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Homology of iterated semidirect products of free groups

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Abstract

Let G be a group which admits the structure of an iterated semidirect product of finitely generated free groups. We construct a finite, free resolution of the integers over the group ring of G. This resolution is used to define representations of groups which act compatibly on G, generalizing classical constructions of Magnus, Burau, and Gassner. Our construction also yields algorithms for computing the homology of the Milnor fiber of a fiber-type hyperplane arrangement, and more generally, the homology of the complement of such an arrangement with coefficients in an arbitrary local system. © 1998 Elsevier Science B.V.

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0. Introduction

Let $G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$ be a group which admits the structure of an iterated semidirect product of finitely generated free groups. For any such group, we construct an explicit finite, free resolution $C_\bullet(G) \to \mathbb{Z}$ over the group ring of G (Theorem 2.10). Topologically, this resolution may be viewed as the equivariant chain complex of the universal cover of an Eilenberg–MacLane space of type $K(G,1)$. The boundary maps of the chain complex $C_\bullet(G)$ are computed recursively by means of Fox derivatives from the various actions of F_{d_p} on F_{d_q}, $p < q$, dictated by the semidirect product structure of G. Independent of these actions, each term, $C_k(G)$, of $C_\bullet(G)$ is a free $\mathbb{Z}G$-module of rank $\sum d_{p_1} d_{p_2} \cdots d_{p_k}$ (the sum being over all $1 \leq p_1 < \cdots < p_k \leq \ell$).

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Perhaps the most famous groups of this type are Artin's pure braid groups. The pure braid group on ℓ strings may be realized as $P_\ell = F_{\ell-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$ [4]. A natural generalization also belongs to this class of groups. The fundamental group of the complement of any (affine) fiber-type hyperplane arrangement admits the structure of an iterated semidirect product of free groups [18, 25, 43]. Examples include Brieskorn's generalized pure braid groups, $\mathrm{PB}(W)$, where W is a Coxeter group of type A_ℓ, B_ℓ, G_2, or $\mathrm{I}_2(p)$, see [8]. Groups of the form $P_{n,\ell} = \ker(P_n \to P_\ell)$ also admit this structure. These latter groups arise in the studies of representations of braid groups, generalized hypergeometric functions, and the Knizhnik–Zamolodchikov equations, see e.g. [29, 2, 47].

Much is known about many groups of this type. Arnol'd [3] and Cohen [14] computed the cohomology of the pure braid group $P_\ell = \mathrm{PB}(\mathrm{A}_{\ell-1})$, and showed that the Poincaré polynomial factors into linear terms. The lower central series of P_ℓ was found by Kohno [27]. These two results combine to yield the "LCS formula" relating the Betti numbers, b_j, the exponents, $d_q = q$, and the ranks of the lower central series quotients, ϕ_k, of P_ℓ:

$$\sum_{j=0}^{\ell-1} b_j(-t)^j = \prod_{q=1}^{\ell-1}(1 - d_q t) = \prod_{k\geq 1}(1 - t^k)^{\phi_k}.$$

These results were subsequently generalized to the group of an arbitrary fiber-type arrangement by Falk and Randell [18, 19] and Jambu [25]. See [30] for analogous results on the pure braid group of a Riemann surface, and see [8, 41, 42, 48] for further results on the cohomology and factorization of the Poincaré polynomial of an arrangement. We obtain a further generalization here. The LCS formula holds for any iterated semidirect product of free groups G for which the split extensions arising in the semidirect product structure give rise to IA-automorphisms of the free groups comprising G (Theorem 3.5).

The aforementioned results on the homology of an iterated semidirect product of free groups, G, apply only in the constant coefficient case (that is, homology with coefficients in a trivial G-module). A desire to compute the homology of G with coefficients in an arbitrary G-module led to the construction of this paper. Let $\nu : G \to \mathrm{Aut}(V)$ be a representation of G, and denote by $V = V_\nu$ the corresponding $\mathbb{Z}G$-module. Then the homology of G with coefficients in V is equal to the homology of the chain complex $C_\bullet(G)\otimes_G V$ (see [9]). In this manner, we obtain an algorithm for computing the homology of G with arbitrary coefficients. In particular, we can use this construction to compute the homology of the complement of a fiber-type arrangement with coefficients in any local system. This type of problem has been the focus of a great deal of recent activity. In [16], Esnault et al. present an algorithm for computing the cohomology of the complement of an (arbitrary) arrangement with coefficients in certain complex local systems (see also [47]). Refinements of the results of [16] may be found in [20, 46, 51]. Note, however, that none of these results hold for arbitrary local systems. See [24, 28, 45, 10] for other results along similar lines.

The construction of the chain complex $C_\bullet(G)$ has certain functorial properties that allow us to define representations of groups acting compatibly on an iterated semidirect product of free groups G. The resulting representations generalize the classical Magnus representations [36, 5]. Given a compatible automorphism $\psi \in \mathrm{Aut}^{\rtimes}(G)$, we explicitly construct the chain equivalence $\Psi_\bullet : C_\bullet(G) \to C_\bullet(G)$ by means of "higher-order" Fox Jacobians of ψ. Let Γ be a group which acts on G by compatible automorphisms. Such an action $\Phi : \Gamma \to \mathrm{Aut}^{\rtimes}(G)$ gives rise to a map $\Phi_\bullet$ from Γ to the the group of chain automorphisms of $C_\bullet(G)$. However, this map need not be a homomorphism. The failure is measured precisely by the chain rule for higher-order Jacobians (Proposition 4.8). This problem can be overcome by following Magnus' original idea in the case $G = F_n$. Let $\tau : G \to K$ be a Φ-invariant homomorphism. Then extension of scalars via the map induced by τ on group rings yields the desired homomorphism $\Phi_k^\tau : \Gamma \to \mathrm{Aut}_{\mathbb{Z}K}(\mathbb{Z}K \otimes_{\mathbb{Z}G} C_k(G))$, for each $k \geq 1$ (Theorem 4.11).

In the case where $\Gamma = B_\ell$ is the full braid group, acting on $G = P_{n,\ell}$ in a natural fashion, certain choices of homomorphisms $\tau : P_{n,\ell} \to \mathbb{Z}$ yield representations over the ring $\Lambda = \mathbb{Z}\mathbb{Z}$ that generalize the classical Burau representations. Other homomorphisms $\tau : P_{n,\ell} \to \mathbb{Z}^m$ yield representations of B_ℓ that depend on m parameters. We also obtain generalized Gassner representations of P_ℓ from the natural action of P_ℓ on $P_{n,\ell}$.

Linear representations of braid groups have been much studied over the years, starting with the pioneering work of Burau, Magnus, and Gassner (see [5]), and more recently by Jones [26], Kohno [29], Lawrence [31], Moody [40], Lüdde and Toppan [35], Long and Paton [34], Birman et al. [6], and others. The abiding interest in the subject owes a great deal to the strong relationship it has with the theory of knots and links in S^3, see [5, 26]. The representations of B_ℓ we obtain here are powerful enough to detect braids in the kernel of the Burau representation, which is now known to be unfaithful for $\ell \geq 6$, see [40, 34, 6]. Unlike the representations considered in [26, 31], our generalized Burau representations do not factor through the Hecke algebra in general. Analogous behavior is exhibited by the representations constructed in [29, 35, 6] by other means. Varchenko informs us that he has also obtained representations of a similar nature, see [49].

Braid groups are also important in the study of plane algebraic curves and hyperplane arrangements. To a plane algebraic curve of degree d, Moishezon [39] associates a certain "braid monodromy" homomorphism, $\theta : F_s \to B_d$, that depends on a choice of generic linear projection $\mathbb{C}^2 \to \mathbb{C}$. For an arbitrary hyperplane arrangement, there is an analogous "pure braid monodromy," see [12] and the references therein. Given a representation $\rho : B_d \to \mathrm{GL}(N,R)$, Libgober [33] shows that the R-module $H_0(F_s; R^N_{\rho\circ\theta})$ is an invariant of the curve. In particular, using the reduced Burau representation, he recovers the Alexander polynomial (up to a factor). We expect that using the generalized Burau and Gassner representations defined here will lead to invariants of plane curves and arrangements that cannot be explained solely in terms of the homology of the maximal abelian cover of the complement.

We present several other applications of our construction. For example, using it, we obtain algorithms for computing the integral homology of the Milnor fiber of an arbitrary fiber-type hyperplane arrangement, as well as the homology eigenspaces of the algebraic monodromy. We exhibit the results of some of these computations in Section 7. The chain complexes arising in these instances are more manageable than those generated by Aleksandrov [1] and Dimca [15] for the same purposes. The complexes (of differential forms) found in these works are generally infinitely generated. Also the (rank one) complex local systems arising in the computation of the eigenspaces of the monodromy are often among those excluded by the conditions placed on the local system in [16].

The homology of the complement of an arrangement with coefficients in a rank one complex local system is intimately related to the study of generalized hypergeometric functions [2, 50]. In light of the work of Schechtman and Varchenko [47], affine "discriminantal" arrangements obtained from the braid arrangements are of particular interest. The fundamental groups of these discriminantal arrangements are of the form $G = P_{n,\ell}$, and may therefore be realized as iterated semidirect products of free groups. In Section 6, we prove a vanishing theorem pertaining to these groups which generalizes a result of Kohno [29]. We show that if $v : P_{n,\ell} \to \mathrm{Aut}(V)$ is a complex representation (of arbitrary rank) that is "quasi-generic" through rank q, then $H_i(P_{n,\ell}; V) = 0$ for $0 \le i \le \min\{q, n-\ell-1\}$ (Theorem 6.10). Results of this form have implications in the study of the Knizhnik–Zamolodchikov equations, see [47, 16].

Conventions. Unless otherwise specified, we will regard all modules over the group ring $\mathbb{Z}G$ of a group G as *left* modules. Elements of the free module $(\mathbb{Z}G)^n$ are viewed as *row* vectors, and $\mathbb{Z}G$-linear maps $(\mathbb{Z}G)^n \to (\mathbb{Z}G)^m$ are viewed as $n \times m$ matrices which act on the *right* (so that the matrix of $B \circ A$ is $A \cdot B$). We will write $[A]^k$ for the map $\bigoplus_1^k A$ (or, the block-diagonal $kn \times km$ matrix with diagonal blocks A), A^{T} for the transpose of A, and I_n for the $n \times n$ identity matrix.

If U and V are two $\mathbb{Z}G$-modules, $U \otimes_G V$ denotes the $\mathbb{Z}G$-module equal to $U \otimes V$ modulo the diagonal G-action. If $\phi : G \to H$ is a homomorphism, $\tilde{\phi} : \mathbb{Z}G \to \mathbb{Z}H$ denotes its extension to group rings, given by $\tilde{\phi}(\sum n_g g) = \sum n_g \phi(g)$. (We will abuse notation and also write $\tilde{\phi} : (\mathbb{Z}G)^n \to (\mathbb{Z}H)^n$ for the map $\bigoplus_1^n \tilde{\phi}$.) For a $\mathbb{Z}G$-module V, there is a $\mathbb{Z}H$-module $\mathbb{Z}H \otimes_{\mathbb{Z}G} V$ obtained by extension of scalars. This is achieved by imposing on $\mathbb{Z}H$ the structure of a *right* $\mathbb{Z}G$-module via $s \cdot r = s\tilde{\phi}(r)$, and setting $s \cdot (s' \otimes m) = ss' \otimes m$. An excellent reference for all this is Brown's book [9].

1. Semidirect products of free groups

In this section, we introduce the class of groups under consideration (iterated semidirect products of finitely generated free groups), give some topological and geometric interpretations, and provide some examples.

1.1. Let G_1 and G_2 be two groups, and let α be an action of G_1 on G_2, i.e. a homomorphism $\alpha : G_1 \to \mathrm{Aut}(G_2)$ from G_1 to the group of *right* automorphisms of G_2. The *semidirect product* of G_1 and G_2 with respect to α, $G_2 \rtimes_\alpha G_1$, is the set $G_2 \times G_1$, endowed with the group operation $(g_2, g_1) \cdot (g_2', g_1') = (\alpha(g_1')(g_2)g_2', g_1g_1')$. The group $G = G_2 \rtimes_\alpha G_1$ fits into a split exact sequence

$$1 \to G_2 \xrightarrow{\iota_2} G \underset{\iota_1}{\overset{\pi}{\rightleftarrows}} G_1 \to 1,$$

where $\iota_2(g_2) = (g_2, 1)$, $\iota_1(g_1) = (1, g_1)$, and $\pi(g_2, g_1) = g_1$. Identifying the groups G_k with their images in G under ι_k, we see that G is generated by G_1 and G_2, and the following relations hold in G: $g_1^{-1}g_2g_1 = \alpha(g_1)(g_2)$, for every $g_1 \in G_1; g_2 \in G_2$.

This construction can of course be iterated. Assume we are given groups $G_1, \dots, G_\ell$, and, for each $i < j$, homomorphisms $\alpha_j^i : G_i \to \mathrm{Aut}(G_j)$ satisfying the compatibility conditions $\alpha_k^i(g_i)^{-1}\alpha_k^j(g_j)\alpha_k^i(g_i) = \alpha_k^j(\alpha_k^i(g_i)(g_j))$, for each $i < j < k$. Then, we define the *iterated semidirect product* of $G_1, \dots, G_\ell$ with respect to the actions α_j^i to be the group

$$G = G_\ell \rtimes_{\alpha_\ell} G_{\ell-1} \rtimes \cdots \rtimes_{\alpha_3} G_2 \rtimes_{\alpha_2} G_1,$$

where, for each $1 \le q \le \ell$, the partial iteration, $G^q = G_q \rtimes_{\alpha_q} G^{q-1}$, is defined by the homomorphism $\alpha_q : G^{q-1} \to \mathrm{Aut}(G_q)$, whose restriction to G_p, $1 \le p < q$, is α_q^p.

In this paper, we study in detail groups G which may be realized as iterated semidirect products of finitely generated free groups. Such groups can be written as $G = \rtimes_{q=1}^{\ell} G_q$, where $G_q = F_{d_q} = \langle x_{1,q}, \dots, x_{d_q,q} \rangle$ is free on d_q generators. It follows readily that the group G has presentation

$$G = \langle x_{i,q}\ (1 \le i \le d_q, 1 \le q \le \ell) \mid x_{j,p}^{-1}x_{i,q}x_{j,p} = \alpha_q^{j,p}(x_{i,q})\ (p < q) \rangle, \tag{1.1}$$

where $\alpha_q^{j,p} := \alpha_q(x_{j,p}) \in \mathrm{Aut}(F_{d_q})$. Conversely, any group G with presentation as above admits the structure of an iterated semidirect product of free groups in an obvious fashion.

Example 1.2. The principal motivation for our analysis of iterated semidirect products of free groups are Artin's (pure) braid groups. Let

$$B_\ell = \langle \sigma_i\ (1 \le i < \ell) \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ (1 \le i < \ell - 1), \\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (|i - j| > 1) \rangle$$

denote the braid group on ℓ strings, and P_ℓ the subgroup of braids with trivial permutation of the strings, see Artin [4], and the books by Birman [5] and Hansen [23]. The pure braid group, $P_\ell = F_{\ell-1} \rtimes_{\alpha_{\ell-1}} \cdots \rtimes_{\alpha_2} F_1$, admits the structure of an iterated semidirect product of free groups. The monodromy homomorphisms $\alpha_q : P_q \to \mathrm{Aut}(F_q)$, $2 \le q \le \ell - 1$ are given by the restriction to P_q of the Artin representation, $\alpha_q : B_q \to \mathrm{Aut}(F_q)$, defined by

$$\alpha_q(\sigma_i)(x_j) = \begin{cases} x_ix_{i+1}x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases}$$

where $F_q = \langle x_1, \dots, x_q \rangle$. The iterated semidirect product structure is in evidence in the familiar presentation of the pure braid group found in the above references. The group P_ℓ has generators

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, \quad 1 \le i < j \le \ell,$$

and defining relations

$$A_{r,s}^{-1} A_{i,j} A_{r,s} = \begin{cases} A_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j, \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if } r<s=i<j, \\ A_{r,j} A_{s,j} A_{i,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if } r=i<s<j, \\ [A_{r,j}, A_{s,j}] A_{i,j} [A_{r,j}, A_{s,j}]^{-1} & \text{if } r<i<s<j. \end{cases}$$

1.3. Let us give a topological interpretation of iterated semidirect products of (finitely generated) free groups. Associated to a group $G = \rtimes_{i=1}^{\ell} F_{d_i}$ there is a standard CW-complex $X = X_G$, with fundamental group isomorphic to G. This complex is defined inductively as a tower of fibrations,

$$X = X^\ell \xrightarrow{p_\ell} X^{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_3} X^2 \xrightarrow{p_2} X^1,$$

such that each projection $p_q : X^q \to X^{q-1}$ admits a section, and has fiber $K_{d_q} = \bigvee_1^{d_q} S^1$.

