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The Fundamental Group of the Complement of a Union of Complex Hyperplanes

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Let M be the complement of the union of n linear subspaces V_i of $\mathbb{C}^l$, and let N be the quotient space of M by the Hopf action of $\mathbb{C}^* = \mathbb{C}^l - \{0\}$ on M. In this paper we determine $\pi_1(M)$ and $\pi_1(N)$. If $\varphi_1, \ldots, \varphi_n$ are linear forms determining $V_1, \ldots, V_n$, then N is just the complement of the projective variety $W = \{f = \varphi_1 \ldots \varphi_n = 0\} \subset \mathbb{P}^{l-1}$. Furthermore, the Milnor fiber $F = \{f = 1\} \subset \mathbb{C}^l$ is a cyclic n-fold cover of N, and we also determine $\pi_1(F)$.

Such spaces and groups are of interest in several areas of mathematics, including algebraic geometry, Lie algebras, combinatorics, and topology. In 1929 Zariski showed [9] that if the V_i were planes in $\mathbb{C}^3$ in general position, then $\pi_1(N)$ is abelian. (We note that $H_1(M)$ or $H_1(N)$ is easily computed by duality, so that this determines π_1(N).) This was then used to show that if $C \subset \mathbb{P}^2$ is a curve which degenerates to such a configuration, then $\pi_1(\mathbb{P}^2 - \mathrm{C})$ is abelian as well.

Much more recently Arnold [1], Brieskorn [2], and Deligne [5] considered the situation in which G is a finite group acting by reflections on $\mathbb{R}^l$ (e.g. the Weyl group acting on a Lie algebra.) By complexification one gets the above situation, with the V_i just the reflecting hyperplanes. In this case it was shown that M is a $K(\pi, 1)$ where $\pi = \pi_1(M)$ is a "generalized colored braid group". In the case that the Lie group is A_{l-1} so that the Weyl group is the symmetric group $\Sigma(l)$, one has an exact sequence

$$1 \to \pi_1(M) \to B(l) \to \Sigma(l) \to 1,$$

where $B(l)$ is the classical Artin braid group on l strings. In general, however, M is not a $K(\pi, l)$ space.

Finally, Orlik and Solomon considered the general situation in connection with their work on complex reflection groups. In [8], for example, they describe $H^*(M)$, and note (p. 45) that the determination of $H^*(F)$ is an unsolved problem. As a corollary of the determination of $\pi_1(M)$ we obtain $H^*(F)$ in the case $l = 3$ (i.e. the real dimension of F is four).

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1. Statement of the Main Result

We first note that there is a commuting diagram with exact row and column:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ & & & & \pi_1(F) & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & \pi_1(\mathbb{C}^*) & \to & \pi_1(M) & \to & \pi_1(N) & \to & 1. \\ & & & & \downarrow & & & & \\ & & & & \mathbb{Z}/n\mathbb{Z} & & & & \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

Here the row comes from the $\mathbb{C}^*$-bundle (which is trivial for $n \geqq 1$), and the column comes form an n-fold cover, to be described in more detail later. Thus it basically suffices to describe $\pi_1(N)$, which we now do.

First, choose a 3-dimensional linear subspace H of $\mathbb{C}^l$. We require H to be in general position with respect to $V = \bigcup V_i$ in the sense that if $\dim_{\mathbb{C}}(V_{i_1} \cap \ldots \cap V_{i_r}) = l - p$, then $\dim_{\mathbb{C}}(H \cap V_{i_1} \cap \ldots \cap V_{i_r}) = 3 - p$. (Of course, any negative dimension indicates the empty set.) By dividing by the $\mathbb{C}^*$-action restricted to $(H - \{0\}, H \cap V - \{0\})$ we obtain a curve $C \cong H \cap V - \{0\}/\mathbb{C}^* \subset \mathbb{P}^2$. Then a theorem of Zariski [6] shows that $\pi_1(N) \cong \pi_1(\mathbb{P}^2 - C)$. Now the curve C consists of n projective lines, not necessarily in general position. For each line C_i, $i = 1, \ldots, n$, we introduce a group generator $a_i, i = 1, \ldots, n$. For each point $p_j, j = 1, \ldots, k$ of intersection of two or more C_i we introduce a *set* of relations $\mathscr{R}_j$ as follows: Suppose exactly $C_1, \ldots, C_t$ intersect at p_j. The torus link of type (t, t) has the property that the fundamental group of its complement can be presented in a certain way with t generators $\alpha_1, \ldots, \alpha_t$, and a set of relations $\mathscr{R}(t)$. To obtain the relations $\mathscr{R}_j$ we merely set $a_i = \alpha_i$ in $\mathscr{R}(t)$:

