Topological complexity of almost-direct products of free groups

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Let X be a path-connected topological space, and let PX be the space of all continuous paths $\gamma\colon [0,1] \to X$, equipped with the compact-open topology. The map $\pi\colon PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$, defined by sending a path to its endpoints is a fibration.

Definition. The *topological complexity* of X, $\mathsf{TC}(X)$, is the minimal k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_i is open and there exists a continuous section $s_i\colon U_i \to PX$, $\pi \circ s_i = \mathrm{id}_{U_i}$, for each i, $1 \leq i \leq k$. In other words, the topological complexity of X is the sectional category (or Schwarz genus) of the path space fibration, $\mathsf{TC}(X) = \mathrm{secat}(\pi\colon PX \to X \times X)$.

This notion, introduced by Farber, provides a topological approach to the motion planning problem from robotics, see the survey [6] and the references therein.

Assume that X is a finite-dimensional cell complex. We shall make use of the following properties of topological complexity, which may be found in [6].

(1) $\mathsf{TC}(X)$ is an invariant of the homotopy type of X;
(2) $\mathsf{TC}(X) = 1 \iff X$ is contractible;
(3) $\mathsf{TC}(X) \leq 2\dim(X) + 1$;
(4) $\mathsf{TC}(X) > \mathsf{zcl}(H^*(X)) := \text{cup length}\big(\ker\big(H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)\big)\big)$.

For the *zero-divisor cup length* $\mathsf{zcl}(H^*(X))$, use cohomology with field coefficients.

Problem (Farber [6])**.** For a discrete group G, define the topological complexity of G to be the topological complexity of an Eilenberg-Mac Lane space of type $K(G,1)$, $\mathsf{TC}(G) := \mathsf{TC}(K(G,1))$. Determine the topological complexity of G in terms of other invariants of G.

Definition. An *almost-direct product of free groups* is an iterated semi-direct product $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ of finitely generated free groups F_{n_i}, $n_i < \infty$, such that the action of F_{n_i} on the homology $H^*(F_{n_j};\mathbb{Z})$ is trivial for $1 \leq i < j \leq \ell$.

We pursue the topological complexity of groups of this type. This is motivated by the following results.

Theorem (Farber-Yuzvinsky [8])**.** *Let P_ℓ be the Artin pure braid group, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C}$. Then* $\mathsf{TC}(P_\ell) = 2\ell - 2$.

Theorem (Farber-Grant-Yuzvinsky [7])**.** *Let $P_{\ell,k} = \ker(P_\ell \to P_k)$ denote the kernel of the map which forgets the last $\ell - k$ strands of an ℓ-strand pure braid, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C} \setminus \{k$ points$\}$. If $k \geq 2$, then* $\mathsf{TC}(P_{\ell,k}) = 2(\ell - k) + 1$.

The groups $P_\ell = F_{\ell-1} \rtimes \cdots \rtimes F_1$ and $P_{\ell,k} = F_{\ell-1} \rtimes \cdots \rtimes F_k$ are almost direct products of free groups. For $i < j$, the action of F_i on F_j is given by (the restriction of) the Artin representation.

Let $P\Sigma_\ell$ be the group of basis-conjugating automorphisms of the free group $F_\ell = \langle x_1, \ldots, x_\ell \rangle$. McCool [10], found the following presentation for $P\Sigma_\ell$:

$$P\Sigma_\ell = \langle \beta_{i,j}, 1 \leq i \neq j \leq \ell \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta j, k)], \rangle,$$

where the indices in the relations are distinct, and the generators $\beta_{i,j}$ are the automorphisms defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \neq j, \\ x_i^{-1} x_j x_i & \text{if } k = j. \end{cases}$$

The subgroup $P\Sigma_\ell^+$ generated by $\beta_{i,j}$ for $i < j$ is an almost-direct product of free groups, see [4]. One has $P\Sigma_\ell^+ = F_{\ell-1} \rtimes P\Sigma_{\ell-1}^+ = F_{\ell-1} \rtimes \cdots \rtimes F_1$, where $F_\ell = \langle \beta_{1,\ell}, \ldots, \beta_{\ell-1,\ell} \rangle$, and the action of $P\Sigma_{\ell-1}^+$ on F_ℓ may be extracted from the above presentation. The *upper triangular McCool group* $P\Sigma_\ell^+$ is *not* isomorphic to the pure braid group P_ℓ.

