IMRN International Mathematics Research Notices
2003, No. x

Lie Algebras Associated to Fiber-Type Arrangements

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1 Introduction

Two classical constructions of interest in the group theory and topology are

(i) the Lie algebra arising from the filtration quotients associated to the descending central series of a discrete group G;

(ii) the Lie algebra of primitive elements in the singular homology of the loop space of a space X, for certain topological spaces X.

The purpose of this paper is to illustrate that these two a priori unrelated Lie algebras are, in fact, isomorphic in certain natural cases. This work is motivated by recent results relating the Lie algebras of (i) and (ii) arising in the context of classical configuration spaces, and resolves a conjecture of the second and third authors concerning the generalization of these results to spaces arising from certain hyperplane arrangements.

Let P_n denote the Artin pure braid group, the fundamental group of the configuration space $F(\mathbb{C}, n)$ of n ordered points in $\mathbb{C}$. The structure of the descending central series of P_n was determined by Kohno [19]. Shortly thereafter, Falk and Randell [11] determined the descending central series structure of the fundamental group of the complement of an arbitrary fiber-type hyperplane arrangement. This class of arrangements, equivalent to the class of supersolvable arrangements, arises in a number of combinatorial and topological contexts, see Orlik and Terao [23] and below.

More recently, the homology of $\Omega F(\mathbb{C}^{k+1}, n)$, the loop space of the configuration space of n ordered points in $\mathbb{C}^{k+1}$ ($k \geq 1$), was determined by Fadell and Husseini [10]. Subsequently, Cohen and Gitler [4] showed that the Lie algebra of primitive elements in

Received 20 August 2002. Revision received 10 January 2003.

this loop space homology is, apart from grading, isomorphic to the Lie algebra obtained from the descending central series of the pure braid group P_n.

These lines of research are continued here. The main result of this paper asserts that the Lie algebra associated to the fundamental group G of the complement of an arbitrary fiber-type hyperplane arrangement is, up to regrading, isomorphic to the Lie algebra of primitive elements in the homology of the loop space of the complement of a higher-dimensional analogue of the arrangement. The main theorem is, in fact, stronger. The Samelson product for the loop space yields a graded Lie algebra given by the homotopy groups modulo torsion. This Lie algebra is, again up to regrading, also isomorphic to the Lie algebra associated to the descending central series quotients of G. In addition, after looping further, there are natural related Poisson algebras arising from the homology of associated iterated loop spaces.

Given a discrete group G, let G_n be the nth stage of the descending central series, defined inductively by $G_1 = G$ and $G_{n+1} = [G_n, G]$ for $n \geq 1$, and let $E_0^n(G) = G_n/G_{n+1}$ be the nth associated quotient. Let $E_0^*(G) = \bigoplus_{n\geq 1} E_0^n(G)$ be the Lie algebra obtained from the descending central series of G, with Lie algebra structure induced by the commutator map $G \times G \to G$, $(x, y) \mapsto xyx^{-1}y^{-1}$. For each positive integer k, use the ungraded Lie algebra $E_0^*(G)$ to define a related graded Lie algebra as follows.

Definition 1.1. For a group G, let $E_0^*(G)_k$ be the graded Lie algebra given by

$$E_0^q(G)_k = \begin{cases} E_0^n(G), & \text{if } q = 2nk, \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

with Lie bracket structure induced by that of the Lie algebra $E_0^*(G)$ obtained from the descending central series of G in the obvious manner.

A theorem relating the Lie algebras of (i) and (ii) above is described next. Recall that P_n denotes the Artin pure braid group, the fundamental group of the configuration space $F(\mathbb{C}, n)$. The results on configuration spaces alluded to above, due to Fadell and Husseini [10], and Cohen and Gitler [4], may be summarized as follows.

Theorem 1.2. For $k \geq 1$, the homology of the loop space of the configuration space $F(\mathbb{C}^{k+1}, n)$ is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(P_n)_k$. Moreover,

(a) the image of the Hurewicz homomorphism

$$\pi_*(\Omega F(\mathbb{C}^{k+1}, n)) \longrightarrow H_*(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z}) \tag{1.2}$$

is isomorphic to $E_0^*(P_n)_k$;

(b) the Hurewicz homomorphism induces isomorphisms of graded Lie algebras

$$\pi_*\big(\Omega F(\mathbb{C}^{k+1}, n)\big)/\,\text{Torsion} \longrightarrow \text{Prim}\, H_*\big(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z}\big) \cong E_0^*(P_n)_k, \quad (1.3)$$

where Prim denotes the module of primitive elements, and the Lie algebra structure of the source is induced by the classical Samelson product.

□

The Lie algebra arising in this theorem is the "universal Yang-Baxter Lie algebra" $\mathcal{L}(n)$, the quotient of the free Lie algebra on a free abelian group of rank $\binom{n}{2}$ by the "infinitesimal pure braid relations" recorded in (4.1). See also Remark 4.2.