$$\mathscr{R}_j = \{a_1 a_2 \ldots a_t = a_2 a_3 \ldots a_t a_1 = \ldots = a_t a_1 \ldots a_{t-1}\},$$

as may be shown using [4].

Theorem 1. $\pi_1(N) \cong \langle a_1, \ldots, a_n | a_1 a_2 \ldots a_n \colon \mathscr{R}_j, j = 1, \ldots, k\rangle$.

Note that if the V_i are in general position, exactly two C_i intersect at each p_j, and $\mathscr{R}_j$ is the single relation saying that the corresponding a_i commute. In the other extreme, in which all the C_i come together in one point, $\pi_1(N)$ will be seen to be a free group on $n-1$ letters.

The proof of Theorem 1 occupies the next section.

2. Proof of the Main Theorem

Choose one irreducible component of C and label it C_1. We choose also a pencil $\mathscr{P}$ of lines in $\mathbb{P}^2$ satisfying the following two properties:

(i) No line of $\mathscr{P}$ is an irreducible component of C.

(ii) There is a line L of $\mathscr{P}$ which is close to C_1 in the sense that $L \subset U_1$, where U_1 is a fixed neighborhood of C_1 in $\mathbb{P}^2$, and $L \cap C$ is n distinct points.

Here we label the irreducible components of C by $C_1, C_2, \ldots, C_n$. Conditions (i) and (ii) are achieved by choosing the pencil $\mathscr{P}$ to be the set of all lines through a point $P \in \mathbb{P}^2$ where for (i), $P \notin C$, and for (ii), P is a general point close to C_1.

Next we choose a basepoint $x \in L - C$. The classical method of computing $\pi_1(\mathbb{P}^2 - C)$ (see [9], or [3] for a modern, detailed proof) then gives us generators $a_1, a_2, \ldots, a_n$, where a_n is represented by a loop in $L - C$ which runs from x to a point near a point of $L \cap C$, around the point, and back to x. Since $L \cong S^2$, we immediately have $a_1 a_2 \ldots a_n = 1$.

The remaining relations are obtained from lines L_j of $\mathscr{P}$ for which $L_j \cap C$ consists of fewer than n points. In general, this occurs because of (i) singular points of C, or (ii) tangencies of L_j with C. In the present case, since each irreducible component of C is a line, L_j could be tangent to C only if L_j actually coincides with some component. Thus all additional relations come from singular points of C.

Now, given this general set-up we proceed by induction on the degree n. The case $n = 1$ is of course trivial; $\mathbb{P}^2 - C \cong \mathbb{C}^2$ and the group is $\langle a_1 | a_1 \rangle \cong 1$.

We thus suppose the theorem holds when the degree of the curve is less than n. As part of the inductive step we require the following lemma.