The complex X is constructed as follows: Take $X^1 = K_{d_1}$. Inductively assume that the space X^{q-1} with $\pi_1(X^{q-1}) = G^{q-1}$ has been constructed. Let $\mathscr{E}_0(K)$ denote the group of based homotopy classes of based self-homotopy equivalences of a space (K, y_0). If K is a $K(\pi, 1)$ space, the evaluation map ev : $\mathscr{E}_0(K) \to \operatorname{Aut}(\pi_1(K, y_0))$, defined by $\operatorname{ev}(f) = f_\#$, is an isomorphism. Applying this observation to $K = K_{d_q}$, we see that the homomorphism $\alpha_q : G^{q-1} \to \operatorname{Aut}(F_{d_q})$ factors through $\tau_q : G^{q-1} \to \mathscr{E}_0(K_{d_q})$. Now let $\pi : \widetilde{X}^{q-1} \to X^{q-1}$ be the universal cover of X^{q-1}, and identify G^{q-1} as the group of deck transformations. Consider the diagonal action of G^{q-1} on $\widetilde{X}^{q-1} \times K_{d_q}$, defined by $g \cdot (\tilde{x}, y) = (g\tilde{x}, \tau_q(g)(y))$, and let X^q denote the orbit space of this action. By construction, the map $p_q : X^q \to X^{q-1}$ given by $p_q([\tilde{x}, y]) = \pi(\tilde{x})$ is a fibration, with fiber K_{d_q}. Moreover, a canonical section $s_q : X^{q-1} \to X^q$ is given by $s_q(x) = [\tilde{x}, y_0]$. (This is well defined, as $[\tilde{x}, y_0] = [g\tilde{x}, \tau_q(g)(y_0)] = [g\tilde{x}, y_0]$.) Finally, $\pi_1(X^q) = F_{d_q} \rtimes_{\alpha_q} G^{q-1} = G^q$, and this finishes the inductive construction of $X = X^\ell$.

Notice that the CW-complex X_G defined above is a $K(G,1)$ space. This follows from the long exact sequence in homotopy for a fibration and induction. Since X_G is ℓ-dimensional by construction, the group G is of type FL, and its cohomological dimension is ℓ, see [9].

1.4. A more geometric interpretation of the group G is when the above tower of (Serre) fibrations can be replaced (up to homotopy) by a tower of locally trivial bundles,

$$M = M^\ell \xrightarrow{\pi_\ell} M^{\ell-1} \xrightarrow{\pi_{\ell-1}} \cdots \xrightarrow{\pi_3} M^2 \xrightarrow{\pi_2} M^1,$$

such that the fiber Σ_q of $\pi_q : M^q \to M^{q-1}$ is a surface with a number of punctures. In this case, the homomorphism $\alpha_q : G^{q-1} \to \mathrm{Aut}(F_{d_q})$ can be realized by the monodromy of the bundle, $\mu_q : M^{q-1} \to \mathrm{Homeo}(\Sigma_q)$.

Perhaps the simplest situation is where each fiber has genus 0, i.e. $\Sigma_q = \mathbb{C} \setminus \{d_q \text{ points}\}$. This is the case, for example, when M is the complement of a fiber-type arrangement of complex hyperplanes. In this instance, each map $\pi_q : M^q \to M^{q-1}$ is, by definition, the restriction of a linear projection $\mathbb{C}^q \to \mathbb{C}^{q-1}$. This notion was introduced by Falk and Randell in [18], where they prove the LCS formula for $G = \pi_1(M)$, the group of a central fiber-type arrangement. This result was subsequently extended to arbitrary fiber-type arrangements by Jambu [25]. The *exponents* $\{d_1, \dots, d_\ell\}$ arising in the iterated semidirect product structure on G from the iterated bundle structure on M (i.e. the exponents of the fiber-type arrangement itself) are, via the LCS formula, determined by the Betti numbers of G. In the case of the braid arrangement, $\mathscr{A}_\ell = \{H_{i,j} = \ker(z_i - z_j)\}$, whose complement, $M(\mathscr{A}_\ell) = \mathbb{C}^\ell \setminus \bigcup H_{i,j}$, is the configuration space of the set of ℓ (ordered) points in $\mathbb{C}$, the fiber-type structure was first discovered by Fadell and Neuwirth [17]. The resulting decomposition of $P_\ell = \pi_1(M(\mathscr{A}_\ell))$ is precisely the one exhibited in Example 1.2.

1.5. We conclude this section with a few remarks concerning the non-uniqueness of semidirect product structures of groups. First, note that if $\alpha, \beta : G_1 \to \mathrm{Aut}(G_2)$ are homomorphisms that differ by an *inner* automorphism γ of G_2 (i.e. $\alpha(g) = \gamma \cdot \beta(g)$, $\forall g \in G_1$), then the semidirect products $G_2 \rtimes_\alpha G_1$ and $G_2 \rtimes_\beta G_1$ are isomorphic. Thus, the isomorphism class of the group $G = G_2 \rtimes_\alpha G_1$ depends only on the homomorphism $\bar{\alpha} : G_1 \to \mathrm{Out}(G_2)$. Second, let us point out that there is no well-defined notion of exponents of a group in general. That is, for a group G that can be written as $G = \rtimes_{i=1}^{\ell} F_{d_i}$, the "exponents" $\{d_1, \dots, d_\ell\}$ depend on the particular iterated semidirect product structure on G. (On the other hand, the number of exponents, ℓ, depends only on G, since $\mathrm{cd}(G) = \ell$.) We illustrate these phenomena with two examples that are relevant to our general discussion.

Example 1.6. The pure braid group on 3 strings, P_3. It follows from Example 1.2 that $P_3 = F_2 \rtimes_{\alpha_2} F_1$, where $F_1 = \langle A_{1,2} \rangle$, $F_2 = \langle A_{1,3}, A_{2,3} \rangle$, and $\alpha_2(A_{1,2})$ is conjugation by $A_{1,3}A_{2,3}$. Thus, P_3 is isomorphic to the direct product $F_2 \times F_1$.

There is another realization of P_3 as a semidirect product of free groups: $P_3 = F_4 \rtimes_\mu F_1$, corresponding to the Milnor fibration of the braid arrangement in $\mathbb{C}^3$ (see Section 7). A computation shows that $F_1 = \langle A_{1,2} \rangle$, $F_4 = \langle t_1, t_2, t_3, t_4 \rangle$, where $t_1 = A_{1,2}^{-1}A_{1,3}$, $t_2 = A_{1,2}^{-1}A_{2,3}$, $t_3 = A_{1,3}A_{1,2}^{-1}$, $t_4 = A_{2,3}A_{1,2}^{-1}$, and the action $\mu : F_1 \to \mathrm{Aut}(F_4)$ is given by

$$A_{1,2} : \begin{cases} t_1 \mapsto t_1 t_4 t_2^{-1}, \\ t_2 \mapsto t_1 t_4 t_3^{-1}, \\ t_3 \mapsto t_1, \\ t_4 \mapsto t_2. \end{cases}$$

Example 1.7. The pure braid group on 4 strings, P_4. As discussed in Example 1.2, this group may be realized as $P_4 \cong F_3 \rtimes_{\alpha_3} F_2 \rtimes_{\alpha_2} F_1$. Since the Coxeter groups A_3 and D_3 are isomorphic, we have $P_4 = \mathrm{PB}(\mathrm{A}_3) \cong \mathrm{PB}(\mathrm{D}_3)$. This latter group may be realized as $\mathrm{PB}(\mathrm{D}_3) \cong F_5 \rtimes_{\beta_3} F_2 \rtimes_{\alpha_2} F_1$.

The geometric reason for this decomposition is due to Brieskorn [8], who found a (non-linear) bundle map from the complement of the D_ℓ arrangement to a hyperplane complement homotopy equivalent to the complement of the $\mathrm{A}_{\ell-1}$ arrangement. This map was studied by Falk and Randell [18], who noted that the Brieskorn bundle admits a section, and that its fiber is a curve of genus $2^{\ell-2}(\ell-3)+1$ with $2^{\ell-1}$ punctures. It follows that $\mathrm{PB}(\mathrm{D}_\ell) \cong F_k \rtimes_{\beta_\ell} P_\ell$, where $k = 2^{\ell-1}(\ell-2)+1$. The representation $\beta_\ell : P_\ell \to \mathrm{Aut}(F_k)$ was recently identified by Leibman and Markushevich [32]. For $\mathrm{PB}(\mathrm{D}_3)$, the action of P_3, generated by $\{A_{1,2}, A_{1,3}, A_{2,3}\}$, on $F_5 = \langle t_1, \ldots, t_5 \rangle$ is given by

$$A_{1,2}: \begin{cases} t_1 \mapsto t_1, \\ t_2 \mapsto t_2 t_4 t_5^{-1} t_3^{-1} t_2 t_1, \\ t_3 \mapsto t_2 t_4 t_5^{-1} t_1, \\ t_4 \mapsto t_1^{-1} t_2^{-1} t_3 t_5, \\ t_5 \mapsto t_1^{-1} t_5 t_4^{-1} t_2^{-1} t_3 t_5, \end{cases} \quad A_{1,3}: \begin{cases} t_1 \mapsto t_2^{-1} t_5, \\ t_2 \mapsto t_2, \\ t_3 \mapsto t_3 t_1 t_5^{-1} t_2, \\ t_4 \mapsto t_2^{-1} t_4 t_1^{-1} t_5, \\ t_5 \mapsto t_2^{-1} t_5 t_1^{-1} t_5, \end{cases} \quad A_{2,3}: \begin{cases} t_1 \mapsto t_3^{-1} t_1 t_4, \\ t_2 \mapsto t_3^{-1} t_2 t_4, \\ t_3 \mapsto t_3, \\ t_4 \mapsto t_4, \\ t_5 \mapsto t_5. \end{cases}$$

(This is not how the representation $\beta_3 : P_3 \to \mathrm{Aut}(F_5)$ is written in [32]; their formula for $A_{1,3}$ is not correct, but can be fixed by carefully following their algorithm.)

2. The resolution

In this section, we construct a finite, free $\mathbb{Z}G$-resolution of the integers for every group G which admits the structure of an iterated semidirect product of finitely generated free groups. The basis for this construction is the non-commutative differential calculus for words in a free group developed by Fox in [21] (see [5] for an exposition).

2.1. First consider a single free group $F_n = \langle x_1, \ldots, x_n \rangle$. Let $K_n = \bigvee_1^n S^1$ be the standard $K(F_n, 1)$, and let $\widetilde{C}_\bullet$ be the augmented chain complex of the universal cover $\widetilde{K}_n$. Identifying C_0 with $\mathbb{Z}F_n$, and C_1 with $(\mathbb{Z}F_n)^n$ (with basis $\{e_1, \ldots, e_n\}$ given by the lifts of the 1-cells at the basepoint), the resolution $\widetilde{C}_\bullet$ can be written as

$$0 \to (\mathbb{Z}F_n)^n \xrightarrow{\Delta} \mathbb{Z}F_n \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\Delta = (x_1 - 1 \ \cdots \ x_n - 1)^{\mathrm{T}}$ and ε is the augmentation map, given by $\varepsilon(x_i) = 1$. Further, consider an automorphism $\alpha : F_n \to F_n$. The induced chain map $\alpha_\bullet : C_\bullet \to C_\bullet$

can be written as:

$$\begin{array}{ccc} (\mathbb{Z}F_n)^n & \xrightarrow{\Delta} & \mathbb{Z}F_n \\ \big\downarrow{\scriptstyle J(\alpha)\circ\tilde{\alpha}} & & \big\downarrow{\scriptstyle \tilde{\alpha}} \\ (\mathbb{Z}F_n)^n & \xrightarrow{\Delta} & \mathbb{Z}F_n \end{array} \tag{2.1}$$

where $J(\alpha) = (\partial\alpha(x_i)/\partial x_j)$ is the $n \times n$ Jacobian matrix of Fox derivatives of α. Note that $\alpha_\bullet$ is the composition of a $\mathbb{Z}F_n$-linear map ($\mathrm{id}_{\mathbb{Z}F_n}$, resp. $J(\alpha)$), with a non-linear map (the extension $\tilde{\alpha} : \mathbb{Z}F_n \to \mathbb{Z}F_n$, resp. $\tilde{\alpha} : (\mathbb{Z}F_n)^n \to (\mathbb{Z}F_n)^n$). The commutativity of diagram (2.1) is a consequence of the "fundamental formula of Fox calculus".

If $\beta : F_n \to F_n$ is another automorphism, the fact that $(\beta \circ \alpha)_\bullet = \beta_\bullet \circ \alpha_\bullet$ is a consequence of the following "chain rule of Fox Calculus":

Lemma 2.2. $J(\beta \circ \alpha) = \tilde{\beta}(J(\alpha)) \cdot J(\beta)$.

In particular, $J(\alpha^{-1}) = \tilde{\alpha}^{-1}(J(\alpha)^{-1})$.

2.3. Given $F_n = \langle x_1, \dots, x_n \rangle$ and $\alpha \in \mathrm{Aut}(F_n)$ as above, form the semidirect product, $G_\alpha := F_n \rtimes_\alpha F_1 = \langle x_i, t \mid t^{-1}x_it = \alpha(x_i)\rangle$, of $F_1 = \langle t \rangle$ with F_n determined by α. Let $R = \mathbb{Z}G_\alpha$, and define $\lambda_t : R \to R$ by $\lambda_t(g) = t \cdot g$. (Note that λ_t is *not* R-linear with respect to the left-module structure on R, unless G_α is abelian.) Extension of scalars yields the following commuting diagram:

$$\begin{array}{ccc} R \otimes_{\mathbb{Z}F_n} (\mathbb{Z}F_n)^n & \xrightarrow{\mathrm{id}\otimes\Delta} & R \otimes_{\mathbb{Z}F_n} \mathbb{Z}F_n \\ \big\downarrow{\scriptstyle \lambda_t\otimes J(\alpha)\circ\tilde{\alpha}} & & \big\downarrow{\scriptstyle \lambda_t\otimes\tilde{\alpha}} \\ R \otimes_{\mathbb{Z}F_n} (\mathbb{Z}F_n)^n & \xrightarrow{\mathrm{id}\otimes\Delta} & R \otimes_{\mathbb{Z}F_n} \mathbb{Z}F_n \end{array}$$

The map $\lambda_t \otimes J(\alpha) \circ \tilde{\alpha}$, together with the canonical isomorphism $R \otimes_{\mathbb{Z}F_n} (\mathbb{Z}F_n)^n \cong R^n$, define a map $\rho(\alpha) : R^n \to R^n$.

Lemma 2.4. *The map* $\rho(\alpha)$ *belongs to* $\mathrm{Aut}_R(R^n)$, *and its matrix is* $t \cdot J(\alpha)$.

Proof. First we must verify that $\rho(\alpha)$ is R-linear. For that, start by computing $\rho(\alpha)$ on the basis vectors $\{e_1, \dots, e_n\}$:

$$\rho(\alpha)(e_i) = t \sum_{j=1}^{n} \frac{\partial\alpha(x_i)}{\partial x_j} \cdot e_j. \tag{$*$}$$

Clearly, $\rho(\alpha)(t^k \cdot e_i) = t^k \cdot \rho(\alpha)(e_i)$. For an element w of F_n, we have

$$\rho(\alpha)(w \cdot e_i) = t \cdot \alpha(w) \sum_{j=1}^{n} \frac{\partial\alpha(x_i)}{\partial x_j} \cdot e_j = t \cdot t^{-1}wt \sum_{j=1}^{n} \frac{\partial\alpha(x_i)}{\partial x_j} \cdot e_j = w \cdot \rho(\alpha)(e_i),$$

and R-linearity follows from the fact that F_1 and F_n generate $G_\alpha = F_n \rtimes_\alpha F_1$. That the matrix of $\rho(\alpha)$ is as asserted follows from $(*)$.

Finally, we must verify that $\rho(\alpha)$ has an inverse. Note that $G_\alpha \cong G_{\alpha^{-1}}$, the isomorphism being given by $x_i \mapsto x_i$ and $t \mapsto t^{-1}$. Thus, $\mathbb{Z}G_{\alpha^{-1}} = R$. It now follows from Lemma 2.2 that $\rho(\alpha) \circ \rho(\alpha^{-1}) = \mathrm{id}$. □

2.5. We now consider an iterated semidirect product $G \cong G_\ell \rtimes_{\alpha_\ell} \dots \rtimes_{\alpha_3} G_2 \rtimes_{\alpha_2} G_1$, where $G_q = F_{d_q} = \langle x_{1,q}, \dots, x_{d_q,q} \rangle$. Recall that G^q denotes the split extension $G_q \rtimes_{\alpha_q} G^{q-1}$, and $\alpha_q^p : G_p \to \mathrm{Aut}(G_q)$ denotes the restriction of α_q to G_p, $1 \le p < q$. Let $R = \mathbb{Z}G$ denote the integral group ring of G. For each generator $x_{r,p}$ ($1 \le p < q, 1 \le r \le d_p$) of G^{q-1}, we have a commuting diagram

$$\begin{array}{ccc} (\mathbb{Z}G_q)^{d_q} & \xrightarrow{\Delta_q} & \mathbb{Z}G_q \\ \downarrow{\scriptstyle J_q^{r,p} \circ \tilde{\alpha}_q^{r,p}} & & \downarrow{\scriptstyle \tilde{\alpha}_q^{r,p}} \\ (\mathbb{Z}G_q)^{d_q} & \xrightarrow{\Delta_q} & \mathbb{Z}G_q \end{array}$$

where $\Delta_q = (x_{1,q} - 1 \ \cdots \ x_{d_q,q} - 1)^{\mathrm{T}}$, $\tilde{\alpha}_q^{r,p}$ is the extension of the automorphism $\alpha_q^{r,p} := \alpha_q^p(x_{r,p}) : G_q \to G_q$ induced by conjugation by $x_{r,p}$, and $J_q^{r,p} = J(\alpha_q^{r,p})$ is the Jacobian matrix of $\alpha_q^{r,p}$. Let $\lambda_{r,p} : R \to R$ be left multiplication by $x_{r,p}$. By Lemma 2.4, the map

$$\rho(\alpha_q^{r,p}) = \lambda_{r,p} \otimes J_q^{r,p} \circ \tilde{\alpha}_q^{r,p} : R \otimes_{\mathbb{Z}G_q} (\mathbb{Z}G_q)^{d_q} \to R \otimes_{\mathbb{Z}G_q} (\mathbb{Z}G_q)^{d_q},$$

defines an R-linear automorphism of R^{d_q}, with matrix $x_{r,p} \cdot (\partial \alpha_q^{r,p}(x_{i,q}) / \partial x_{j,q})$.