An important ingredient in the proof of Theorem 1.2 is a classical result of Fadell and Neuwirth [9], which shows that configuration spaces admit iterated bundle structure. Similar results are known to hold for certain orbit configuration spaces [3, 8, 29], which admit analogous bundle structure, and are described in more detail below. All of these spaces fit in the following general framework.

For each natural number ℓ, let X_ℓ be a functor from Euclidean spaces, with morphisms restricted to endomorphisms, to topological spaces. For a Euclidean space $\mathbb{E}$, let $\mathcal{Q}_\ell(\mathbb{E})$ be a discrete subset of $\mathbb{E}$ of fixed (possible infinite) cardinality depending on ℓ. Assume that there are natural transformations $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ which satisfy the following conditions:

(1) the space $X_1(\mathbb{E}) = \mathbb{E} \setminus \mathcal{Q}_1(\mathbb{E})$ is the complement of a discrete subset of $\mathbb{E}$;
(2) the map $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ is a fiber bundle projection, with fiber $\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})$;
(3) each bundle $X_\ell(\mathbb{E}) \to X_{\ell-1}(\mathbb{E})$ admits a cross section;
(4) if $\mathbb{E} \cong \mathbb{C}$, the fundamental group of $X_{\ell-1}(\mathbb{E})$ acts trivially on the homology of the fiber $\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})$.

The prototypical examples are given by the configuration spaces $X_\ell(\mathbb{E}) = F(\mathbb{E}, \ell)$, where $\mathbb{E} = \mathbb{C}^k$. Further examples are given below.

It seems likely that for many choices of X_ℓ, the Lie algebras associated to $X_\ell(\mathbb{E})$ as $\mathbb{E}$ varies are related in a manner analogous to those arising in Theorem 1.2. If $\mathbb{E} \cong \mathbb{C}$, conditions (1) and (2) imply that $X_\ell(\mathbb{E})$ is a $K(G, 1)$ space, where $G = \pi_1(X_\ell(\mathbb{E}))$ is the fundamental group of $X_\ell(\mathbb{E})$, as is readily seen from the homotopy sequence of a bundle. In this case, condition (3) further implies that the group G admits the structure of an iterated semidirect product of free groups, and condition (4) restricts the type of free group automorphisms arising in this structure. These conditions determine the additive structure of the Lie algebra $E_0^*(G)$, see [11] and Section 4.

For higher-dimensional $\mathbb{E}$, conditions (1), (2), and (3) imply that the homology of the loop space of $X_\ell(\mathbb{E})$ is isomorphic to the universal enveloping algebra of the Lie algebra $\pi_*(\Omega X_\ell(\mathbb{E}))/$ Torsion, and determine the additive structure of Prim $H_*(\Omega X_\ell(\mathbb{E}); \mathbb{Z})$, see [8] and Section 5. These conditions have analogous implications for the homology of an iterated loop space $\Omega^q X_\ell(\mathbb{E})$ with $q > 1$ and the Poisson algebra structure admitted by this homology, see [6] and Section 6.

A brief indication of how the Lie algebras arising for various choices of $\mathbb{E}$ may be analyzed and compared is as follows. First, there is a variant of the classical Freudenthal suspension relating reduced suspensions and loop spaces as indicated below, where the maps are induced by (homology) suspensions

$$\begin{array}{ccc} H_{2k-2}\big(\Omega X_\ell(\mathbb{C}^k)\big) & \longleftarrow & H_{2k-1}\big(X_\ell(\mathbb{C}^k)\big) & & \\ & & \downarrow & & \\ & & H_{2k}\big(\Sigma X_\ell(\mathbb{C}^k)\big) & \longrightarrow & H_{2k}\big(\Omega X_\ell(\mathbb{C}^{k+1})\big). \end{array} \tag{1.4}$$

If $k \geq 2$, conditions (1), (2), and (3) above imply that these maps are all (additive) isomorphisms. In the case where $k = 1$, these maps yield an additive isomorphism $E_0^1(G) = H_1(X_\ell(\mathbb{C})) \cong H_2(\Omega X_\ell(\mathbb{C}^2))$, where $G = \pi_1(X_\ell(\mathbb{C}))$. While this comparison does not in general preserve the structures of these Lie algebras, it does provide a geometric way to compare indecomposable elements.

