \qquad \lambda_2 \colon \begin{cases} x \mapsto xyx^{-1}, \\ y \mapsto x, \end{cases}$$

and set $\lambda_3 = \lambda_1$. Then, calculations with the definitions of the automorphism β_k and the epimorphisms $f_I \colon P_n \to F_2$ (see (2.7), (3.1), and (3.2)) reveal that

$$f_I \circ \beta_k = \begin{cases} \lambda_j \circ f_{\tau_k(I)} & \text{if } k = i_j = i_{j+1} - 1, \\ f_{\tau_k(I)} & \text{otherwise.} \end{cases}$$

Note that, in the first case above, $\tau_k(I) = I$ and $j < l$. Thus, $K_I = \beta_k(K_{\tau_k(I)})$ for each I, and β_k permutes the subgroups K_I of P_n (for $|I| = 3$ and $|I| = 4$ respectively). It follows that $\beta_k(K_n) = K_n$.

Finally, if $\beta = \beta_n$, calculations with (2.7), (3.1), and (3.2)) reveal that

$$f_I \circ \beta_n = \begin{cases} f_{I \cup \{n\}} & \text{if } |I| = 3 \text{ and } n \notin I, \\ \lambda_1 \circ f_I & \text{if } |I| = 3 \text{ and } n \in I, \\ f_I & \text{if } |I| = 4 \text{ and } n \notin I, \\ f_{I \smallsetminus \{n\}} & \text{if } |I| = 4 \text{ and } n \in I. \end{cases}$$

Thus, $\beta_n(K_I) = K_I$ if either $|I| = 3$ and $n \in I$ or $|I| = 4$ and $n \notin I$, while $\beta_n(K_I) = K_{I \cup \{n\}}$ and $\beta_n(K_{I \cup \{n\}}) = K_I$ if $|I| = 3$ and $n \notin I$. It follows that $\beta_n(K_n) = K_n$, and K_n char P_n. $\square$

Theorem 5.2. *For $n \geq 4$, the pure braid group P_n is not residually free.*

Proof. Let $f : P_n \to F$ be a homomorphism from the pure braid group to a nonabelian free group. Since P_n is finitely generated and subgroups of free groups are free, we may assume without loss that $f : P_n \to F_k$ is a surjection onto a finitely generated nonabelian free group. We claim that K_n is contained in the kernel of f. By Theorem 3.5, $k = 2$ and $f \sim f_{[3]}$. Thus there are automorphisms $\alpha \in \mathrm{Aut}(P_n)$ and $\lambda \in \mathrm{Aut}(F_2)$ so that $\lambda \circ f = f_{[3]} \circ \alpha$. Let $x \in K_n$. Then $\alpha(x) \in K_n$ since K_n is characteristic in P_n by Proposition 5.1. Since $K_n \subset K_{[3]} = \ker(f_{[3]})$ by definition, we have $\alpha(x) \in \ker(f_{[3]})$. Hence, $\lambda \circ f(x) = f_{[3]} \circ \alpha(x) = 1$, and $x \in \ker(f)$.

To complete the proof, it suffices to exhibit a nontrivial element of K_n that is in the kernel of every homomorphism $g : P_n \to \mathbb{Z}$ from the pure braid group to an abelian free group. This is straightforward since is it easy to see that the intersection $K_n \cap [P_n, P_n]$ of K_n with the commutator subgroup of P_n is nontrivial. For instance, a calculation reveals that $x = [[A_{1,2}, A_{2,3}], [A_{2,3}, A_{3,4}]] \in K_n \cap [P_n, P_n]$. The pure braid x is nontrivial (one can check that the braids $[A_{1,2}, A_{2,3}][A_{2,3}, A_{3,4}]$ and $[A_{2,3}, A_{3,4}][A_{1,2}, A_{2,3}]$ are distinguished by the Artin representation). Thus $x \neq 1$ is in the kernel of every homomorphism from P_n to a free group, and P_n is not residually free. □

Remark 5.3. When viewed as an element of the 4-strand pure braid group, the braid $[[A_{1,2}, A_{2,3}], [A_{2,3}, A_{3,4}]] \in P_4$ is an example of a Brunnian braid. The deletion of any strand trivializes the braid, see Figure 5.1. (Compare [CFR10, Section 3].)