Lemma. *Suppose $C = C' \cup C''$, where C' and C'' consist of n' and n'' lines, respectively. If Theorem* 1 *holds for both C' and C'' and if $C' \cap C''$ consists of $n' n''$ distinct points, then Theorem* 1 *holds for C.*

Proof. This is essentially the main result of [7]. Choose a line L in general position with respect to C' and C''. Thus $L \cap C' \cap C'' = \emptyset$. We regard $\mathbb{P}^2 - L \cong \mathbb{C}^2$ as usual, and note that $C' \cap C'' \subset \mathbb{C}^2$ consists of $n' n''$ points. We conclude from [7] that

$$\pi_1(\mathbb{C}^2 - (C' \cup C'')) \cong \pi_1(\mathbb{C}^2 - C') \times \pi_1(\mathbb{C}^2 - C'').$$

As in [7, Sect. 3], we may replace L and obtain a central extension

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - (C' \cup C'')) \to \pi_1(\mathbb{P}^2 - (C' \cup C'')) \to 1,$$

where the subgroup $\mathbb{Z}$ is generated by a large circle slightly pushed off L.

But this is exactly Theorem 1 says in this situation. Generators from C' commute with those from C'', and there is a relation $a_1 a_2 \ldots a_n = 1$ from the infinite cyclic kernel. Notice that $\pi_1(\mathbb{C}^2 - C')$ is the same as $\pi_1(\mathbb{P}^2 - C')$ expect for the deletion of the relation $a_1 a_2 \ldots a_{n'}$, and similarly for C''.

To complete the induction step, we now deform C slightly, apply the above lemma and the induction hypothesis, and deform back to C. Choose a family of lines $C_1(\tau)$ such that $C_1(0) = C_1$ and so that $C_1(\tau) \cap (C_2 \cup \ldots \cup C_n)$ is $n-1$

points for small non-zero τ. Using $C_1(\tau)$ we form the family of degree n curves $C(\tau)=C_1(\tau)\cup C_2\cup\ldots\cup C_n$.

By our inductive assumption and the above Lemma, Theorem 1 holds for $C(\tau)$ when τ is small and non-zero. Thus, if,

$$\pi_1(\mathbb{P}^2-(C_2\cup\ldots\cup C_n))\cong\langle a_2,\ldots,a_n|a_2\ldots a_n;\mathcal{R}_j, j=1,\ldots,k'\rangle,$$

then

$$\pi_1(\mathbb{P}^2-C(\tau))\cong\langle a_1,\ldots,a_n|a_1\ldots a_n;[a_1,a_i], i=2,\ldots,n;\mathcal{R}_j, j=1,\ldots,k'\rangle.$$

Finally, we examine what happens when τ becomes 0. The crucial point is that everything here is happening inside the neighborhood U_1 of C_1 chosen initially. Thus problems of conjugation are avoided.

Now if $C_1\cap(C_2\cup\ldots\cup C_n)$ is in fact $n-1$ distinct points, there is nothing left to do. So suppose there is a point p of intersection of C_1 and $t-1$ other C_i, say $C_2, C_3,\ldots,C_t$. For $\tau\neq 0$, $C_1(\tau)\cap(C_2\cup\ldots\cup C_t)=\{p_1,\ldots,p_{t-1}\}$, where p_i is near p, $i=1,\ldots,t-1$.

But it is well-known what happens in this situation when τ becomes 0. (See [9] for a detailed discussion): The set of relations for the $(t-1)$-fold intersection of $C(\tau)$ at p $(\tau\neq 0)$ and the relations $[a_1,a_i]$, $i=2,3,\ldots,t$ are replaced by the set of relations for the torus link $L(t,t)$. Since this is exactly what Theorem 1 says, we have completed the inductive step and the proof of the theorem.

Notice that the relations for the $L(t,t)$ link are consequences of the relations for the $L(t-1,t-1)$ link and the commuting relations $[a_1,a_i]$. This is the well-known "semicontinuity" principle: $\pi_1(\mathbb{P}^2-C(0))$ maps *onto* $\pi_1(\mathbb{P}^2-C(\tau))$.

3. Consequences of the Main Theorem

We note what Theorem 1 implies for some special cases.