Lemma 2.6. *For each q, $1 < q \le \ell$, there is a (unique) representation $\rho_q : G^{q-1} \to \mathrm{Aut}_R(R^{d_q})$ with the property that $\rho_q(x) = \lambda_x \otimes J(\alpha_q(x)) \circ \tilde{\alpha}_q(x)$ for every $x \in G^{q-1}$.*

Proof. Specifying the automorphisms $\rho_q(x_{r,p})$ for each $1 \le r \le d_p$ defines a representation $\rho_q^p : G_p \to \mathrm{Aut}_R(R^{d_q})$ for each of the free groups $G_p = F_{d_p}$, $1 \le p < q \le \ell$. We are left with showing that these representations are compatible with the iterated semidirect product structure on G, i.e. $\rho_s^p(x_{j,p}^{-1})\rho_s^q(x_{i,q})\rho_s^p(x_{j,p}) = \rho_s^q(\alpha_q^{j,p}(x_{i,q}))$, for $p < q < s$. This follows from the fact that $\lambda_x \circ \lambda_y = \lambda_{xy}$ and Lemma 2.2. □

2.7. The above representation extends to a representation $\rho_q : G \to \mathrm{Aut}_R(R^{d_q})$ via the convention $\rho_q(x_{r,p}) = I_{d_q}$ if $p \ge q$. We denote by $\tilde{\rho}_q : R \to \mathrm{End}_R(R^{d_q})$ the extension of ρ_q to the group ring R. Replacing each entry x of an $m \times n$ matrix by $\tilde{\rho}_q(x)$ defines a homomorphism $\mathrm{Hom}_R(R^m, R^n) \to \mathrm{Hom}_R(R^{md_q}, R^{nd_q})$ that we still denote by $\tilde{\rho}_q$. By restriction, we also get a homomorphism $\tilde{\rho}_q : \mathrm{Aut}_R(R^n) \to \mathrm{Aut}_R(R^{nd_q})$.

2.8. We now construct a free resolution $\varepsilon : C_\bullet \to \mathbb{Z}$ over the ring $R = \mathbb{Z}G$. While this is done in a purely algebraic fashion, the reader may find it useful to keep in mind the topological underpinnings of the construction, as explained in 1.3 and Remark 2.11.

Let $C_0 = R$, and, for $1 \le k \le \ell$, let

$$C_k = \bigoplus_{1 \le p_1 < \cdots < p_k \le \ell} R^{d_{p_1} d_{p_2} \cdots d_{p_k}}.$$

The augmentation map, $\varepsilon : C_0 \to \mathbb{Z}$, is the usual augmentation of the group ring, given by $\varepsilon(g) = 1$, for $g \in G$. We define the boundary maps of the complex $C_\bullet$ by specifying their restrictions $\Delta^{p_1,p_2,\dots,p_k}$ to the summands $R^{d_{p_1}d_{p_2}\cdots d_{p_k}}$. This is done recursively as follows: Define $\Delta_p : R^{d_p} \to R$ by $\Delta_p = (x_{1,p} - 1 \ \cdots \ x_{d_p,p} - 1)^{\mathrm{T}}$. For $p_1 < p_2$, define $\Delta_{p_1,p_2} : R^{d_{p_1}d_{p_2}} \to R^{d_{p_2}}$ by $\Delta_{p_1,p_2} = -\tilde{\rho}_{p_2}(\Delta_{p_1})$. In general, for $1 \le p_1 < \cdots < p_k \le \ell$, define $\Delta_{p_1,\dots,p_k} : R^{d_{p_1}\cdots d_{p_k}} \to R^{d_{p_2}\cdots d_{p_k}}$ by

$$\Delta_{p_1,\dots,p_k} = -\tilde{\rho}_{p_k}(\Delta_{p_1\dots,p_{k-1}}).$$

Now define $\Delta^{p_1,\dots,p_k} : R^{d_{p_1}\cdots d_{p_k}} \to \bigoplus_{i=1}^{k} R^{d_{p_1}\cdots \hat{d}_{p_i}\cdots d_{p_k}}$ by

$$\Delta^{p_1,\dots,p_k} = \left(\Delta_{p_1\dots,p_k}, \left[\Delta_{p_2,\dots,p_k}\right]^{d_{p_1}}, \dots, \left[\Delta_{p_i,\dots,p_k}\right]^{d_{p_1}\cdots d_{p_{i-1}}}, \dots, \left[\Delta_{p_k}\right]^{d_{p_1}\cdots d_{p_{k-1}}}\right).$$

Finally, define $\Delta : C_k \to C_{k-1}$ by

$$\Delta = \bigoplus_{1\le p_1<\dots<p_k\le\ell} \Delta^{p_1,\cdots,p_k}.$$

In the context of the above construction, the fundamental formula of Fox calculus has the following consequences.

Lemma 2.9. *For $x \in G_i$ and $1 \le i < p_1 < \cdots < p_k < q \le \ell$, we have*

$$\tilde{\rho}_q \circ \tilde{\rho}_{p_k} \circ \cdots \circ \tilde{\rho}_{p_1}(x) \cdot [\Delta_q]^{d_{p_1}\cdots d_{p_k}} = [\Delta_q]^{d_{p_1}\cdots d_{p_k}} \cdot \tilde{\rho}_{p_k} \circ \cdots \circ \tilde{\rho}_{p_1}(x).$$

Proof. First consider the case $k = 0$. Let $\alpha \in \mathrm{Aut}(G_q)$ be the automorphism induced by conjugation by x. Then the matrix of $\rho_q(x)$ is $x \cdot J(\alpha)$ (see 2.5). Using the fundamental formula of Fox calculus as in (2.1), we have

$$\rho_q(x) \cdot \Delta_q = x \cdot J(\alpha) \cdot \Delta_q = x \cdot \tilde{\alpha}(\Delta_q) = \Delta_q \cdot x.$$

In general, write $A = \tilde{\rho}_{p_k} \circ \cdots \circ \tilde{\rho}_{p_1}(x)$ and note that A is a square matrix of size $d = d_{p_1} \cdots d_{p_k}$. For each entry a of A, we have $\tilde{\rho}_q(a) \cdot \Delta_q = \Delta_q \cdot a$ by (2.1) as above. It then follows from some elementary matrix manipulations that $\tilde{\rho}_q(A) \cdot [\Delta_q]^d = [\Delta_q]^d \cdot A$. □

We now come to the main theorem of this section.

Theorem 2.10. *Given a group G which admits the structure of an iterated semidirect product of finitely generated free groups, the system of R-modules and homomorphisms $\{C_\bullet, \Delta\}$ is a finite, free resolution of $\mathbb{Z}$ over $R = \mathbb{Z}G$.*

Proof. The proof is by induction on ℓ with the case $\ell = 1$ clear.

Let $G = \rtimes_{p=1}^{\ell} G_p$, where $G_p = F_{d_p}$, and consider the (normal) subgroup $\mathscr{G} < G$ given by $\mathscr{G} = \rtimes_{p=2}^{\ell} G_p$. By induction, the construction of 2.8 yields a free resolution $\varepsilon : C_\bullet(\mathscr{G}) \to \mathbb{Z}$ over $\mathscr{R} = \mathbb{Z}\mathscr{G}$.

Let $\widehat{C}_\bullet = C_\bullet(\mathscr{G}) \otimes_{\mathscr{G}} R$, and let $D_\bullet$ denote the chain complex of R-modules with terms $D_k = (\widehat{C}_k)^{d_1}$ and differentials $\partial_D = -[\partial_{\widehat{C}}]^{d_1}$. That is, $D_\bullet$ is the direct sum of d_1 copies of $\widehat{C}_\bullet$, with the sign of the differential reversed. Note that $\widehat{C}_\bullet$ and $D_\bullet$ are complexes of free R-modules, and that $H_*(\mathscr{G};R) = H_*(\widehat{C}_\bullet)$.

Define a map $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ by setting the restriction of $\Xi_\bullet$ to the summand $R^{d_1 d_{p_1} \cdots d_{p_k}}$ of D_k to be equal to

$$\Delta_{1,p_1,\dots,p_k} : R^{d_1 d_{p_1} \cdots d_{p_k}} \to R^{d_{p_1} \cdots d_{p_k}} \subset \widehat{C}_k.$$

In particular, $\Xi_0 : D_0 \to \widehat{C}_0$ is given by $\Xi_0 = \Delta_1 : R^{d_1} \to R$. We claim that $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ is a chain map. To prove this assertion, it suffices to verify that

$$\Delta_{1,p_1,\dots,p_k} \cdot [\Delta_{p_j,\dots,p_k}]^{d_{p_1} \cdots d_{p_{j-1}}} = -[\Delta_{p_j,\dots,p_k}]^{d_1 d_{p_1} \cdots d_{p_{j-1}}} \cdot \Delta_{1,p_1,\dots,\hat{p}_j,\dots,p_k}$$

for $1 \le j \le k$ (where $d_{p_1} \cdots d_{p_{j-1}} = 1$ if $j = 1$). These equalities all follow easily from Lemma 2.9. Furthermore, it is clear from the construction in 2.8 that $C_\bullet = C_\bullet(G)$ is the mapping cone of the chain map $\Xi_\bullet$. Thus $C_\bullet$ is a chain complex of free R-modules.

We now show that $C_\bullet \to \mathbb{Z}$ is a resolution, i.e. that $\widetilde{C}_\bullet$ is acyclic. It is clear from the construction that $H_0(\widetilde{C}_\bullet) = 0$. Since $C_\bullet$ is the mapping cone of $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$, we have a long exact sequence in homology

$$\cdots \to H_{i+1}(C_\bullet) \to H_i(D_\bullet) \xrightarrow{H_i(\Xi_\bullet)} H_i(\widehat{C}_\bullet) \to H_i(C_\bullet) \to \cdots$$

with connecting homomorphisms induced by the chain map $\Xi_\bullet$. Now R is free as a $\mathscr{G}$-module and $H_*(\mathscr{G};R) = H_*(\widehat{C}_\bullet)$, so

$$H_i(\widehat{C}_\bullet) = \begin{cases} R_{\mathscr{G}} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \quad \text{and therefore} \quad H_i(D_\bullet) = \begin{cases} (R_{\mathscr{G}})^{d_1} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Thus the long exact sequence above reduces to

$$0 \to H_1(C_\bullet) \to (R_{\mathscr{G}})^{d_1} \xrightarrow{H_0(\Xi_\bullet)} R_{\mathscr{G}} \to H_0(C_\bullet) \to 0,$$

and we have $H_i(C_\bullet) = 0$ for $i \ge 2$. It remains to show that $H_1(C_\bullet) = 0$, i.e. that the connecting homomorphism $H_0(\Xi_\bullet)$ is injective.

Now $H_0(\widehat{C}_\bullet) = R_{\mathscr{G}} = \mathbb{Z}G \otimes_{\mathscr{G}} \mathbb{Z} = \mathbb{Z}[G/\mathscr{G}] = \mathbb{Z}F_{d_1}$, so $H_0(D_\bullet) = (\mathbb{Z}F_{d_1})^{d_1}$. Under this identification we have $H_0(\Xi_\bullet) = (x_{1,1} - 1 \ \cdots \ x_{d_1,1} - 1)^{\mathrm{T}}$, where $F_{d_1} = \langle x_{1,1}, \dots, x_{d_1,1} \rangle$. Hence $H_0(\Xi_\bullet)$ is injective, and the proof is complete. □

Remark 2.11. The chain complex $C_\bullet(G)$ is, by acyclic models, chain-equivalent to $C_\bullet(\widetilde{X}_G)$, the equivariant chain complex of the universal cover of the $K(G,1)$ space X_G constructed in 1.3. Note that $\widetilde{X}_G \cong \times_{p=1}^{\ell} \widetilde{X}_{G_p}$, and that $C_\bullet(G)$ and $\bigotimes_{p=1}^{\ell} C_\bullet(X_{G_p})$ are isomorphic as graded groups. As in the case of a direct product (see Example 3.2), the boundary of a k-chain is computed from the boundaries of 1-chains. However, the boundary maps of $C_\bullet(G)$ are not equal to those of $\bigotimes_{p=1}^{\ell} C_\bullet(X_{G_p})$ in general, but rather, are "twisted" by the representations ρ_q, according to the recursion formulas in 2.8.

For example, if $G = F_n \rtimes_\alpha F_m$ is the semidirect product of *two* free groups, the boundary maps of $C_\bullet(G)$ are determined by those of $C_\bullet(F_n)$ and $C_\bullet(F_m)$, and the representation $\rho : F_m \to \mathrm{GL}(n, \mathbb{Z}G)$. Note that, in this instance, it is immediate that $C_\bullet(G)$ is the chain complex resulting from application of the Fox calculus to the presentation $G = \langle x_i, y_j \mid y_j x_i = x_i \alpha(x_i)(y_j) \rangle$, where $F_m = \langle x_i \rangle$ and $F_n = \langle y_j \rangle$. Thus $C_\bullet(G)$ is equal to the chain complex of the universal cover of the "presentation two-complex," X_G, associated to this presentation of the group G.

Remark 2.12. After completing this work, we became aware of a construction of Brady [7]. Given a semidirect product $G = G_2 \rtimes_\alpha G_1$, a free resolution of G_1, and a free resolution of G_2 which admits an action of G_1 compatible with α, Brady describes in [7] an algorithm for producing a free resolution of G. Carrying out this algorithm inductively in our situation, and making use of the lemmas from 2.1–2.7, it is possible to show that the resulting resolution coincides with the one constructed in 2.8. Although this argument is somewhat shorter than the one presented here, the explicit construction and arguments presented above will be of further use in the remainder of the paper.

3. Some consequences

In this section we derive some consequences of Theorem 2.10, and illustrate the construction of the previous section by means of several examples.

3.1. Direct products

We first consider the simplest situation – that of a direct product.

Proposition 3.1. *Let G and H be two iterated semidirect products of free groups, and let $C_\bullet(G)$ and $C_\bullet(H)$ be the corresponding chain complexes. Then $G \times H$ is also an iterated semidirect product of free groups, and $C_\bullet(G \times H) = C_\bullet(G) \otimes C_\bullet(H)$.*

Proof. The iterated semidirect product structure on $G \times H$ is obtained in an obvious fashion. The structure of $C_\bullet(G \times H)$ follows from the construction and standard facts about tensor products of resolutions (see e.g. [9, p. 107]). □

This result has the following topological interpretation. Recall the $K(G,1)$ space X_G defined in Section 1. It follows from the construction that $X_{G\times H} \simeq X_G \times X_H$. Passing to equivariant chain complexes of universal covers recovers the above result.

Example 3.2. Let $G = \times_{p=1}^{\ell} G_p$, where $G_p = F_{d_p}$. Then $\widetilde{C}_\bullet(G) = \bigotimes_{p=1}^{\ell} \widetilde{C}_\bullet(G_p)$, where $\widetilde{C}_\bullet(G_p) : (\mathbb{Z}G_p)^{d_p} \xrightarrow{\Delta^{(p)}} \mathbb{Z}G_p \xrightarrow{\varepsilon^{(p)}} \mathbb{Z} \to 0$ is as in 2.1. Explicitly, $C_k(G)$ is the direct sum of $C_{i_1}(G_1) \otimes \cdots \otimes C_{i_\ell}(G_\ell)$, over all indices $i_r \in \{0,1\}$ such that $i_1 + \cdots + i_\ell = k$, and the restriction of the differential $\Delta : C_k(G) \to C_{k-1}(G)$ to such a summand is given by

$$\Delta(c_1 \otimes \cdots \otimes c_\ell) = \sum_{r=1}^{\ell} (-1)^{i_1+\cdots+i_{r-1}} c_1 \otimes \cdots \otimes c_{r-1} \otimes \delta_r(c_r) \otimes c_{r+1} \otimes \cdots \otimes c_\ell,$$

where $\delta_r = \Delta^{(r)}$ if $i_r = 1$ and $\delta_r = \varepsilon^{(r)}$ if $i_r = 0$.