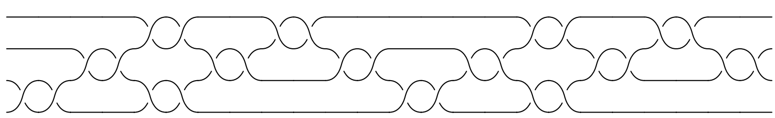

Figure 5.1. The braid $[[A_{1,2}, A_{2,3}], [A_{2,3}, A_{3,4}]]$ in P_4.

Remark 5.4. I. Marin showed us an argument he credited to L. Paris, showing the P_5 is not residually free, implying that P_n is not residually free for $n \geq 5$. Paris' argument uses the solution of the Tits conjecture for B_5 due to Droms, Lewin, and Servatius [DLS90] (see also Collins [Col94]) to produce a subgroup of P_5 isomorphic to the free product $\mathbb{Z} * (\mathbb{Z} \times F_2)$, as explained in [Mar11, Proposition 1.1]. This latter group is not residually free, see [Bau67, Theorems 6 and 3].

Let Σ be an orientable surface, possibly with punctures. Let $\Sigma^{\times n} = \Sigma \times \cdots \times \Sigma$ denote the n-fold Cartesian product. The pure braid group

$P_n(\Sigma)$ of the surface Σ is the fundamental group of the configuration space

$$F(\Sigma, n) = \{(x_1, \dots, x_n) \in \Sigma^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\}.$$

of n distinct ordered points in Σ.

Corollary 5.5. *For $n \geq 4$, the pure braid group $P_n(\Sigma)$ is not residually free.*

Proof. If $\Sigma \neq S^2$, the pure braid group P_4 embeds in $P_n(\Sigma)$, see Paris and Rolfsen [PR99]. If $\Sigma = S^2$, then $P_4 < P_n(S^2)$ for $n \geq 5$, and $P_4(S^2) \cong P_4/Z(P_4)$ (see, for instance, [Bir75]). So the result follows from Theorem 5.2. □

Remark 5.6. The fundamental group of the orbit space $F(\Sigma, n)/\Sigma_n$, where Σ_n denotes the symmetric group, is the (full) braid group $B_n(\Sigma)$ of the surface Σ. Recall from the Introduction that B_3 is not residually free, and that the only two-generator residually free groups are $\mathbb{Z}$, $\mathbb{Z}^2$, and F_2. If $\Sigma \neq S^2$ and $n \geq 3$, then $B_n(\Sigma)$ has a B_3 subgroup, so is not residually free. If $\Sigma = S^2$, then $B_3 < B_n(S^2)$ for $n \geq 4$, and $B_3(S^2) \cong B_3/Z(B_3)$. So $B_n(S^2)$ is not residually free for $n \geq 3$.

A complex hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ is a finite collection of codimension one subspaces of $\mathbb{C}^n$. Fix coordinates $(z_1, \dots, z_n)$ on $\mathbb{C}^n$, and for $1 \leq i \leq m$, let $\ell_i(z_1, \dots, z_n)$ be a linear form with $\ker(\ell_i) = H_i$. The product $Q = Q(\mathcal{A}) = \prod_{i=1}^m \ell_i$ is a defining polynomial for $\mathcal{A}$. The group $G(\mathcal{A})$ of the arrangement is the fundamental group of the complement $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{i=1}^m H_i = \mathbb{C}^n \setminus Q^{-1}(0)$.