Corollary 1. *Suppose* $V=\bigcup_{i=1}^{n} V_i$ *satisfies*

a) No three V_i *intersect in a subspace of dimension* $l-2$. *Then* $\pi_1(\mathbb{C}^l-V)$ *is abelian.*

We may paraphrase a) by saying "V is in general position in codimension 2." Corollary 1 is just Zariski's result [9] that if $C\subset\mathbb{P}^2$ consists of n lines in general position, $\pi_1(\mathbb{P}^2-C)$ is abelian.

At the other extreme, we have

Corollary 2. *Suppose* $\bigcap_{i=1}^{n} V_i=H^{l-2}$, *a subspace of dimension* $l-2$. *Then* $\pi_1(\mathbb{C}^l-\bigcup V_i)\cong\mathbb{Z}\times F_{n-1}$.

Here F_{n-1} denotes the free group on $(n-1)$ letters. This may be seen directly as follows. We must consider $\mathbb{P}^2-C$, where C consists of n lines meeting in a single point. But $\mathbb{P}^2 - C = (\mathbb{P}^2 - C_1) - \left(\bigcup_{i=2}^{n} C_i\right)$, which is just $\mathbb{C}^2$ with $n-1$ parallel planes removed. Thus $\pi_1(\mathbb{P}^2 - C) \cong F_{n-1}$, and the corollary follows.

Next we consider the associated Milnor fiber. Recall that F was a cyclic n-fold cover of $\mathbb{P}^{l-1} - W = N$. More precisely, by duality one easily sees that $H_1(\mathbb{P}^{l-1} - W) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n \text{ times}} / (1, 1, \ldots, 1) \cong \mathbb{Z}^{n-1}$, where each copy of $\mathbb{Z}$ is represented by a small loop linking one of the irreducible components of W. There is a homomorphism $s: H_1(N) \to \mathbb{Z}/n\mathbb{Z}$ given by taking the algebraic sum of these linkings modulo n, and F is just the cover corresponding to the kernel of the composite homomorphism Φ.

$$\pi_1(N) \xrightarrow{\varphi} H_1(N) \xrightarrow{s} \mathbb{Z}/n\mathbb{Z}, \qquad \Phi = s \circ \varphi$$

Clearly $\Phi(a_{i_1}^{\varepsilon_{i_1}} \ldots a_{i_m}^{\varepsilon_{i_m}}) = \sum_{j=1}^{m} \varepsilon_{i_j} \pmod{n}$.

For example, if V is as in Corollary 2, we note that $\pi_1(F) \subset \pi_1(\mathbb{P}^2 - C) \cong F_{n-1}$, and $\pi_1(F)$ has index n. But it is well known that a subgroup of finite index n in a free group of rank $n-1$ is free of easily computed rank $r = n(n-1) - n + 1 = (n-1)^2$. Thus $\pi_1(F) \cong F_{(n-1)^2}$.

Example 1. Take V as in the paragraph above, with the addition of one hyperplane V_{n+1} such that $\dim_{\mathbb{C}}(V_{n+1} \cap V_i \cap V_j) = l-3$, for all $1 \leqq i < j \leqq n$. Then $\pi_1(N) \cong F_{n-1} \times \mathbb{Z}$, the factors generated by $a_2, \ldots, a_n$ and a_{n+1} respectively. Then

$$\pi_1(F) \cong \ker \Phi \cong \{(a_{i_1}^{\varepsilon_{i_1}} \ldots a_{i_m}^{\varepsilon_{i_m}}, a_{n+1}) | \rho + \Sigma\, \varepsilon_{i_m} \equiv 0 \pmod{n+1}\}.$$

But then $\ker \Phi \cong F_{n-1} \times \mathbb{Z}$ as well.