The simplest such instance is when all the exponents d_i are equal to 1, in which case we have $G = \mathbb{Z}^\ell = \langle x_i \mid [x_i, x_j] = 1 \rangle$. Then $\widetilde{C}_\bullet(\mathbb{Z}^\ell)$ is the usual free $\mathbb{Z}\mathbb{Z}^\ell$-resolution of $\mathbb{Z}$. That is, X_G is the ℓ-torus T^ℓ, and $\widetilde{C}_\bullet = \widetilde{C}_\bullet(\mathbb{Z}^\ell)$ is the equivariant augmented chain complex of the universal cover of T^ℓ. Specifically, $C_0 = R = \mathbb{Z}\mathbb{Z}^\ell$, $C_1 = R^\ell$ may be identified with a free R-module with basis $\{e_1, \dots, e_\ell\}$, and $C_k = R^{\binom{\ell}{k}} \cong \bigwedge^k C_1$. With these identifications, the differential $\Delta : C_k \to C_{k-1}$ may be expressed as

$$\Delta(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{r=1}^{k} (-1)^{r-1} (x_{i_r} - 1)(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_r} \wedge \cdots \wedge e_{i_k}).$$

3.2. IA-products

An automorphism of a group G is said to be an *IA-automorphism* if it induces the identity automorphism on the abelianization $H_1(G; \mathbb{Z}) = G/G'$. The IA-automorphisms of G form a (normal) subgroup $\mathrm{IA}(G)$ of $\mathrm{Aut}(G)$. The groups $\mathrm{IA}(F_n)$ have been much studied; for example, it is known from the work of Nielsen and Magnus that the natural map $\mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$ is surjective, and that its kernel, $\mathrm{IA}(F_n)$, is finitely generated and torsion-free. Let us say that a group G is an *IA-product of free groups* if it can be written as $G = G_\ell \rtimes_{\alpha_\ell} \cdots \rtimes_{\alpha_3} G_2 \rtimes_{\alpha_2} G_1$, with $G_q = F_{d_q}$, and $\alpha_q : G^{q-1} \to \mathrm{IA}(F_{d_q})$.

Proposition 3.3. *Let G be an IA-product of free groups. Then the chain complex $C_\bullet(G) \otimes_G \mathbb{Z}$ has trivial boundary maps.*

Proof. It suffices to show that each of the matrices $\Delta_{p_1,\dots,p_k}$ reduces to the zero matrix upon applying the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ to every entry. This is accomplished by an inductive argument. $\square$

Let $b_j(G) = \operatorname{rank} H_j(G; \mathbb{Z})$ be the jth Betti number of G. The following generalize the results of Kohno [27] and Falk and Randell [18].

Corollary 3.4 (Factorization). *If G is an IA-product of free groups, then the homology groups of G are torsion free, and the Poincaré polynomial of G factors into linear terms:*

$$\sum_{j=0}^{\ell} b_j(G) t^j = \prod_{q=1}^{\ell} (1 + d_q t).$$

This result may also be obtained using a spectral sequence argument as in [18].

Let $\phi_k = \operatorname{rank} \Gamma_k(G)/\Gamma_{k+1}(G)$ denote the rank of the kth lower central series quotient of G. Noting that the group theoretic results of [18] or [30] apply in our situation, we obtain:

Theorem 3.5 (LCS Formula). *If G is an IA-product of free groups, then in $\mathbb{Z}[[t]]$ we have*

$$\sum_{j=0}^{\ell} b_j(G)(-t)^j = \prod_{q=1}^{\ell}(1-d_q t) = \prod_{k\geq 1}(1-t^k)^{\phi_k}.$$

3.3. An example

We conclude this section with a detailed example of the construction of the chain complex from Section 2. Recall that B_ℓ denotes the group of braids on ℓ strings. Let B_ℓ^1 be the subgroup of braids that fix the endpoint of the last string. The group B_ℓ^1 is the semidirect product of $B_{\ell-1}$ with $F_{\ell-1}$, determined by the Artin representation $\alpha_{\ell-1} : B_{\ell-1} \to \operatorname{Aut}(F_{\ell-1})$ (see Example 1.2).

Example 3.6. Let $G = B_4^1 = F_3 \rtimes_{\alpha_3} B_3$. This group admits the structure of an iterated semidirect product of free groups, $G = F_3 \rtimes_{\alpha_3} F_2 \rtimes_{\mu_2} F_1$. To see this, first note that $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$ admits a semidirect product structure $B_3 = F_2 \rtimes_{\mu_2} F_1$, coming from the Milnor fibration of the discriminant singularity $\mathscr{D}_3$ in $\mathbb{C}^3$ (see 7.5). The group F_1 is generated by $x_{1,1} = \sigma_1$, the group F_2 is generated by $x_{1,2} = \sigma_1\sigma_2^{-1}, x_{2,2} = \sigma_2^{-1}\sigma_1$, and the action $\mu_2 : F_1 \to \operatorname{Aut}(F_2)$ is given by

$$x_{1,1} : \begin{cases} x_{1,2} \mapsto x_{2,2}, \\ x_{2,2} \mapsto x_{1,2}^{-1}x_{2,2}. \end{cases}$$

Now let $F_3 = \langle x_{1,3}, x_{2,3}, x_{3,3}\rangle$. The Artin representation $\alpha_3 : B_3 \to \operatorname{Aut}(F_3)$ is given by

$$x_{1,1} : \begin{cases} x_{1,3} \mapsto x_{1,3}x_{2,3}x_{1,3}^{-1}, \\ x_{2,3} \mapsto x_{1,3}, \\ x_{3,3} \mapsto x_{3,3}, \end{cases}$$

$$x_{1,2} : \begin{cases} x_{1,3} \mapsto x_{1,3}x_{3,3}x_{1,3}^{-1}, \\ x_{2,3} \mapsto x_{1,3}, \\ x_{3,3} \mapsto x_{3,3}^{-1}x_{2,3}x_{3,3}, \end{cases} \qquad x_{2,2} : \begin{cases} x_{1,3} \mapsto x_{1,3}x_{2,3}x_{1,3}^{-1}, \\ x_{2,3} \mapsto x_{3,3}, \\ x_{3,3} \mapsto x_{3,3}^{-1}x_{1,3}x_{3,3}. \end{cases}$$

This establishes the iterated semidirect product structure on $G = B_4^1$. Carrying out the procedure described in Section 2, we see that the chain complex $C_\bullet(G)$ is given by

$$R^6 \xrightarrow{(\Delta_{1,2,3}\ \ \Delta_{2,3}\ \ [\Delta_3]^2)} R^6 \oplus R^3 \oplus R^2 \xrightarrow{\begin{pmatrix} \Delta_{2,3} & [\Delta_3]^2 & 0 \\ \Delta_{1,3} & 0 & \Delta_3 \\ 0 & \Delta_{1,2} & \Delta_2 \end{pmatrix}} R^3 \oplus R^2 \oplus R \xrightarrow{\begin{pmatrix} \Delta_3 \\ \Delta_2 \\ \Delta_1 \end{pmatrix}} R,$$

where $R = \mathbb{Z}G$, and

$$\Delta_1 = (x_{1,1} - 1), \qquad \Delta_2 = \begin{pmatrix} x_{1,2} - 1 \\ x_{2,2} - 1 \end{pmatrix}, \qquad \Delta_3 = \begin{pmatrix} x_{1,3} - 1 \\ x_{2,3} - 1 \\ x_{3,3} - 1 \end{pmatrix},$$

$$\Delta_{1,2} = \begin{pmatrix} 1 & -x_{1,1} \\ x_{1,1}x_{1,2}^{-1} & 1 - x_{1,1}x_{1,2}^{-1} \end{pmatrix},$$

$$\Delta_{1,3} = \begin{pmatrix} 1 + (x_{1,3} - 1)x_{1,1} & -x_{1,1}x_{1,3} & 0 \\ -x_{1,1} & 1 & 0 \\ 0 & 0 & 1 - x_{1,1} \end{pmatrix},$$

$$\Delta_{2,3} = \begin{pmatrix} 1 + (x_{1,3} - 1)x_{1,2} & 0 & -x_{1,2}x_{1,3} \\ -x_{1,2} & 1 & 0 \\ 0 & -x_{1,2}x_{3,3}^{-1} & 1 - x_{1,2}x_{3,3}^{-1}(x_{2,3} - 1) \\ 1 + (x_{1,3} - 1)x_{2,2} & -x_{2,2}x_{1,3} & 0 \\ 0 & 1 & -x_{2,2} \\ -x_{2,2}x_{3,3}^{-1} & 0 & 1 - x_{2,2}x_{3,3}^{-1}(x_{1,3} - 1) \end{pmatrix},$$

and $\Delta_{1,2,3} = \begin{pmatrix} -I_3 & I_3 - \Delta_{1,3} \\ -A & -I_3 + A \end{pmatrix}$, where

$$A = \begin{pmatrix} (1 - x_{1,3})x_{1,1}x_{1,2}^{-1} & (1 - x_{1,3})x_{1,1}x_{1,2}^{-1}x_{1,3} & x_{1,1}x_{1,2}^{-1}x_{1,3}x_{2,3} \\ 0 & x_{1,1}x_{1,2}^{-1} & 0 \\ x_{1,1}x_{1,2}^{-1}x_{2,3}^{-1} & x_{1,1}x_{1,2}^{-1}x_{2,3}^{-1}(x_{1,3} - 1) & 0 \end{pmatrix}.$$

Remark 3.7. This chain complex can be used to compute the homology of G with various (twisted) coefficients. For example, the homology $H_* = H_*(G; \mathbb{Z})$ with trivial $\mathbb{Z}$ coefficients is given by $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^2, H_2 = \mathbb{Z}^2, H_3 = \mathbb{Z}$. Note that both aspects of the LCS formula – the factorization of the Poincaré polynomial and the recursion for the ranks of the LCS quotients – fail for this group. The former does not hold since the Poincaré polynomial equals $(1+t)(1+t+t^2)$. The latter would imply that $\phi_2 = -1$. Since the monodromy automorphisms α_2 and α_3 defining G are *not* IA-automorphisms, these facts do not contradict Theorem 3.5.

4. Generalized Magnus representations

Let G be a group that admits the structure of an iterated semidirect product of finitely generated free groups. In this section, we use the construction of the chain complex $C_\bullet(G)$ to define representations of groups Γ which act compatibly on G.

Definition 4.1. An automorphism $\psi \in \mathrm{Aut}(G)$ is said to be *compatible* with the iterated semidirect product structure $G = G_\ell \rtimes_{\alpha_\ell} \cdots \rtimes_{\alpha_3} G_2 \rtimes_{\alpha_2} G_1$ if it satisfies the following conditions:

(1) $\forall g \in G_p$, we have $\psi(g) \in G_p$; and
(2) $\forall g_p \in G_p$, $\forall g_q \in G_q$, $p < q$, we have $\alpha_q^p(\psi(g_p))(\psi(g_q)) = \psi(\alpha_q^p(g_p)(g_q))$ in G_q.

Given a compatible automorphism ψ of G, let $\psi_p \in \mathrm{Aut}(G_p)$ be the restriction of ψ to G_p (well defined by condition (1) above). Condition (2) can be restated as:

(2′) $\forall x \in G_p$, $p < q$, we have $\alpha_q^p(\psi_p(x)) = \psi_q \circ \alpha_q^p(x) \circ \psi_q^{-1}$ in $\mathrm{Aut}(G_q)$.

It is easily checked that the compatible automorphisms of G form a subgroup of $\mathrm{Aut}(G)$, which we shall denote by $\mathrm{Aut}^{\rtimes}(G)$. The compatibility conditions are quite restrictive. For example, if $G = \times_{p=1}^{\ell} G_p$, then $\mathrm{Aut}^{\times}(G) = \times_{p=1}^{\ell} \mathrm{Aut}(G_p)$; in particular, $\mathrm{Aut}^{\times}(\mathbb{Z}^\ell) \cong (\mathbb{Z}_2)^\ell$.

4.2. Now assume $G_p = F_{d_p}$, for each p, $1 \le p \le \ell$, and let $C_\bullet = C_\bullet(G)$ be the chain complex of free modules over $R = \mathbb{Z}G$ constructed in Section 2. Any automorphism $\psi : G \to G$ gives rise to a chain equivalence $\Psi_\bullet : C_\bullet \to C_\bullet$. We shall explicitly describe the chain map $\Psi_\bullet$ in the case where ψ belongs to $\mathrm{Aut}^{\rtimes}(G)$.

Let $J(\psi_p) \in \mathrm{Aut}_{\mathbb{Z}G_p}((\mathbb{Z}G_p)^{d_p})$ be the Jacobian matrix of Fox derivatives of the automorphism $\psi_p \in \mathrm{Aut}(G_p)$. Let $J_p(\psi) := \mathrm{id}_R \otimes J(\psi_p) \in \mathrm{Aut}_R(R^{d_p})$ be the automorphism obtained from $J(\psi_p)$ by extension of scalars. Define the *higher-order Jacobians* of ψ recursively as follows:

$$J_{p_1,\dots,p_k}(\psi) := \left[J_{p_k}(\psi)\right]^{d_{p_1}\cdots d_{p_{k-1}}} \cdot \tilde{\rho}_{p_k}(J_{p_1,\dots,p_{k-1}}(\psi)) : R^{d_{p_1}\cdots d_{p_k}} \to R^{d_{p_1}\cdots d_{p_k}}.$$

Now define the map $\Psi_k : C_k \to C_k$ by specifying its restriction to the summand $R^{d_{p_1}\cdots d_{p_k}}$ of C_k to be the composition of the higher-order Jacobian $J_{p_1,\dots,p_k}(\psi)$ with the extension $\tilde{\psi} : R^{d_{p_1}\cdots d_{p_k}} \to R^{d_{p_1}\cdots d_{p_k}}$:

$$\Psi_k = \bigoplus_{1 \le p_1 < \cdots < p_k \le \ell} J_{p_1,\dots,p_k}(\psi) \circ \tilde{\psi}.$$

(The map $\Psi_0 : C_0 \to C_0$ is just $\tilde{\psi} : R \to R$.) In order to prove that $\Psi_\bullet : C_\bullet \to C_\bullet$ is a chain map, we first need a lemma.

Lemma 4.3. *For all $x \in G_p$ and $p < q$, we have $\rho_q(\psi_p(x)) = J_q(\psi)^{-1} \cdot \tilde{\psi}(\rho_q(x)) \cdot J_q(\psi)$.*

Proof. Recall from Lemma 2.4 that, if $G_\alpha = F_n \rtimes_\alpha F_1 = \langle x_i, t \mid t^{-1}x_i t = \alpha(x_i)\rangle$, and $R = \mathbb{Z}G_\alpha$, then $\rho(\alpha)$ is an R-linear map, with matrix $t \cdot J(\alpha)$. Furthermore, note that, if A is an arbitrary matrix with entries in $\mathbb{Z}F_n$, then $t^{-1} \cdot A \cdot t = \tilde{\alpha}(A)$.

Using these observations, together with Lemma 2.2, and condition (2′) of Definition 4.1, we obtain:

$$\begin{aligned}
\rho_q(\psi_p(x)) &= \psi_p(x)\cdot J(\alpha_q^p(\psi_p(x)))\\
&= \psi_p(x)\cdot J(\psi_q\circ\alpha_q^p(x)\circ\psi_q^{-1})\\
&= \psi_p(x)\cdot \tilde{\psi}_q\circ\widetilde{\alpha_q^p(x)}(J(\psi_q^{-1}))\cdot J(\psi_q\circ\alpha_q^p(x))\\
&= \psi_p(x)\cdot \tilde{\psi}_q\circ\widetilde{\alpha_q^p(x)}\circ\tilde{\psi}_q^{-1}(J(\psi_q)^{-1})\cdot J(\psi_q\circ\alpha_q^p(x))\\
&= \psi_p(x)\cdot \alpha_q^p(\widetilde{\psi_p(x)})(J_q(\psi)^{-1})\cdot J(\psi_q\circ\alpha_q^p(x))\\
&= \psi_p(x)\cdot\psi_p(x)^{-1}\cdot J_q(\psi)^{-1}\cdot\psi_p(x)\cdot J(\psi_q\circ\alpha_q^p(x))\\
&= J_q(\psi)^{-1}\cdot\psi(x)\cdot\tilde{\psi}(J(\alpha_q^p(x)))\cdot J_q(\psi)\\
&= J_q(\psi)^{-1}\cdot\tilde{\psi}(\rho_q(x))\cdot J_q(\psi). \quad\square
\end{aligned}$$

Proposition 4.4. *The map $\Psi_\bullet : C_\bullet \to C_\bullet$ is a chain equivalence.*

Proof. To show that $\Psi_\bullet : C_\bullet \to C_\bullet$ is a chain map, we must show that $\Psi_{k-1}\circ\Delta = \Delta\circ\Psi_k$, for $1\le k\le \ell$. We accomplish this by induction on k, with the case $k=1$ following from diagram 2.1.