The arrangement $\mathcal{A}_{r,1,n}$ with defining polynomial

$$Q = Q(\mathcal{A}_{r,1,n}) = z_1 \cdots z_n \prod_{1 \leq i < j \leq n} (z_i^r - z_j^r)$$

is known as the full monomial arrangement (it is the reflection arrangement corresponding to the full monomial group $G(r, 1, n)$). Note that the arrangement $\mathcal{A}_{2,1,n}$ is the Coxeter arrangement of type B_n. The complement $M(\mathcal{A}_{r,1,n})$ of the full monomial arrangement may be realized as the orbit configuration space

$$F_\Gamma(\mathbb{C}^*, n) = \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^{\times n} \mid \Gamma \cdot x_i \cap \Gamma \cdot x_j = \emptyset \text{ if } i \neq j\}$$

of ordered n-tuples of points in $\mathbb{C}^*$ which lie in distinct orbits of the free action of $\Gamma = \mathbb{Z}_r$ on $\mathbb{C}^*$ by multiplication by the primitive r-th root of unity $\exp(2\pi\sqrt{-1}/r)$.

Call the fundamental group $P(r,1,n) = G(\mathcal{A}_{r,1,n})$ the pure monomial braid group. For $n = 1$, $P(r,1,1) \cong \mathbb{Z}$, and for $n = 2$, it is well known that $P(r,1,2) \cong \mathbb{Z} \times F_{r+1}$. Hence, $P(r,1,n)$ is residually free for $n \leq 2$.

Corollary 5.7. *For $n \geq 3$, the pure monomial braid group $P(r,1,n)$ is not residually free.*

Proof. For $n \geq 3$, it follows from [Coh01] that the pure braid group P_4 embeds in $P(r,1,n)$. So the result follows from Theorem 5.2. $\square$

Remark 5.8. The fundamental group of the orbit space $M(\mathcal{A}_{r,1,n})/G(r,1,n)$ is the (full) monomial braid group $B(r,1,n)$. This group admits a presentation with generators $\rho_0, \rho_1, \ldots, \rho_{n-1}$ and relations

$$\begin{aligned} (\rho_0\rho_1)^2 &= (\rho_1\rho_0)^2, \ \rho_i\rho_{i+1}\rho_i \\ &= \rho_{i+1}\rho_i\rho_{i+1}\ (1 \leq i < n), \\ \rho_i\rho_j &= \rho_j\rho_i\ (|j-i| > 1). \end{aligned}$$

Observe that $B(r,1,n)$ is independent of r, and is the Artin group of type B_n. For $n \geq 3$, the group $B(r,1,n)$ has a B_3 subgroup, so is not residually free. The group $B(r,1,2)$ is not residually free, since it is a two-generator group which is not free or free abelian.

Let Γ be a Coxeter graph, with associated Artin group A_Γ and pure Artin group P_Γ. We say that Γ contains an A_k subgraph if it contains a path of length k with unlabelled edges as a vertex-induced subgraph.

Corollary 5.9. *If Γ contains an A_3 subgraph, then the associated pure Artin group P_Γ is not residually free.*

Proof. If Γ contains an A_3 subgraph, it follows from van der Lek [Lek83] (see also [Par97]) that P_Γ has a P_4 subgroup. So the result follows from Theorem 5.2. $\square$

If Γ is a connected Coxeter graph of finite type different from B_3, H_3, or F_4, then Γ contains an A_3 subgraph, hence P_Γ is not residually free by Corollary 5.9. If Γ is of type B_3, then P_Γ is the pure monomial braid group $P(2,1,3)$, so is not residually free by Corollary 5.7. If Γ is of type F_4, then Γ contains a B_3 subgraph, hence P_Γ is not residually free by the same argument as in the proof of Corollary 5.9.

Remark 5.10. If Γ contains an A_2 subgraph, then the (full) Artin group A has a B_3 subgroup, so is not residually free. This includes all irreducible Artin groups of finite type and rank at least 2, except type $I_2(m)$.

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