This example is an illustration of the joint dependence of $\pi_1(F)$ on the number of hyperplanes of V and the cardinality of the subsets intersecting in codimension two subspaces. We will not worry here about a general presentation for $\pi_1(F)$, which is a routine but tedious exercise using the Reidemeister-Schreier algorithm. Notice that this suffices to determine $H^*(F, \mathbb{Z})$ when F has real dimension four, as follows. Of course, $\pi_1(F)$ determines $H_1(F; \mathbb{Z})$. Further, we can compute $\chi(\mathbb{P}^2 - C)$ easily, and since F is an n-fold cover of $\mathbb{P}^2 - C$, $\chi(F) = n\,\chi(\mathbb{P}^2 - C)$. Then we invoke the fact that F has the homotopy type of a 2-dimensional CW complex to conclude that $H_2(F; \mathbb{Z})$ is free and to compute its rank. Clearly, $H^q(F; \mathbb{Z}) = 0$ if $q > 2$.

We conclude with a final example from the theory of Lie algebras. The complexification for the Lie algebra of type D_3 yields 6 hyperplanes in $\mathbb{C}^3$: $z_i = \pm z_j$, for $1 \leqq i < j \leqq 3$. The line $z_1 = z_2 = z_3$ is in the intersection of three, as are $z_1 = -z_2 = z_3$, $z_1 = -z_2 = -z_3$, and $z_1 = z_2 = -z_3$. Furthermore there are the z_k axes which are the intersections of two ($z_i = \pm z_j$), but not three, hyperplanes.

Thus

$$\pi_1(N) \cong \langle a_1, \ldots, a_6 | a_1 \ldots a_6 = 1, [a_1, a_2], [a_3, a_4], [a_5, a_6],$$
$$a_2 a_3 a_6 = a_3 a_6 a_2 = a_6 a_2 a_3, \quad a_1 a_3 a_5 = a_3 a_5 a_1 = a_5 a_1 a_3,$$
$$a_2 a_4 a_5 = a_4 a_5 a_2 = a_5 a_2 a_4, \quad a_1 a_4 a_6 = a_4 a_6 a_1 = a_6 a_1 a_4 \rangle,$$

where a_1 corresponds to $z_1 = z_2$, a_2 to $z_1 = -z_2$, a_3 to $z_1 = z_3$, etc. Further, $P_N(t) = 1 + 5t + 6t^2 = (1+2t)(1+3t)$, as expected from [1, 2, 8]. Since $D_3 = A_3$, the group above is just the colored braid group $CB(4)$ which fits into the exact sequence

$$1 \to CB(4) \to B(4) \to \Sigma(4) \to 1.$$

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Inventiones mathematicae

The fundamental group of the complement of a union of complex hyperplanes: correction

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In this note we correct the main result (Theorem 1) of [1]. The error involves the local description in the neighborhood of a singularity. In order to correct this we must assume that the arrangement is real and specify an ordering of its hyperplanes.

Thus we consider a collection of linear forms $\phi_1, \ldots, \phi_n, \phi_{n+1}$ in the complex variables (z_1, z_2, z_3), with zero loci $V_1, \ldots, V_n, V_{n+1}$ respectively. Without loss of generality we may assume that $\phi_{n+1}(z_1, z_2, z_3) = z_3$, so that if we identify $\mathbb{C}^2$ (with coordinates $Z_1 = z_1/z_3$, $Z_2 = z_2/z_3$) with $\mathbb{CP}^2 - V_{n+1}$, we have $\mathbb{CP}^2 - (V_1 \cup \ldots \cup V_n \cup V_{n+1}) \cong \mathbb{C}^2 - (V_1 \cup \ldots \cup V_n)$. We henceforth work with the latter space, which we call N. Notice that if $\phi_1, \ldots, \phi_n, \phi_{n+1}$ are real forms (their coefficients are real), we may make the appropriate change of coordinates as above so that $V_1, \ldots, V_n$ are defined by real forms.

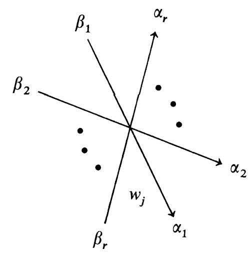
We now specialize to the real case. First write $Z_1 = u_1 + iv_1$, $Z_2 = u_2 + iv_2$. Then we have the canonical $\mathbb{R}^2 \subset \mathbb{C}^2$ given by $v_1 = v_2 = 0$, and letting $L_i = V_i \cap \mathbb{R}^2$, $i = 1, \ldots, n$ we have the representation of the arrangement by n lines in $\mathbb{R}^2$.