By virtue of the direct sum decompositions of $C_\bullet$ and $\Psi_\bullet$, it is enough to show that diagrams of the form

$$\begin{array}{ccc}
R^{d_{p_1}\cdots d_{p_k}} & \xrightarrow{\Delta^{p_1,\dots,p_k}} & \bigoplus_{i=1}^{k} R^{d_{p_1}\cdots\hat{d}_{p_i}\cdots d_{p_k}}\\
\downarrow{\Psi_k} & & \downarrow{\Psi_{k-1}}\\
R^{d_{p_1}\cdots d_{p_k}} & \xrightarrow{\Delta^{p_1,\dots,p_k}} & \bigoplus_{i=1}^{k} R^{d_{p_1}\cdots\hat{d}_{p_i}\cdots d_{p_k}}
\end{array}$$

commute. Since

$$\Delta^{p_1,\dots,p_k} = \left(\Delta_{p_1,\dots,p_k}, [\Delta_{p_2,\dots,p_k}]^{d_{p_1}}, \dots, [\Delta_{p_i,\dots,p_k}]^{d_{p_1}\cdots d_{p_{i-1}}}, \dots, [\Delta_{p_k}]^{d_{p_1}\cdots d_{p_{k-1}}}\right),$$

this amounts to showing that

$$J_{p_1,\dots,p_k}(\psi)\cdot[\Delta_{p_i,\dots,p_k}]^{d_{p_1}\cdots d_{p_{i-1}}} = \tilde{\psi}\left([\Delta_{p_i,\dots,p_k}]^{d_{p_1}\cdots d_{p_{i-1}}}\right)\cdot J_{p_1,\dots,\hat{p}_i,\dots,p_k}(\psi)$$

for $1\le i\le k$. These equalities all follow from the definitions using induction, together with Lemma 4.3.

To complete the proof, we must show that the chain map $\Psi_\bullet$ is, in fact, a chain equivalence. This follows directly from the definitions. $\square$

Example 4.5. The simplest example of this construction is in the case of a direct product $G = \times_{p=1}^{\ell} G_p$, where $G_p = F_{d_p}$. Using the decomposition $C_k(G) = \bigoplus C_{i_1}(G_1)\otimes\cdots\otimes C_{i_\ell}(G_\ell)$ from Example 3.2, we can write the chain map induced by $\psi = \times_{p=1}^{\ell}\psi_p$ as $\Psi_k = \bigoplus \Psi_{i_1}^{(1)}\otimes\cdots\otimes\Psi_{i_\ell}^{(\ell)}$, where $\Psi_{i_r}^{(r)} = J(\psi_r)\circ\tilde{\psi}_r$ if $i_r = 1$ and $\Psi_{i_r}^{(r)} = \tilde{\psi}_r$ if $i_r = 0$.

Example 4.6. We further illustrate the construction using the notations and computations of Example 3.6. Recall the group $G = B_4^1 = F_3 \rtimes_{\alpha_3} F_2 \rtimes_{\mu_2} F_1$. Consider the normal subgroup $\mathscr{G} = F_3 \rtimes_{\alpha_3} F_2$, where $F_2 = \langle x_{1,2}, x_{2,2} \rangle$, $F_3 = \langle x_{1,3}, x_{2,3}, x_{2,3} \rangle$, and define the automorphism $\psi \in \mathrm{Aut}^{\rtimes}(\mathscr{G})$ to be conjugation by $x_{1,1} \in F_1$.

The chain complex $C_\bullet(\mathscr{G})$ is of the form

$$\mathscr{R}^6 \xrightarrow{(\,\Delta_{2,3}\ [\Delta_3]^2\,)} \mathscr{R}^3 \oplus \mathscr{R}^2 \xrightarrow{\begin{pmatrix}\Delta_3\\ \Delta_2\end{pmatrix}} \mathscr{R},$$

where $\mathscr{R} = \mathbb{Z}\mathscr{G}$, and the boundary maps are given by restricting the boundary maps of $C_\bullet(G)$. Carrying out the construction described in 4.2, the chain equivalence $\Psi_\bullet : C_\bullet(\mathscr{G}) \to C_\bullet(\mathscr{G})$ induced by ψ can be written as

$$\begin{array}{ccccc}
\mathscr{R}^6 & \longrightarrow & \mathscr{R}^3 \oplus \mathscr{R}^2 & \longrightarrow & \mathscr{R} \\
\Big\downarrow{\scriptstyle J_{2,3}\circ\tilde{\psi}} & & \Big\downarrow{\scriptstyle (J_3 \oplus J_2)\circ\tilde{\psi}} & & \Big\downarrow{\scriptstyle \tilde{\psi}} \\
\mathscr{R}^6 & \longrightarrow & \mathscr{R}^3 \oplus \mathscr{R}^2 & \longrightarrow & \mathscr{R}
\end{array}$$

where $J_2 = J(\psi|_{F_2})$, $J_3 = J(\psi|_{F_3})$, and $J_{2,3} = [J_3]^2 \cdot \tilde{\rho}_2(J_2)$ are given by

$$J_2 = x_{1,1}^{-1} \cdot (I_2 - \Delta_{1,2}) = \begin{pmatrix} 0 & 1 \\ -x_{1,2}^{-1} & x_{1,2}^{-1} \end{pmatrix},$$

$$J_3 = x_{1,1}^{-1} \cdot (I_3 - \Delta_{1,3}), \qquad J_{2,3} = x_{1,1}^{-1} \cdot (I_6 - \Delta_{1,2,3}).$$

4.7. In order to proceed with the construction of generalized Magnus representations, we need to see how the higher-order Jacobians behave under composition. The following result can be viewed as a generalization of the chain rule of Fox calculus (Lemma 2.2). Let $G = \rtimes_{p=1}^{\ell} F_{d_p}$, and consider $\phi, \psi \in \mathrm{Aut}^{\rtimes}(G)$.

Proposition 4.8 (chain rule). $J_{p_1,\dots,p_k}(\psi \circ \phi) = \tilde{\psi}(J_{p_1,\dots,p_k}(\phi)) \cdot J_{p_1,\dots,p_k}(\psi)$.

Proof. This is proved by induction on k, with the case $k = 1$ following from Lemma 2.2.

For the inductive step we need Lemma 4.3, which, we recall, states that for $x \in G^{q-1}$, we have $\rho_q(\psi(x)) = J_q(\psi)^{-1} \cdot \tilde{\psi}(\rho_q(x)) \cdot J_q(\psi)$. It follows immediately that $\tilde{\rho}_q(\tilde{\psi}(\sum n_x x)) = J_q(\psi)^{-1} \cdot \tilde{\psi}\left(\tilde{\rho}_q(\sum n_x x)\right) \cdot J_q(\psi)$. If A is a $d \times d$ matrix with entries in R, this implies that

$$\tilde{\rho}_q\left(\tilde{\psi}(A)\right) = \left[J_q(\psi)^{-1}\right]^d \cdot \tilde{\psi}(\tilde{\rho}_q(A)) \cdot \left[J_q(\psi)\right]^d.$$

Using the definition of higher-order Jacobians, the case $k = 1$, the inductive hypothesis, the fact that $\tilde{\rho}_q$ is a homomorphism (see 2.7), and the above formula, we get

$$\begin{aligned}
J_{p_1,\dots,p_k}(\psi\circ\phi) &= \left[J_{p_k}(\psi\circ\phi)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\cdot\tilde\rho_{p_k}(J_{p_1,\dots,p_{k-1}}(\psi\circ\phi))\\
&= \left[\tilde\psi(J_{p_k}(\phi))\cdot J_{p_k}(\psi)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\cdot\tilde\rho_{p_k}\left(\tilde\psi(J_{p_1,\dots,p_{k-1}}(\phi))\cdot J_{p_1,\dots,p_{k-1}}(\psi)\right)\\
&= \left[\tilde\psi\left(J_{p_k}(\phi)\right)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\cdot\left[J_{p_k}(\psi)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\cdot\tilde\rho_{p_k}\left(\tilde\psi(J_{p_1,\dots,p_{k-1}}(\phi))\right)\\
&\quad\cdot\tilde\rho_{p_k}\left(J_{p_1,\dots,p_{k-1}}(\psi)\right)\\
&= \tilde\psi\left(\left[J_{p_k}(\phi)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\right)\cdot\tilde\psi\left(\tilde\rho_{p_k}(J_{p_1,\dots,p_{k-1}}(\phi))\right)\cdot\left[J_{p_k}(\psi)\right]^{d_{p_1}\cdots d_{p_{k-1}}}\\
&\quad\cdot\tilde\rho_{p_k}\left(J_{p_1,\dots,p_{k-1}}(\psi)\right)\\
&= \tilde\psi(J_{p_1,\dots,p_k}(\phi))\cdot J_{p_1,\dots,p_k}(\psi). \qquad\square
\end{aligned}$$

4.9. Now consider a group Γ that acts compatibly on G. That is, we are given a homomorphism $\Phi:\Gamma\to\operatorname{Aut}^{\rtimes}(G)$. For each $\gamma\in\Gamma$, the construction of 4.2 yields a chain map $\Phi(\gamma)_\bullet : C_\bullet(G)\to C_\bullet(G)$. But these chain maps are not R-linear in general, so a judicious extension of scalars is required in order to define a representation of Γ. The idea is suggested by the original approach followed by Magnus [36] (see [5, p. 115]).

Definition 4.10. A homomorphism $\tau : G\to K$ is said to be *Φ-invariant* if $\tau(\Phi(\gamma)(g)) = \tau(g)$ for all $g\in G$ and $\gamma\in\Gamma$.

Let $R=\mathbb{Z}G$, $S=\mathbb{Z}K$, $\tilde\tau : R\to S$ the extension of τ to group rings, and $S\otimes_R -$ the extension of scalars functor defined by $\tilde\tau$. Applying this functor, we obtain a chain complex of free S-modules, $S\otimes_R C_\bullet(G)$. We are now ready to state the main theorem of this section.

Theorem 4.11. *Suppose $G=\rtimes_{p=1}^{\ell} G_p$ is an iterated semidirect product of free groups, $\Phi:\Gamma\to\operatorname{Aut}^{\rtimes}(G)$ is a compatible action of a group Γ on G, and $\tau : G\to K$ is a Φ-invariant homomorphism. Given $\gamma\in\Gamma$, let $\Phi^\tau_\bullet(\gamma)=\mathrm{id}_S\otimes\Phi(\gamma)_\bullet : S\otimes_R C_\bullet(G)\to S\otimes_R C_\bullet(G)$. Then, for each k, $1\le k\le\ell$,*
(i) *the map $\Phi^\tau_k(\gamma) : S\otimes_R C_k(G)\to S\otimes_R C_k(G)$ is S-linear;*
(ii) *the map $\Phi^\tau_k(\gamma)$ is a chain equivalence; and*
(iii) *the map $\Phi^\tau_k : \Gamma\to\operatorname{Aut}_S(S\otimes_R C_k(G)), \gamma\mapsto\Phi^\tau_k(\gamma)$, is a homomorphism.*

Proof. (i) For a fixed $\gamma\in\Gamma$, write $\psi=\Phi(\gamma)\in\operatorname{Aut}^{\rtimes}(G)$, and let $\Psi_\bullet : C_\bullet\to C_\bullet$ be the chain map induced by ψ. Recall that $\Psi_0 : C_0\to C_0$ is identified with $\tilde\psi : R\to R$. Recall also that $\Psi_k : C_k\to C_k$ is the composition of a certain R-linear map with the non-linear map $\tilde\psi : C_k\to C_k$, which is a direct sum of copies of $\tilde\psi : R\to R$. Thus, it is enough to show that $\Phi^\tau_0=\mathrm{id}_S\otimes\tilde\psi : S\otimes_R R\to S\otimes_R R$ is an S-linear map. We will show more, namely

$$\mathrm{id}_S\otimes\tilde\psi=\mathrm{id}_{S\otimes_R R}\,. \qquad (**)$$

Let $\omega : S \otimes_R R \xrightarrow{\sim} S$ be the canonical isomorphism given by $\omega(s \otimes r) = s\tilde{\tau}(r)$. The Φ-invariance condition, $\tau \circ \psi = \tau$, yields

$$\omega(\Phi_0^\tau(s \otimes r)) = \omega(s \otimes \tilde{\psi}(r)) = s\tilde{\tau}(\tilde{\psi}(r)) = s\tilde{\tau}(r) = \omega(s \otimes r),$$

proving the claim.

(ii) This follows from Proposition 4.4.

(iii) This follows from Proposition 4.8 and (∗∗). □

In the special case where $G = F_n$ and $\Gamma < \mathrm{Aut}(F_n)$, representations such as the above were introduced by Magnus [36] (see [5] for details). We therefore refer to the homomorphisms $\Phi_k^\tau : \Gamma \to \mathrm{Aut}_S(S \otimes_R C_k(G))$ as *generalized Magnus representations.* Since the maps $\Phi_\bullet^\tau(\gamma)$ are chain maps, we also obtain *homological Magnus representations* $\bar{\Phi}_k^\tau : \Gamma \to \mathrm{Aut}_S H_k(S \otimes_R C_\bullet(G))$. However, note that these homology groups need not be free S-modules in general. In such a situation, one may still be able to "reduce" Φ_k^τ by restricting to a free, invariant submodule of $S \otimes_R C_k(G)$ – see e.g. Example 5.7.

Remark 4.12. There is an alternate way to interpret these representations. Recall that, in forming the tensor product $S \otimes_R C_\bullet(G)$, we view S as a right R-module via $s \cdot r = s\tilde{\tau}(r)$. Using the involution of the group ring $R = \mathbb{Z}G$ given by $\overline{\sum n_g g} = \sum n_g g^{-1}$, we can turn S into a left R-module by setting $r \cdot s = s\tilde{\tau}(\bar{r})$, and form the tensor product $C_\bullet(G) \otimes_G S$. We then have a chain equivalence $S \otimes_R C_\bullet(G) \xrightarrow{\sim} C_\bullet(G) \otimes_G S$ given by $s \otimes c \mapsto c \otimes s$, inducing an isomorphism between $H_\bullet(S \otimes_R C_\bullet(G))$ and $H_\bullet(C_\bullet(G) \otimes_G S)$. Thus, we can view the generalized Magnus representations as $\Phi_k^\tau : \Gamma \to \mathrm{Aut}_S(C_k(G) \otimes_G S)$, respectively $\bar{\Phi}_k^\tau : \Gamma \to \mathrm{Aut}_S H_k(G; S)$. When K is an abelian group, the coefficients module $S = \mathbb{Z}K$ is determined by the representation $\hat{\tau} : G \to \mathrm{Aut}\, S$, given by $\hat{\tau}(g)(s) = \tau(g^{-1})s$.

Example 4.13. The simplest example is where $\tau : G \to \{1\}$ is the trivial homomorphism. If Γ acts compatibly on G, the resulting representations, $\Phi_k^\tau : \Gamma \to \mathrm{Aut}(C_k(G) \otimes_G \mathbb{Z})$ and $\bar{\Phi}_k^\tau : \Gamma \to \mathrm{Aut}\, H_k(G; \mathbb{Z})$, can be non-trivial, even when $G = F_n$, see [5, p. 117]. However, if Γ acts by IA-automorphisms of G, then obviously $\bar{\Phi}_k^\tau$ is trivial.

Example 4.14. Let Γ be a group that acts compatibly on G, and assume that the action Φ of Γ on G factors as $\Phi : \Gamma \to \mathrm{Aut}^{\rtimes}(G) \cap \mathrm{IA}(G) \hookrightarrow \mathrm{Aut}^{\rtimes}(G)$. Then the abelianization map $\mathrm{ab} : G \twoheadrightarrow G/G'$, and more generally, maps of the form $\tau : G \xrightarrow{\mathrm{ab}} G/G' \twoheadrightarrow K$ (where K is abelian) are Φ-invariant. See Section 5.2 and Example 5.7 for examples of generalized Magnus representations obtained this way.

5. Representations of braid groups

We use the techniques developed above to define new linear representations of braid groups. Detailed discussion of these representations is deferred to [13].

5.1. Generalized Burau representations

For $1 \le \ell < n$, let $P_{n,\ell} = \ker(P_n \to P_\ell)$ denote the kernel of the homomorphism from P_n to P_ℓ defined by forgetting the last $n-\ell$ strands. Then $P_{n,\ell} = \rtimes_{p=\ell}^{n-1} F_p$ is generated by $\{A_{i,j}\}$ with $\ell < j \le n$ and $1 \le i < j$. (Note that $P_n = P_{n,1}$.) The braid group B_ℓ acts on $P_{n,\ell}$ in a natural fashion. On each free factor $F_p = F_\ell * F_{p-\ell}$ of $P_{n,\ell}$, it acts by the Artin representation on F_ℓ (see Example 1.2), and acts trivially on $F_{p-\ell}$. It is readily checked that the action so defined, $\Phi_{\ell,n} : B_\ell \to \mathrm{Aut}(P_{n,\ell})$, is compatible with the iterated semidirect product structure of $P_{n,\ell}$. The semidirect product $P_{n,\ell} \rtimes_{\Phi_{\ell,n}} B_\ell$ is the group $B_n^{n-\ell}$ of braids that fix the endpoints of the last $n-\ell$ strings.