Theorem (Cohen-Pruidze [2])**.** *Let $P\Sigma_\ell^+$ be the upper triangular McCool group. Then $\mathsf{TC}(P\Sigma_\ell^+) = 2\ell - 2$.*

Remark. The pure braid group P_ℓ and triangular McCool group $P\Sigma_\ell^+$ each have infinite cyclic center. Denoting the center of a group G by $Z(G)$, and writing $\overline{G} = G/Z(G)$, we have $P_\ell = \overline{P}_\ell \times \mathbb{Z}$ and $P\Sigma_\ell^+ = \overline{P\Sigma}_\ell^+ \times \mathbb{Z}$, where $\overline{P}_\ell$ and $\overline{P\Sigma}_\ell^+$ are almost-direct products of free groups, each of which has rank at least two.

The above results are unified by the following:

Theorem. *Let $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ be an almost-direct product of free groups. If $n_j \geq 2$ for each j and m is a non-negative integer, then $\mathsf{TC}(G \times \mathbb{Z}^m) = 2\ell + m + 1$.*

Problem. If $\mathbb{Z}^m$ acts nontrivially on G, what is $\mathsf{TC}(G \rtimes \mathbb{Z}^m)$?

For the sake of brevity, we focus on the case $m = 0$. Let $X = K(G, 1)$ be an Eilenberg-Mac Lane space of type $K(G, 1)$. The above result may be established using the bounds $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \leq 2\dim(X) + 1$ noted previously. First, it is not difficult to show that the cohomological dimension of G is equal to the geometric dimension of G, which in turn, is equal to ℓ,

$$\mathrm{cd}(G) = \mathrm{geom\ dim}(G) = \ell \implies \mathsf{TC}(G) \leq 2\ell + 1.$$

The lower bound $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G)$ may be established through analysis of the (integral) cohomology ring $H^*(G)$.

Theorem. *The cohomology ring $H^*(G)$ is a quadratic algebra. That is, $H^*(G) \cong E/J$, where E is an exterior algebra generated in degree 1, and J is an ideal generated in degree 2.*

The integral homology $H_*(G)$ is torsion-free and the Poincaré polynomial is given by $P(G, t) = \sum_{k=0}^{\ell} b_k(G) \cdot t^k = \prod_{j=1}^{\ell}(1 + n_j t)$, where $b_k(G)$ is the k-th Betti number of G, see [5]. A minimal, free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, which we denote by $C_\bullet(G) \xrightarrow{\epsilon} \mathbb{Z}$, is constructed in [3].

Let $N = b_1(G)$. The abelianization map $G \to \mathbb{Z}^N$ induces a chain map $\Phi_\bullet \colon C_\bullet \to K_\bullet$, where $C_\bullet = C_\bullet(G) \otimes_{\mathbb{Z}G} \mathbb{Z}\mathbb{Z}^N$ and $K_\bullet \to \mathbb{Z}$ is the standard $\mathbb{Z}\mathbb{Z}^N$-resolution of $\mathbb{Z}$. One can show that the induced map in cohomology $\Phi_2^* \colon H^2(\mathbb{Z}^N) \to H^2(G)$ is surjective, and that $H^*(G) \cong E/J$, where $E = H^*(\mathbb{Z}^N)$ and $J = \ker(\Phi_2^*)$.

The identification $H^*(G) \cong E/J$ may be used to show that $\mathsf{zcl}(H^*(G)) = 2\ell$. For each i, $1 \le i \le \ell$, let x_i and y_i be classes in $H^1(G)$ corresponding to distinct generators of the free group F_{n_i}. Then one can show that the product

$$\prod_{i=1}^{\ell}(x_i \otimes 1 - 1 \otimes x_i)(y_i \otimes 1 - 1 \otimes y_i)$$

is non-zero. So we have $2\ell = \mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \le 2\dim(K(G,1)) + 1 = 2\ell + 1$.

As another consequence of the calculation of the cohomology ring of an almost-direct product of free groups G, one can show that $H^*(G;\mathbb{Q})$ is, in fact, Koszul. If, additionally, the group G is 1-formal, then the space $K(G,1)$ is formal. This is the case for the groups P_ℓ, $P_{\ell,k}$, and $P\Sigma_\ell^+$, see [9] and [1]. When G is 1-formal, the calculation of $\mathsf{TC}(G)$ above may be viewed as evidence in support of the conjecture that, for a formal space X, one has $\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X;\Bbbk))$ for some field $\Bbbk$.

The basis-conjugating automorphism group $P\Sigma_\ell$ is also known to be 1-formal, see [1]. Moreover, we have $\mathsf{TC}(P\Sigma_\ell) = 2\ell - 1 = 1 + \mathsf{zcl}(H^*(P\Sigma_\ell;\mathbb{Q}))$, see [2].

Problem.

(1) Presumably, $P\Sigma_\ell$ is not an almost-direct product of free groups. Prove (or disprove) this.
(2) Determine if the cohomology ring $H^*(P\Sigma_\ell;\mathbb{Q})$ is Koszul.

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