To specify the algorithm, we first order and orient these lines. Then we specify generators and relations for $\pi_1(N)$, based upon this choice of order and orientation:

By a change of coordinates we may assume without loss of generality that no line L_i is vertical or horizontal in $\mathbb{R}^2$. Thus each line L_i is defined in $\mathbb{R}^2$ by a linear equation $u_2 = m_i u_1 + d_i$, where $m_i \in \mathbb{R} - \{0\}$, $d_i \in \mathbb{R}$. We order the lines by the lexicographical order for (m_i, d_i). Thus $L_i < L_j$ if and only if $i < j$ if and only if $m_i < m_j$ or $m_i = m_j$ and $d_i < d_j$. We orient the line L_i by taking the positive direction to be that of increasing u_1.

Next we specify generators for $\pi_1(N)$. Note that $\mathbb{R}^2 \cap (V_1 \cup \ldots \cup V_n) = \Gamma \subset \mathbb{R}^2$ is a planar "graph" (allowing rays). Let $W = \{w_1, \ldots, w_k\}$ denote the set of vertices of Γ. Then $\Gamma - W$ has several components. For each component we introduce a generator of $\pi_1(N)$ as follows. For the components of L_i we will have generators $a_i, b_i, c_i, \ldots$ where a_i corresponds to the component of $L_i - W$ which is farthest to the right along L_i, b_i corresponds to the next farthest to the right, etc. Let G denote the set of such generators.

Finally, we specify relations for $\pi_1(N)$. They are of three types, all of which arise from vertices of Γ. Suppose we consider a vertex w_j of Γ:



Then we have relations:

$$R_{1j}\colon\ \alpha_1^{-1}\alpha_2^{-1}\dots\alpha_r^{-1}\beta_1\beta_2\dots\beta_r=1,$$

$$\begin{aligned} R_{2j}\colon\ &\beta_1=\alpha_1\\ &\beta_2=\alpha_1^{-1}\alpha_2\alpha_1\\ &\ \vdots\\ &\beta_r=\alpha_1^{-1}\alpha_2^{-1}\dots\alpha_{r-1}^{-1}\alpha_r\alpha_{r-1}\dots\alpha_2\alpha_1,\end{aligned}$$

$$R_{3j}\colon\ \alpha_1\alpha_2\dots\alpha_r=\alpha_2\alpha_3\dots\alpha_r\alpha_1=\alpha_3\dots\alpha_r\alpha_1\alpha_2=\dots=\alpha_r\alpha_1\dots\alpha_{r-1}.$$

Relations R_{1j} arise from a Wirtinger-type presentation of $\pi_1(\mathbb{R}^3-\Gamma)$ where $\mathbb{R}^3=\{v_2=0\}\subset\mathbb{C}^2$. Since β_1 is a conjugate of α_1, relations R_{2j} specify precisely how this occurs. And the relations R_{3j} are the relations for the group of the Hopf link as in [1].

The correct statement is then:

Theorem 1. *Suppose N is the complement of a real arrangement of lines in $\mathbb{CP}^2$ as above. Then*

$$\pi_1(N)\cong\langle G\,|\,R_{1j},R_{2j},R_{3j},j=1,\dots,k\rangle.$$

Comment. The relation R_{1j} is a consequence of the set of R_{2j}, and R_{2j}, $j=1,2,\dots,k$ may be used to obtain a presentation of $\pi_1(N)$ with generators $a_1,\dots,a_n$ and relations R'_{3j}, where R'_{3j} is simply R_{3j} written in terms of $a_1,\dots,a_n$.

In [1] the relations R_{2j} were incorrectly given (in other notation) as $\beta_1=\alpha_1$, $\beta_2=\alpha_2,\dots,\beta_r=\alpha_r$.

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