Let $\mathbb{Z} = \langle t \rangle$, and identify the group ring, $\mathbb{Z}\mathbb{Z}$, with the ring of Laurent polynomials in t, $\Lambda = \mathbb{Z}[t, t^{-1}]$. Fix m, $1 \le m \le n-\ell$, and define a homomorphism $\tau : P_{n,\ell} \to \mathbb{Z}$ by $\tau(A_{r,s}) = t$ if $n-m+1 \le s \le n$, and $\tau(A_{r,s}) = 1$ if $\ell+1 \le s \le n-m$. Since $\tau(A_{r,s}) = \tau(A_{p,q})$ if $s = q$, the homomorphism τ is invariant with respect to the action $\Phi_{\ell,n}$ of B_ℓ on $P_{n,\ell}$. We thus obtain by Theorem 4.11 generalized Magnus representations of the braid group,

$$\beta^m_{\ell,n-\ell,k} = (\Phi_{\ell,n})^\tau_k : B_\ell \to \mathrm{Aut}_\Lambda\left(\Lambda \otimes_{\mathbb{Z}P_{n,\ell}} C_k(P_{n,\ell})\right),$$
$$\bar{\beta}^m_{\ell,n-\ell,k} : B_\ell \to \mathrm{Aut}_\Lambda H_k(\Lambda \otimes_{\mathbb{Z}P_{n,\ell}} C_\bullet(P_{n,\ell}))$$

for each k, $1 \le k \le n-\ell$. If $\ell = n-1$, we have $P_{n,\ell} = F_\ell$, $k = 1$, and the action $\Phi_{\ell,\ell+1} = \alpha_\ell$ is the Artin representation. In this instance, it is easy to see that $\beta_\ell = \beta^1_{\ell,1,1} : B_\ell \to \mathrm{GL}(\ell, \Lambda)$ is the Burau representation, and $\bar{\beta}_\ell : B_\ell \to \mathrm{GL}(\ell-1, \Lambda)$ is the reduced Burau representation (see [5, 26, 34]).

Example 5.1. In the case $\ell = 3$, $n = 5$, $m = 1$, we have $H_2(\Lambda \otimes_{\mathbb{Z}P_{5,3}} C_\bullet(P_{5,3})) = \Lambda^6$. Thus, the above construction yields a generalized Burau representation $\bar{\beta}^1_{3,2,2} : B_3 \to \mathrm{GL}(6, \Lambda)$. This representation is given by

$$\sigma_1 \mapsto \begin{pmatrix} 0 & 0 & -t-t^2 & t & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1-t & -1 & 0 & 0 & 0 & 0 \\ 1 & -1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & -t-t^2 & 1 \end{pmatrix},$$

$$\sigma_2 \mapsto \begin{pmatrix} 1 & -t-t^2 & 0 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & -t^2-t^3 & t^2 \\ 0 & 0 & -t^{-1} & t^{-1}-1 & 0 & 0 \\ 0 & 0 & -1-t^{-1} & t^{-1} & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of each of these matrices is $(\lambda-1)^2(\lambda+1)(\lambda+t)(\lambda^2-t^3)$. It follows that, unlike the Burau representation, this representation does not factor through the Hecke algebra $H(3,t)$, see [26].

Remark 5.2. In [29], Kohno uses a vanishing theorem involving the groups $P_{n,\ell}$ to construct representations of the braid group generalizing the (reduced) Burau representation. See Section 6 for a generalization of this vanishing theorem. Like the above representations, those generated by Kohno do not, in general, factor through the Hecke algebra. The construction of the generalized Magnus representations presented here was, in part, motivated by Kohno's work.

Remark 5.3. Our generalized Burau representations are powerful enough to detect braids which belong to the kernel of the Burau representation. Answering a long-standing question, Moody [40] showed that $\beta_\ell : B_\ell \to \mathrm{GL}(\ell,\Lambda)$ is not faithful for $\ell \geq 9$. This was sharpened to $\ell \geq 6$ by Long and Paton [34], who found the following braid in $\ker(\beta_6)$: $\xi = [\zeta^{-1}\sigma_5\zeta, (\sigma_2\sigma_3\sigma_4\sigma_5)^5]$, where $\zeta = \sigma_4^{-1}\sigma_5\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_3^{-3}\sigma_1^3\sigma_5\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1$. A computation shows that the representation $\bar{\beta}^1_{6,2,2} : B_6 \to \mathrm{GL}(30,\Lambda)$ detects the braid ξ: $\bar{\beta}^1_{6,2,2}(\xi) \neq I_{30}$.

5.4. The above construction may be applied in a more general context so as to yield representations of the braid group with several parameters. Fix m, $1 \leq m \leq n-\ell$, and consider the free abelian group $\mathbb{Z}^m$ generated by $\{t_s \mid n-m+1 \leq s \leq n\}$. Identify the group ring $\mathbb{Z}\mathbb{Z}^m$ with the ring Λ_m of Laurent polynomials in the variables $\{t_s\}$. Let $\tau : P_{n,\ell} \to \mathbb{Z}^m$ be the homomorphism defined by $\tau(A_{r,s}) = t_s$ if $n-m+1 \leq s \leq n$ and $\tau(A_{r,s}) = 1$ otherwise. As before, the homomorphism τ is invariant with respect to the action $\Phi_{\ell,n} : B_\ell \to \mathrm{Aut}^{\rtimes}(P_{n,\ell})$. Thus we obtain generalized Magnus representations,

$$\eta^m_{\ell,n-\ell,k} = (\Phi_{\ell,n})^\tau_k : B_\ell \to \mathrm{Aut}_{\Lambda_m}\left(\Lambda_m \otimes_{\mathbb{Z}P_{n,\ell}} C_k(P_{n,\ell})\right),$$

$$\bar{\eta}^m_{\ell,n-\ell,k} : B_\ell \to \mathrm{Aut}_{\Lambda_m} H_k(\Lambda_m \otimes_{\mathbb{Z}P_{n,\ell}} C_\bullet(P_{n,\ell}))$$

for each k, $1 \leq k \leq n-\ell$, depending on m parameters. Clearly, $\eta^1_{\ell,q,r} = \beta^1_{\ell,q,r}$ and $\bar{\eta}^1_{\ell,q,r} = \bar{\beta}^1_{\ell,q,r}$.

Example 5.5. In the case $\ell = 3$, $n = 5$, $m = 2$, we have $H_2(\Lambda_2 \otimes_{\mathbb{Z}P_{5,3}} C_\bullet(P_{5,3})) = \Lambda_2^6$. We therefore obtain a representation $\bar{\eta}^2_{3,2,2} : B_3 \to \mathrm{GL}(6,\Lambda_2)$. This representation is given below, where we denote the generators of $\mathbb{Z}^2$ by s and t (as opposed to t_4 and t_5) to simplify notation.

$$\sigma_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & s & -s & -s \\ 0 & 1 & 0 & 1+st & 0 & -1 \\ 0 & 0 & 0 & -t & 0 & 0 \\ -st & 0 & -s^2 & -t & -s^2t & s^2-s^2t \\ 0 & 0 & 1 & -1 & 1 & t \\ 0 & 0 & s+st & -t-t^2 & 0 & -s \end{pmatrix},$$

$$\sigma_2 \mapsto \begin{pmatrix} -t-st & st & -s^2+s^2t & 0 & 0 & 0 \\ -t & 0 & -s+st & 0 & 0 & 0 \\ t^2 & 0 & -s+st & 0 & st & 0 \\ s & 0 & -1 & 1 & s^2 & 0 \\ t & 0 & s & 0 & 0 & 0 \\ -1-t & 0 & -1-s-t & 0 & -s & 1 \end{pmatrix}.$$

If one sets s equal to t in the above representation (i.e. applies the map $\tilde{\phi} : \Lambda_2 \to \Lambda$ defined by $\phi(s) = t$, $\phi(t) = t$), the resulting representation $B_3 \to \mathrm{GL}(6, \Lambda)$ is precisely the one parameter representation $\bar{\beta}^2_{3,2,2} : B_3 \to \mathrm{GL}(6, \Lambda)$ defined in Section 5.1.

Remark 5.6. Methods for "lifting" representations from $B_{\ell+1}$ to B_ℓ are given by Lüdde and Toppan in [35], and by Birman et al. in [6]. Either of these techniques may be used to generate braid group representations which depend on several parameters. For instance, Lüdde and Toppan obtain an m parameter representation v^m_ℓ of B_ℓ by successively lifting the trivial representation of $B_{\ell+m}$. We have checked that the (reduced) representation $\bar{v}^2_3$ is equivalent to $\bar{\eta}^2_{3,2,2}$. We conjecture that $v^m_\ell \cong \eta^m_{\ell,m,m}$ and $\bar{v}^m_\ell \cong \bar{\eta}^m_{\ell,m,m}$ for all ℓ and m.

5.2. Generalized Gassner representations

For each $n > \ell$, the pure braid group P_ℓ acts on the group $P_{n,\ell} = F_{n-1} \rtimes \cdots \rtimes F_\ell$ by restriction of the action $\Phi_{\ell,n}$. This (compatible) action is the "usual" one, discussed in Example 1.2 (see also Section 6). The semidirect product $P_{n,\ell} \rtimes_{\Phi_{\ell,n}} P_\ell$ is, of course, the (entire) pure braid group P_n.

Fix m, $1 \le m \le n - \ell$, and let $N = \binom{n}{2} - \binom{n-m}{2}$. Consider the free abelian group $\mathbb{Z}^N$ generated by $\{t_{r,s} \mid 1 \le r < s, n-m+1 \le s \le n\}$, and let $\tau : P_{n,\ell} \to \mathbb{Z}^N$ be the homomorphism defined by $\tau(A_{r,s}) = t_{r,s}$ if $n-m+1 \le s \le n$ and $\tau(A_{r,s}) = 1$ otherwise. Checking that τ is invariant with respect to the action $\Phi_{\ell,n}$ of P_ℓ on $P_{n,\ell}$, we obtain generalized Magnus representations of the pure braid group,

$$\theta^m_{\ell,n-\ell,k} = (\Phi_{\ell,n})^\tau_k : P_\ell \to \mathrm{Aut}_{\Lambda_N}\left(\Lambda_N \otimes_{\mathbb{Z}P_{n,\ell}} C_k(P_{n,\ell})\right),$$

$$\bar{\theta}^m_{\ell,n-\ell,k} : P_\ell \to \mathrm{Aut}_{\Lambda_N} H_k(\Lambda_N \otimes_{\mathbb{Z}P_{n,\ell}} C_\bullet(P_{n,\ell}))$$

for each k, $1 \le k \le n - \ell$. In the special case $\ell = n-1$ (and thus $m = 1$), we have $N = \ell$ and $P_{n,\ell} = F_\ell$. In this instance, it is easy to see that $\theta_\ell = \theta^1_{\ell,1,1} : P_\ell \to \mathrm{GL}(\ell, \Lambda_\ell)$ is the Gassner representation. Note, however, that $\bar{\theta}_\ell : P_\ell \to \mathrm{Aut}_{\Lambda_\ell} H_1(\Lambda_\ell \otimes_{\mathbb{Z}F_\ell} C_\bullet(F_\ell))$

is not the reduced Gassner representation for $\ell > 2$, as $H_1(\Lambda_\ell \otimes_{\mathbb{Z}F_\ell} C_\bullet(F_\ell))$ is not a free module.

Example 5.7. Consider the case $\ell = 3$, $n = 5$, $m = 1$. Denote the generators of $\Lambda_4 \otimes_{\mathbb{Z}P_{5,3}} C_2(P_{5,3}) = (\Lambda_4)^{12}$ by $\{e_1, \dots, e_{12}\}$, and write $t_r = t_{r,5}$. The elements

$$\begin{array}{ll} t_4(t_2-1)e_1 + (1-t_1t_4)e_2 + (t_2-1)e_4, & (1-t_3)e_2 + (t_2-1)e_3, \\ t_4(t_3-1)e_5 + (1-t_1)e_7 + (1-t_1)(t_3-1)e_8, & t_4(t_3-1)e_6 + (1-t_2t_4)e_7 + (t_3-1)e_8, \\ (1-t_2)e_9 + (t_1-1)e_{10}, & (1-t_3t_4)e_9 + (t_1-1)e_{11} + t_3(t_1-1)e_{12}, \end{array}$$

generate a free, rank 6 submodule M of $\Lambda_4 \otimes_{\mathbb{Z}P_{5,3}} C_2(P_{5,3})$. Checking that this submodule is invariant under the action of the representation $\theta^1_{3,2,2} : P_3 \to \mathrm{GL}(12, \Lambda_4)$, we obtain a subrepresentation $\hat{\theta}^1_{3,2,2} : P_3 \to \mathrm{GL}(6, \Lambda_4)$, given by

$$A_{1,2} \mapsto \begin{pmatrix} t_1t_2t_4 & 0 & 0 & 0 & 0 & 0 \\ t_3-1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-t_1+t_1t_2t_4 & t_1(1-t_1) & 0 & 0 \\ 0 & 0 & 1-t_2t_4 & t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1t_2 & 0 \\ 0 & 0 & 0 & 0 & t_1(t_3t_4-1) & 1 \end{pmatrix},$$

$$A_{1,3} \mapsto \begin{pmatrix} t_3 & 1-t_1t_4 & 0 & 0 & 0 & 0 \\ t_3(1-t_3) & 1-t_3+t_1t_3t_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1t_3 & 0 & 0 & 0 \\ 0 & 0 & t_3(t_2t_4-1) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_2-1 \\ 0 & 0 & 0 & 0 & 0 & t_1t_3t_4 \end{pmatrix},$$

$$A_{2,3} \mapsto \begin{pmatrix} 1 & t_2(t_1t_4-1) & 0 & 0 & 0 & 0 \\ 0 & t_2t_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t_1-1 & 0 & 0 \\ 0 & 0 & 0 & t_2t_3t_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-t_2+t_2t_3t_4 & t_2(1-t_2) \\ 0 & 0 & 0 & 0 & 1-t_3t_4 & t_2 \end{pmatrix}.$$

If one sets all t_i equal to t in the above representation (i.e. applies the map $\tilde{\phi} : \Lambda_4 \to \Lambda$ defined by $\phi(t_i) = t$), the resulting representation $P_3 \to \mathrm{GL}(6, \Lambda)$ is merely the restriction to P_3 of the representation $\bar{\beta}^1_{3,2,2} : B_3 \to \mathrm{GL}(6, \Lambda)$ discussed in Example 5.1.

The submodule $M \cong (\Lambda_4)^6$ considered above is *not* isomorphic to the homology group $H_2(\Lambda_4 \otimes_{\mathbb{Z}P_{5,3}} C_\bullet(P_{5,3}))$, which is not a free Λ_4-module. This discrepancy disappears if one works over the complex numbers. Let $v : \mathbb{Z}^4 \to \mathbb{C}^*$ be a complex representation, and $M \otimes_{\mathbb{Z}^4} \mathbb{C}_v \cong \mathbb{C}^6$ the corresponding vector space. If the complex parameters $v(t_r)$ are sufficiently generic (simply all different from 1 in this instance), and $\bar{v}(t_r) := v(t_r)^{-1}$, this vector space *is* isomorphic to $H_2((\Lambda_4 \otimes_{\mathbb{Z}P_{5,3}} C_\bullet(P_{5,3})) \otimes_{\mathbb{Z}^4} \mathbb{C}_v) \cong$

$H_2(P_{5,3};\mathbb{C}_{\bar{v}}) \cong \mathbb{C}^6$ (see 4.12 for the first isomorphism). Similar genericity conditions are addressed in detail in Section 6.

Notice that the representation $\hat{\theta}^1_{3,2,2}$ decomposes into a sum of 3 rank 2 representations of P_3. More generally, one can find a free, rank $\ell(\ell-1)$ submodule of $\Lambda_{\ell+1} \otimes_{\mathbb{Z}P_{\ell+2,\ell}} C_2(P_{\ell+2,\ell})$ which is invariant under the action of the representation $\theta^1_{\ell,2,2} : P_\ell \to \mathrm{GL}(\ell(\ell+1), \Lambda_{\ell+1})$. The resulting subrepresentation $\hat{\theta}^1_{\ell,2,2}$ decomposes into a sum of ℓ rank $\ell-1$ representations of P_ℓ for any ℓ, see [13].

6. A vanishing theorem

Let P_ℓ denote the pure braid group on ℓ strings. In this section, we carry out a detailed analysis of the representations which dictate the structure of the boundary maps of the free resolution $C_\bullet(P_\ell)$ of $\mathbb{Z}$ over the integral group ring $R = \mathbb{Z}P_\ell$ given by the construction of Section 2.

6.1. The boundary maps of the chain complex $C_\bullet(P_\ell)$ are comprised of homomorphisms of the form $\Delta_{p_1,\dots,p_k} : R^{p_1\cdots p_k} \to R^{p_2\cdots p_k}$ defined recursively by $\Delta_{p_1,\dots,p_k} = -\tilde{\rho}_{p_k}(\Delta_{p_1\dots,p_{k-1}})$, where $1 \le p_1 < p_2 < \cdots < p_k \le \ell-1$. Since

$$\Delta_p = (A_{1,p+1} - 1 \quad \cdots A_{p,p+1} - 1)^{\mathrm{T}},$$

we have

$$\Delta_{p_1,\dots,p_k} = (-1)^{k-1}(\phi(A_{1,p_1+1}) - I \quad \cdots \quad \phi(A_{p_1,p_1+1}) - I)^{\mathrm{T}},$$

where $\phi = \tilde{\rho}_{p_k} \circ \cdots \circ \tilde{\rho}_{p_2}$ is the composition of the representations $\tilde{\rho}_{p_j}$, $2 \le j \le k$, and I is the identity matrix of size $p_2 \cdots p_k$. Thus, to understand the chain complex $C_\bullet(P_\ell)$, we must make sense of the iterated compositions of the representations $\tilde{\rho}_{p_j}$.

We first consider a single representation ρ_p. To simplify notation, we assume that $p = \ell - 1$ and $r < s < \ell$. This representation is (essentially) given by the Jacobian matrix of Fox derivatives of the action of $A_{r,s}$ on the free group $F_{\ell-1} = \langle A_{1,\ell}, \dots, A_{\ell-1,\ell}\rangle$ exhibited in Example 1.2. By Lemma 2.4, the matrix of $\rho_{\ell-1}(A_{r,s})$ is given by

$$A_{r,s} \cdot \left(\frac{\partial A_{r,s}^{-1} A_{i,\ell} A_{r,s}}{\partial A_{j,\ell}} \right).$$

For our immediate purposes (see below), rather than using the "standard" presentation of the pure braid group given in Example 1.2, it is useful to work with a different generating set for the free group $F_{\ell-1}$. Let

$$y_i = \begin{cases} A_{i,\ell} & \text{if } i \le r \text{ or } i > s, \\ A_{r,\ell} A_{s,\ell} & \text{if } i = s, \\ A_{r,\ell} A_{i,\ell} A_{r,\ell}^{-1} & \text{if } r < i < s. \end{cases}$$

A computation yields:

Lemma 6.2. *The action of $A_{r,s}$ on the free group $F_{\ell-1} = \langle y_1, \dots, y_{\ell-1} \rangle$ is given by*

$$A_{r,s}^{-1} y_i A_{r,s} = \begin{cases} y_i & \text{if } i \neq r, \\ y_s y_r y_s^{-1} & \text{if } i = r. \end{cases}$$

Computing Fox derivatives, we obtain:

Proposition 6.3. *With respect to the generating set $\{y_1, \dots, y_{\ell-1}\}$ of $F_{\ell-1}$, the matrix $U = (u_{i,j})$ of $\rho_{\ell-1}(A_{r,s})$ is upper triangular with non-zero entries $u_{r,r} = A_{r,s}A_{r,\ell}A_{s,\ell} = A_{r,s}y_s$, $u_{r,s} = (1 - y_r)A_{r,s}$, and $u_{k,k} = A_{r,s}$ for $k \neq r$.*

6.4. The above result may be viewed as a first step in an inductive description of the matrix of $\tilde{\rho}_{p_k} \circ \cdots \circ \tilde{\rho}_{p_1}(A_{r,s})$. For example, with respect to the generating set $\{A_{1,\ell}, \dots, A_{\ell-1,\ell}\}$ of $F_{\ell-1}$, the matrix of $\tilde{\rho}_{\ell-1} \circ \tilde{\rho}_{k-1}(A_{r,s})$ is, by Proposition 6.3, similar to the matrix obtained by applying $\rho_{\ell-1}$ to the entries of the matrix

$$\begin{pmatrix} A_{r,s} \cdot I_{r-1} & 0 & 0 \\ 0 & U_{r,s} & 0 \\ 0 & 0 & A_{r,s} \cdot I_{k-s-1} \end{pmatrix},$$

where $U_{r,s}$ is an $(s-r+1) \times (s-r+1)$ upper triangular matrix with diagonal entries $A_{r,s}A_{r,k}A_{s,k}, A_{r,s}, \dots, A_{r,s}$. Thus, to understand the composition $\tilde{\rho}_{\ell-1} \circ \tilde{\rho}_{k-1}(A_{r,s})$, we must understand not only $\rho_{\ell-1}(A_{r,s})$, but $\rho_{\ell-1}(A_{r,s}A_{r,k}A_{s,k})$ as well.

Let $J = \{j_1, j_2, \dots, j_p\}$, where $1 \leq j_1 < j_2 < \cdots < j_p \leq \ell$, and consider the pure braid A_J defined by

$$A_J = (A_{j_1,j_2})(A_{j_1,j_3}A_{j_2,j_3})(A_{j_1,j_4}A_{j_2,j_4}A_{j_3,j_4}) \cdots (A_{j_1,j_p} \cdots A_{j_{p-1},j_p}).$$

If $j_p \leq \ell - 1$, then A_J acts on the free group $F_{\ell-1} = \langle A_{1,\ell}, \dots, A_{\ell-1,\ell} \rangle$ by conjugation. We once again work with a different generating set of the free group $F_{\ell-1}$. Let

$$y_i = \begin{cases} A_{i,\ell} & \text{if } i \leq j_1 \text{ or } i > j_p, \\ A_{j_1,\ell} \cdots A_{j_k,\ell} & \text{if } i = j_k,\ 2 \leq k \leq p, \\ (A_{j_1,\ell} \cdots A_{j_k,\ell})A_{i,\ell}(A_{j_1,\ell} \cdots A_{j_k,\ell})^{-1} & \text{if } j_k < i < j_{k+1},\ 1 \leq k \leq p-1. \end{cases}$$

Lemma 6.5. *The action of A_J on the free group $F_{\ell-1} = \langle y_1, \dots, y_{\ell-1} \rangle$ is given by*

$$A_J^{-1} y_i A_J = \begin{cases} y_i & \text{if } i \neq j_k,\ 1 \leq k \leq p-1, \\ y_{j_p} y_i y_{j_p}^{-1} & \text{if } i = j_k,\ 1 \leq k \leq p-1. \end{cases}$$

Proof. If $p = 2$, the above is merely the reformulation of the defining relations of the pure braid group P_ℓ considered in Lemma 6.2. Using induction on p and the defining relations in the pure braid group, one checks that the action of A_J on $F_{\ell-1}$ is as asserted. □

Computing Fox derivatives once again, we obtain:

Proposition 6.6. *With respect to the generating set $\{y_1,\dots,y_{\ell-1}\}$ of $F_{\ell-1}$, the matrix $U=(u_{i,j})$ of $\rho_{\ell-1}(A_J)$ is upper triangular with non-zero entries $u_{j_k,j_k}=A_J y_{j_p}$ and $u_{j_k,j_p}=(1-y_{j_k})A_J$ for $1\le k\le p-1$, and $u_{i,i}=A_J$ for $i\ne j_k$, $1\le k\le p-1$.*

6.7. We now use these results to study the homology of the pure braid group (and related groups) with non-trivial coefficients. Recall that, for $1\le \ell<n$, $P_{n,\ell}=\ker(P_n\to P_\ell)$ denotes the kernel of the homomorphism given by $A_{r,s}\mapsto 1$ if $s\le\ell$ and $A_{r,s}\mapsto A_{r,s}$ if $s>\ell$. We consider complex representations, $v: P_{n,\ell}\to \mathrm{GL}(m,\mathbb{C})$, of these groups. For $J=\{j_1,\dots,j_p\}$ such that $A_J=(A_{j_1,j_2})\cdot(A_{j_1,j_3}A_{j_2,j_3})\cdots(A_{j_1,j_p}\cdots A_{j_{p-1},j_p})$ is an element of $P_{n,\ell}$, write $v(A_J)=v_J$. Let V denote the $(\mathbb{Z}P_{n,\ell})$-module (resp. local coefficient system on $K(P_{n,\ell},1)$) corresponding to v.

Definition 6.8. Fix $q\ge 0$. The representation $v: P_{n,\ell}\to\mathrm{GL}(m,\mathbb{C})$ is said to be *quasi-generic through rank q* if, for each J such that $2\le |J|\le q+2$ and $A_J\in P_{n,\ell}$, the eigenvalues of the matrix v_J are all different from 1.

Remark 6.9. If v is quasi-generic through rank q, repeated application of Proposition 6.6 shows that the eigenvalues of the matrix of $\tilde v\circ\phi(A_{i,j})$ are all different from 1, where $\tilde v:\mathbb{Z}P_{n,\ell}\to\mathrm{End}(\mathbb{C}^m)$ denotes the linear extension of v, $\phi=\tilde\rho_{p_m}\circ\cdots\circ\tilde\rho_{p_1}$, and $m\le q+1$. Thus, under these conditions, the endomorphism $I-\tilde v\circ\phi(A_{i,j})$ is in fact an isomorphism. This observation motivates the following, which generalizes the vanishing theorem found in [29] (see also [46, 51]).

Theorem 6.10. *If $v: P_{n,\ell}\to\mathrm{GL}(m,\mathbb{C})$ is quasi-generic through rank q, then the homology groups of $P_{n,\ell}$ with coefficients in V vanish for $0\le i\le\min\{q,n-\ell-1\}$.*

Proof. The proof is by induction on the cohomological dimension, $d=\mathrm{cd}(P_{n,\ell})=n-\ell$, of the group $P_{n,\ell}$.

If $d=n-\ell=1$, we have $P_{n,\ell}=F_{n-1}$ and $\min\{q,n-\ell-1\}=0$. In this instance, the hypothesis of the theorem merely states that the eigenvalues of the matrices $v(A_{i,\ell})$ are different from 1, and it follows easily that $H_0(F_{n-1};V)=0$.

In the general case, write $C_\bullet(n,\ell)=C_\bullet(P_{n,\ell})\otimes_{P_{n,\ell}}V$, and denote the boundary maps of this complex by $\partial_i(n,\ell)$. The restriction of the representation v to the subgroup $P_{n,\ell+1}$ of $P_{n,\ell}$ gives rise to a $(\mathbb{Z}P_{n,\ell+1})$-module which we continue to denote by V. By induction, we have $H_i(P_{n,\ell+1};V)=0$ for $0\le i\le\min\{q,n-\ell-2\}$.

As in the proof of Theorem 2.10, we exploit the fact that the complex $C_\bullet(P_{n,\ell})$ may be realized as the mapping cone of $\Xi_\bullet:[C_\bullet(n,\ell+1)]^\ell\to C_\bullet(n,\ell+1)$, where $C_\bullet(n,\ell+1)=C_\bullet(P_{n,\ell+1})\otimes_{P_{n,\ell+1}}V$. Thus we have a short exact sequence of chain complexes

$$0\to C_\bullet(n,\ell+1)\to C_\bullet(n,\ell)\to[C_{\bullet-1}(n,\ell+1)]^\ell\to 0.$$

It follows immediately from the corresponding long exact sequence in homology:

$$\dots\to[H_i(P_{n,\ell+1};V)]^\ell\to H_i(P_{n,\ell+1};V)\to H_i(P_{n,\ell};V)\to[H_{i-1}(P_{n,\ell+1};V)]^\ell\to\dots$$

that $H_i(P_{n,\ell}; V) = 0$ for $0 \le i \le \min\{q, n-\ell-2\}$. This completes the proof if $q \le n-\ell-2$.

If $q \ge n-\ell-1$, since we have $H_i(P_{n,\ell}; V) = 0$ for $i \le n-\ell-2$ by the previous paragraph, it remains to show that $H_{n-\ell-1}(P_{n,\ell}; V) = 0$. For this, it suffices to show that $\dim \operatorname{im} \partial_{n-\ell}(n,\ell) = \dim \ker \partial_{n-\ell-1}(n,\ell)$. The boundary map $\partial_{n-\ell}(n,\ell)$ is of the form

$$\partial_{n-\ell}(n,\ell) = (\Delta_{\ell,\ell+1,\dots,n-1} \quad [\partial_{n-\ell-1}(n,\ell+1)]^{\ell}).$$

The map $\Delta_{\ell,\ell+1,\dots,n-1}$ is given by

$$\Delta_{\ell,\ell+1,\dots,n-1} = (-1)^{n-\ell}(\tilde{v}\circ\phi(A_{1,\ell+1}) - I \quad \cdots \quad \tilde{v}\circ\phi(A_{\ell,\ell+1}) - I)^{\mathrm{T}},$$

where $\phi = \tilde{\rho}_{n-1} \circ \cdots \circ \tilde{\rho}_{\ell+1}$ is the composition of the representations $\tilde{\rho}_j$, $\ell+1 \le j \le n-1$, and I is the identity matrix of size $m\cdot(\ell+1)\cdots(n-1)$. Thus $\partial_{n-\ell}(n,\ell)$ has a submatrix of the form

$$\begin{pmatrix} (-1)^{n-\ell}(\tilde{v}\circ\phi(A_{1,\ell+1}) - I) & 0 \\ * & [\partial_{n-\ell-1}(n,\ell+1)]^{\ell-1} \end{pmatrix},$$

and

$$\dim \operatorname{im}(\partial_{n-\ell}(n,\ell)) \ge (\ell-1)\cdot \dim \operatorname{im}(\partial_{n-\ell-1}(n,\ell+1)) + \operatorname{rank}(\tilde{v}\circ\phi(A_{1,\ell+1}) - I).$$

Now the conditions on the representation v assure that $\tilde{v}\circ\phi(A_{1,\ell+1}) - I$ is an invertible matrix, hence has rank $m\cdot(\ell+1)\cdots(n-1)$. Since $H_i(P_{n,\ell}; V) = 0$ for $i \le n-\ell-2$, we compute $\dim \ker \partial_{n-\ell-1}(n,\ell)$ using an Euler characteristic argument. It then follows easily that

$$\begin{aligned}&(\ell-1)\cdot \dim \operatorname{im}(\partial_{n-\ell-1}(n,\ell+1)) + \operatorname{rank}(\tilde{v}\circ\phi(A_{1,\ell+1}) - I)\\ &\quad = \dim \ker(\partial_{n-\ell-1}(n,\ell)). \qquad \square\end{aligned}$$

Corollary 6.11. *If $q \ge n-\ell-1$ and $v : P_{n,\ell} \to \mathrm{GL}(m,\mathbb{C})$ is quasi-generic through rank q, then the homology group $H_{n-\ell}(P_{n,\ell}; V)$ has rank $m\cdot(n-2)!/(\ell-2)!$ if $\ell \ge 2$, and is trivial if $\ell = 1$.*

Remark 6.12. Note that if $v : P_{n,1} \to \mathrm{GL}(m,\mathbb{C})$ is a quasi-generic representation through rank q of the (entire) pure braid group, and $q \ge n-2$, then all homology groups $H_i(P_{n,1}; V)$ vanish. That is, the chain complex $C_\bullet(n,1) = C_\bullet(P_n) \otimes_{P_n} V$ is acyclic.

Remark 6.13. The group $P_{n,\ell}$ arises as the fundamental group of the complement of a discriminantal arrangement of Schechtman and Varchenko. Denote this arrangement by $\mathscr{A}_{n,\ell}$, let $M_{n,\ell}$ be its complement, and note that $M_{n,\ell}$ is a $K(P_{n,\ell}, 1)$ space. In [47], for certain rank one local coefficient systems V, cycles in $H_{n-\ell}(M_{n,\ell}; V)$, together with differential forms in the (de Rham) cohomology group $H^{n-\ell}(M_{n,\ell}; V^*)$ with coefficients in the dual local system, are used to generate solutions of the Knizhnik–Zamolodchikov equations in terms of generalized hypergeometric functions (see also [16, 31]). In the instances where the local coefficient systems constructed in [47] arise from representations which are quasi-generic through rank $n-\ell-1$, it follows from Corollary 6.11 that there are $(n-2)!/(\ell-2)!$ linearly independent solutions of the corresponding KZ equations.

Remark 6.14. Let $\lambda_{r,s}$ be real numbers, and consider the unitary, rank one representation v of $P_{n,\ell}$ defined by $v(A_{r,s}) = \exp(2\pi i \lambda_{r,s})$. It is easy to see that v is quasi-generic through rank q if, for each $J = \{j_1,\dots,j_p\}$ with $2 \le p \le q+2$ and $A_J \in P_{n,\ell}$, we have $\lambda_J := \sum_{1\le r<s\le p} \lambda_{j_r,j_s} \notin \mathbb{Z}$.

This condition may be expressed in terms of (the lattice of) the discriminantal arrangement $\mathscr{A}_{n,\ell}$. Note that sets $J = \{r,s\}$ with $A_J \in P_{n,\ell}$ correspond to hyperplanes $H = H_{r,s} \in \mathscr{A}_{n,\ell}$. Write $\lambda_H = \lambda_{r,s}$. In general, sets $J = \{j_1,\dots,j_p\}$ as above correspond to codimension $p-1$ "dense edges" $X = X_J = \bigcap_{r,s\in J} H_{r,s}$ of $\mathscr{A}_{n,\ell}$ (using the terminology of [46]). Thus, as noted by the referee, the representation v is quasi-generic through rank q if $\lambda_X := \sum_{H\supseteq X} \lambda_H \notin \mathbb{Z}$ for all dense edges X of $\mathscr{A}_{n,\ell}$ of codimension less than $q+2$.

Notice that replacing $\lambda_{r,s}$ with $\lambda_{r,s}+m_{r,s}$ does not alter the representation v if $m_{r,s} \in \mathbb{Z}$. It follows that we can assume that $\lambda_J > 0$ for all J (resp. $\lambda_X > 0$ for all X), and restate the above quasi-genericity condition accordingly (cf. [16, 20, 46, 51]).

7. Milnor fibrations

In this section, we discuss how the chain complex constructed in Section 2 and the vanishing theorem of the previous section may be used in the study of Milnor fibrations.

7.1. Let $\mathscr{A}$ be a central fiber-type arrangement in $\mathbb{C}^\ell$ with complement $M = M(\mathscr{A})$ and exponents $\{1 = d_1, d_2, \dots, d_\ell\}$ (see [18, 43]). Then the fundamental group G of M may be realized as an iterated semidirect product of free groups, $G \cong F_{d_\ell} \rtimes \cdots \rtimes F_{d_1}$, and M is an Eilenberg–MacLane space of type $K(G,1)$. Let $Q = Q(\mathscr{A})$ be a defining polynomial of $\mathscr{A}$. Then, since $\mathscr{A}$ is central, Q is homogeneous of degree $n = \sum d_q = |\mathscr{A}|$ and we have a (global) Milnor fibration $Q : M \to \mathbb{C}^*$, with fiber $F = Q^{-1}(1)$, and monodromy $h: F \to F$ given by multiplication by $\xi = \exp(2\pi i/n)$ [38]. The Milnor fiber $F = F(\mathscr{A})$ has the homotopy type of an $(\ell-1)$-dimensional $K(\pi,1)$ space, where $\pi = \pi_1(F)$.

Since F, M, and $\mathbb{C}^*$ are Eilenberg–MacLane spaces, the homotopy exact sequence of the Milnor fibration reduces to $1 \to \pi \to G \rightleftarrows \mathbb{Z} \to 1$. Therefore, by Shapiro's Lemma (see [9]), we have

$$H_*(F;\mathbb{Z}) = H_*(\pi;\mathbb{Z}) = H_*(G;\mathbb{Z}G \otimes_{\mathbb{Z}\pi} \mathbb{Z}) = H_*(G;\mathbb{Z}\mathbb{Z}).$$

Thus the construction of Section 2 provides an algorithm for computing the integral homology of the Milnor fiber of an arbitrary fiber-type arrangement. This algorithm also applies to certain non-linearly fibered arrangements, such as the Coxeter arrangements of type D_ℓ mentioned in Example 1.7.

7.2. We have carried out this computation for the braid arrangements $\mathscr{A}_\ell$, for $\ell \leq 5$. The results are tabulated below. For $\ell \leq 4$, these explicit results have been obtained by other means (see e.g. [11] and the references therein). To the best of our knowledge, the results for $\mathscr{A}_5$ were previously unknown. Since there is no torsion in the homology of these Milnor fibers, we list only the Betti numbers. We conjecture that the homology of the Milnor fiber of the braid arrangement $\mathscr{A}_\ell$ is torsion free for any ℓ.

	$b_0(F)$	$b_1(F)$	$b_2(F)$	$b_3(F)$
$\mathscr{A}_3$	1	4		
$\mathscr{A}_4$	1	7	18	
$\mathscr{A}_5$	1	9	28	80

For arbitrary ℓ, we can compute the Euler characteristic of $F(\mathscr{A}_\ell)$. Indeed, the Milnor fiber of $\mathscr{A}_\ell$ is a cyclic $\binom{\ell}{2}$-sheeted cover of the complement of the projectivized braid arrangement, $M(\mathscr{A}_\ell^*)$, (see e.g. [43]). Since $M(\mathscr{A}_\ell^*)$ is a $K(P_{\ell,2},1)$ space, it follows from the LCS formula that $\chi(M(\mathscr{A}_\ell^*)) = (-1)^\ell(\ell-2)!$. Thus $\chi(F(\mathscr{A}_\ell)) = (-1)^\ell \ell!/2$.

7.3. The construction of Section 2 may also be used to compute the homology eigenspaces of the algebraic monodromy of the Milnor fibration. In [11], it is shown that, for an arbitrary central arrangement $\mathscr{A}$, the ξ^k-eigenspace of the monodromy is isomorphic to $H_*(M(\mathscr{A}^*); V_k)$, the homology of the complement of the projectivization of $\mathscr{A}$ with coefficients in a complex rank one local system V_k. This local system is induced by the representation $v_k : \pi_1(M(\mathscr{A}^*)) \to \mathbb{C}^*$ which sends each meridian of $\mathscr{A}^*$ to ξ^k.

We have computed the homology eigenspaces of the monodromy of the Milnor fibration of the braid arrangement $\mathscr{A}_\ell$, for $\ell \leq 5$. The characteristic polynomials, $p_i(t)$, of the maps induced in homology by the monodromy, $h_* : H_i(F) \to H_i(F)$, are given below.

	$p_0(t)$	$p_1(t)$	$p_2(t)$	$p_3(t)$
$\mathscr{A}_3$	$1-t$	$(1-t)(1-t^3)$		
$\mathscr{A}_4$	$1-t$	$(1-t)^4(1-t^3)$	$(1-t)^3(1-t^3)(1-t^6)^2$	
$\mathscr{A}_5$	$1-t$	$(1-t)^9$	$(1-t)^{26}(1-t^2)$	$(1-t)^{18}(1-t^2)(1-t^{10})^6$

For arbitrary ℓ, the zeta function of the monodromy is given by

$$\zeta(t) = p_0(t)^{-1} \cdot p_1(t) \cdot p_2(t)^{-1} \cdot p_3(t) \cdots p_{\ell-2}(t)^{\pm 1} = (1-t^n)^{(-1)^{\ell+1}(\ell-2)!},$$

where $n = \binom{\ell}{2}$, see [38].

Remark 7.4. Since the complement of the projectivized braid arrangement $M(\mathscr{A}_\ell^*)$ is a $K(P_{\ell,2},1)$ space, Theorem 6.10 may be used to obtain partial results on the homology eigenspaces of the monodromy of the Milnor fibration of the braid arrangement for general ℓ. For instance, if $n = \binom{\ell}{2}$ is not divisible by 2 or 3, then for $1 \le k \le n$, the rank one representation v_k of $\pi_1(M(\mathscr{A}_\ell^*)) = P_{\ell,2}$ is quasi-generic through rank 2. It follows from Theorem 6.10 (see [11] for details) that $H_i(F(\mathscr{A}_\ell)) = H_i(M(\mathscr{A}_\ell^*))$ for $i \le 2$.

For arbitrary ℓ, if ξ^k is a primitive nth root of unity, then the representation v_k is quasi-generic through rank $\ell - 3$. Thus the ξ^k-eigenspace of the monodromy is "concentrated" in dimension $\ell - 2$, that is $H_i(M(\mathscr{A}_\ell^*); V_k) = 0$ if $i \ne \ell - 2$.

7.5. The approach outlined above also gives information on the Milnor fibration of the discriminant singularity $\mathscr{D}_\ell$ in $\mathbb{C}^\ell$. (See [44] for another possible way to attack this problem.) As noticed by Arnol'd, the complement, $M(\mathscr{D}_\ell)$, is the configuration space of the set of ℓ (unordered) points in $\mathbb{C}$, and thus is a $K(B_\ell,1)$-space, see e.g. [23]. The Milnor fibration $M(\mathscr{D}_\ell) \to \mathbb{C}^*$ induces on π_1 the abelianization map $\mathrm{ab} : B_\ell \to \mathbb{Z}$, see [22]. Hence, the Milnor fiber, $F(\mathscr{D}_\ell)$, is a $K(B'_\ell,1)$-space.

For arbitrary ℓ we can compute the Euler characteristic of $F(\mathscr{D}_\ell)$. Indeed, the usual symmetric group covering $M(\mathscr{A}_\ell) \to M(\mathscr{D}_\ell)$ restricts on Milnor fibers to an alternating group covering $F(\mathscr{A}_\ell) \to F(\mathscr{D}_\ell)$. Thus

$$\chi(F(\mathscr{D}_\ell)) = \frac{(-1)^\ell \ell!/2}{\ell!/2} = (-1)^\ell.$$

However, computing the homology of $F(\mathscr{D}_\ell)$ is, in general, a substantially harder task.

The case $\ell = 3$ is well known: $\mathscr{D}_3$ is the product of the cusp singularity with $\mathbb{C}$. Hence, $\pi_1(F(\mathscr{D}_3)) = F_2$ and the Milnor number, $b_1(F(\mathscr{D}_3))$, is 2.

The case $\ell = 4$ is not as well known. In [36], Massey uses Lê numbers to find upper bounds for the Betti numbers of the Milnor fiber of $\mathscr{D}_4$, the product of the swallowtail singularity with $\mathbb{C}$. He finds $b_1(F(\mathscr{D}_4)) = b_2(F(\mathscr{D}_4)) \le 5$. Our approach gives the exact answer. Let $G := B'_4 = \pi_1(F(\mathscr{D}_4))$. It is shown in [22] that G is a semidirect product $G_2 \rtimes G_1$. The action of $G_1 = F_2 = \langle x_{1,1}, x_{2,1}\rangle$ on $G_2 = F_2 = \langle x_{1,2}, x_{2,2}\rangle$ is given by

$$x_{1,1} : \begin{cases} x_{1,2} \mapsto x_{1,2}x_{2,2}^{-1}x_{1,2}^2, \\ x_{2,2} \mapsto x_{1,2}, \end{cases} \qquad x_{2,1} : \begin{cases} x_{1,2} \mapsto x_{1,2}x_{2,2}^{-1}x_{1,2}^3, \\ x_{2,2} \mapsto x_{1,2}x_{2,2}^{-1}x_{1,2}^4. \end{cases}$$

A computation reveals that $H_1(G) = \mathbb{Z}^2$, $H_2(G) = \mathbb{Z}^2$. Thus $b_1(F(\mathscr{D}_4)) = b_2(F(\mathscr{D}_4)) = 2$.

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References

[1] A. Aleksandrov, Milnor numbers of nonisolated Saito singularities, Functional Anal. Appl. 21 (1987) 1–9.

[2] K. Aomoto, Gauss–Manin connection of integral of difference products, J. Math. Soc. Japan 39 (1987) 191–208.

[3] V.I. Arnol'd, The cohomology ring of the colored braid group, Math. Notes 5 (1969) 138–140.

[4] E. Artin, Theory of braids, Ann. Math. 48 (1947) 101–126.

[5] J. Birman, Braids, Links and Mapping Class Groups, Ann. of Math. Stud., Vol. 82 (Princeton Univ. Press, Princeton, NJ, 1975).

[6] J. Birman, D.D. Long and J. Moody, Finite-dimensional representations of Artin's braid group, in: The Mathematical Legacy of Wilhelm Magnus: Groups, Geometry and Special Functions, Contemporary Math., Vol. 169 (Amer. Math. Soc., Providence, RI, 1994) 123–132.

[7] T. Brady, Free resolutions for semi-direct products, Tôhoku Math. J. 45 (1993) 535–537.

[8] E. Brieskorn, Sur les groupes de tresses, in: Séminaire Bourbaki, 1971/72, Lecture Notes in Mathematics 317 (Springer, Berlin, 1973) 21–44.

[9] K. Brown, Cohomology of Groups, Graduate Texts in Mathematics, Vol. 87 (Springer, Berlin, 1982).

[10] D. Cohen, Cohomology and intersection cohomology of complex hyperplane arrangements, Adv. Math. 97 (1993) 231–266.

[11] D. Cohen and A. Suciu, On Milnor fibrations of arrangements, J. London Math. Soc. 51 (1995) 105–119.

[12] D. Cohen and A. Suciu, The braid monodromy of plane algebraic curves and hyperplane arrangements, Preprint, 1996.

[13] D. Cohen and A. Suciu, Homology representations of braid groups, in preparation.

[14] F. Cohen, The homology of $\mathscr{C}_{n+1}$-spaces, $n \geq 0$, in: The Homology of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 533 (Springer, Berlin, 1976) 207–352.

[15] A. Dimca, On the Milnor fibration of weighted homogeneous polynomials, Compositio Math. 76 (1990) 19–47.

[16] H. Esnault, V. Schechtman and E. Viehweg, Cohomology of local systems on the complement of hyperplanes, Invent. Math. 109 (1992) 557–561; Erratum, ibid. 112 (1993) 447.

[17] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand. 10 (1962) 111–118.

[18] M. Falk and R. Randell, The lower central series of a fiber-type arrangement, Invent. Math. 82 (1985) 77–88.

[19] M. Falk and R. Randell, Pure braid groups and products of free groups, in: Braids, Contemporary Math., Vol. 78 (Amer. Math. Soc., Providence, RI, 1988) 217–228.

[20] M. Falk and H. Terao, β**nbc**-bases for cohomology of local systems on hyperplane complements, Trans. Amer. Math. Soc., to appear.

[21] R. Fox, Free differential calculus I, Ann. Math. 57 (1953) 547–560; II, Ann. Math. 59 (1954) 196–210; III, Ann. Math. 64 (1956) 407–419.

[22] E. Gorin and V. Lin, Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids, Math. USSR Sbornik 7 (1969) 569–596.

[23] V.L. Hansen, Braids and Coverings, London Math. Soc. Student Texts, Vol. 18 (Cambridge Univ. Press, Cambridge, 1989).

[24] A. Hattori, Topology of $\mathbb{C}^n$ minus a finite number of affine hyperplanes in general position, J. Fac. Sci. Univ. Tokyo 22 (1975) 205–219.

[25] M. Jambu, Fiber-type arrangements and factorization properties, Adv. Math. 80 (1990) 1–21.

[26] V. Jones, Hecke algebra representations of braid groups and link polynomials, Ann. Math. 126 (1987) 335–388.

[27] T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pures, Invent. Math. 82 (1985) 57–75.

[28] T. Kohno, Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. Ser. A 62 (1986) 144–147.

[29] T. Kohno, One-parameter family of linear representations of Artin's braid groups, in: Galois Representations and Arithmetic Algebraic Geometry, Advanced Studies in Pure Math., Vol. 12 (North-Holland, Amsterdam, 1987) 189–200.

[30] T. Kohno and T. Oda, The lower central series of the pure braid group of an algebraic curve, in: Galois Representations and Arithmetic Algebraic Geometry, Advanced Studies in Pure Math., Vol. 12 (North-Holland, Amsterdam, 1987) 201–219.
[31] R. Lawrence, Homological representations of the Hecke algebra, Commun. Math. Phys. 135 (1990) 141–191.
[32] A. Leibman and D. Markushevich, The monodromy of the Brieskorn bundle, in: Geometric Topology, Contemporary Math., Vol. 164 (Amer. Math. Soc., Providence, RI, 1994) 91–117.
[33] A. Libgober, Invariants of plane algebraic curves via representations of the braid groups, Invent. Math. 95 (1989) 25–30.
[34] D.D. Long and M. Paton, The Burau representation is not faithful for $n \geq 6$, Topology 32 (1993) 439–447.
[35] M. Lüdde and F. Toppan, Matrix solutions of Artin's braid relations, Phys. Lett. 288 (1992) 321–330.
[36] W. Magnus, On a theorem of Marshall Hall, Ann. Math. 40 (1939) 764–768.
[37] D. Massey, Lê Cycles and Hypersurface Singularities, Lecture Notes in Mathematics, Vol. 1615 (Springer, Berlin, 1995).
[38] J. Milnor, Singular Points of Complex Hypersurfaces, Ann. of Math. Stud., Vol. 61 (Princeton Univ. Press, Princeton, NJ, 1968).
[39] B. Moishezon, Stable branch curves and braid monodromies, in: Algebraic Geometry, Lecture Notes in Mathematics, Vol. 862 (Springer, Berlin, 1981) 107–192.
[40] J. Moody, The Burau representation is unfaithful for large n, Bull. Amer. Math. Soc. 25 (1991) 379–384.
[41] P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (1980) 167–189.
[42] P. Orlik and L. Solomon, Unitary reflection groups and cohomology, Invent. Math. 59 (1980) 77–94.
[43] P. Orlik and H. Terao, Arrangements of Hyperplanes, Grundlehren, Vol. 300 (Springer, Berlin, 1992).
[44] M. Salvetti, The homotopy type of Artin groups, Math. Res. Lett. 1 (1994) 565–577.
[45] M. Salvetti and M. Prati, On local systems over complements to arrangements of hyperplanes associated to Grassman strata, Ann. Mat. Pura Appl. (IV) CLIX (1991) 341–355.
[46] V. Schechtman, H. Terao and A. Varchenko, Cohomology of local systems and the Kac-Kazhdan condition for singular vectors, J. Pure Appl. Algebra 100 (1995) 93–102.
[47] V. Schechtman and A. Varchenko, Arrangements of hyperplanes and Lie algebra homology, Invent. Math. 106 (1991) 139–194.
[48] H. Terao, Modular elements of lattices and topological fibrations, Adv. Math. 62 (1986) 135–154.
[49] A. Varchenko, Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups, Advanced Series in Mathematical Phyics, Vol. 21, (World Scientific, Singapore, 1995).
[50] V. Vassiliev, I. Gelfand and A. Zelevinsky, General hypergeometric functions on complex Grassmannians, Functional Anal. Appl. 21 (1987) 19–31.
[51] S. Yuzvinsky, Cohomology of the Brieskorn-Orlik-Solomon algebras, Comm. Algebra 23 (1995) 5339–5354.