To determine the Lie algebra structure, let S be a sphere of appropriate dimension and $A : S \to X_\ell(\mathbb{E})$ a map representing a (reduced) homology generator in a minimal degree. Consider the pullback $\xi(\mathbb{E})$ of the bundle $X_{\ell+1}(\mathbb{E}) \to X_\ell(\mathbb{E})$ along the map A,

$$\begin{array}{ccccc} \mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E}) & \longrightarrow & \xi(\mathbb{E}) & \longrightarrow & S \\ \| & & \downarrow & & \downarrow A \\ \mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E}) & \longrightarrow & X_{\ell+1}(\mathbb{E}) & \longrightarrow & X_\ell(\mathbb{E}). \end{array} \tag{1.5}$$

These bundles admit compatible cross sections by condition (3). There is consequently a morphism of extensions of Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}\big(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})\big) & \longrightarrow & \mathcal{L}\big(\xi(\mathbb{E})\big) & \longrightarrow & \mathcal{L}(S) & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow A_* & & \\ 0 & \longrightarrow & \mathcal{L}\big(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})\big) & \longrightarrow & \mathcal{L}\big(X_{\ell+1}(\mathbb{E})\big) & \longrightarrow & \mathcal{L}\big(X_\ell(\mathbb{E})\big) & \longrightarrow & 0, \end{array} \tag{1.6}$$

where $\mathcal{L}(\cdot)$ denotes the Lie algebra obtained from the descending central series of the fundamental group if $\mathbb{E} \cong \mathbb{C}$, and the graded Lie algebra of primitive elements in the homology of the loop space for higher-dimensional $\mathbb{E}$. Knowledge of the extension $0 \to \mathcal{L}(\mathbb{E} \setminus \mathcal{Q}_\ell(\mathbb{E})) \to \mathcal{L}(\xi(\mathbb{E})) \to \mathcal{L}(S) \to 0$ and the map $A_* : \mathcal{L}(S) \to \mathcal{L}(X_\ell(\mathbb{E}))$ for all homology generators completely determines the structure of the Lie algebra $\mathcal{L}(X_{\ell+1}(\mathbb{E}))$. In favorable situations, we can show that the extensions of Lie algebras which arise as $\mathbb{E}$ varies are, apart from grading, isomorphic by carefully combining these considerations with the aforementioned comparison of indecomposables.

A natural family of examples which fit in the above framework is given next. Let M be a manifold, and Γ a group which acts properly discontinuously on M. The orbit configuration space $F_\Gamma(M, \ell)$ consists of all ordered ℓ-tuples of points in M, no two of which lie in the same Γ-orbit. The case where $\Gamma = \mathbb{Z}/p\mathbb{Z}$ acts on $M = \mathbb{C}^k \setminus \{0\}$ by rotations was considered in [3, 29], the results of which combine to show that the analogue of Theorem 1.2 holds for these orbit configuration spaces. The spaces $F_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{C}^k \setminus \{0\}, \ell)$ and the classical configuration spaces $F(\mathbb{C}^k, \ell)$ may be realized as complements of finite hyperplane or subspace arrangements. This led to speculation in [8] that similar results may hold for fiber-type arrangements whose complements, like configuration spaces, admit iterated bundle structure.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, a finite collection of codimension one affine subspaces, with complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. See Orlik and Terao [23] as a general reference on the arrangements. Given a hyperplane $H \subset \mathbb{C}^\ell$, let H^k be the codimension k affine subspace of $\mathbb{C}^{k\ell} = (\mathbb{C}^\ell)^k$ consisting of all k-tuples of points in $\mathbb{C}^\ell$, each of which lies in H. For each positive integer k, the elements of the hyperplane arrangement $\mathcal{A}$ may be used in this way to obtain a *redundant* arrangement $\mathcal{A}^k$ of a complex codimension k subspaces in $\mathbb{C}^{k\ell}$, with complement $M(\mathcal{A}^k) = \mathbb{C}^{k\ell} \setminus \bigcup_{H \in \mathcal{A}} H^k$.

When $\mathcal{A}$ is a fiber-type hyperplane arrangement, the behavior of the family of spaces $\{X_\ell(\mathbb{C}^k) = M(\mathcal{A}^k),\ k \geq 1\}$ is reminiscent of that of the family $\{F(\mathbb{C}^k, n),\ k \geq 1\}$ of configuration spaces. Let $G = \pi_1(M(\mathcal{A}))$ be the fundamental group of the complement of the fiber-type arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, and let $E_0^*(G)$ be the Lie algebra obtained from the descending central series of G. The main result of this paper is as follows.

Theorem 1.3. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement and $G = \pi_1(M(\mathcal{A}))$ the fundamental group of the complement of $\mathcal{A}$. For $k \geq 1$, the homology of $\Omega M(\mathcal{A}^{k+1})$, the loop space of the complement of the redundant subspace arrangement $\mathcal{A}^{k+1}$ in $\mathbb{C}^{(k+1)\ell}$, is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(G)_k$. Moreover,

(a) the image of the Hurewicz homomorphism

$$\pi_*\big(\Omega M(\mathcal{A}^{k+1})\big) \longrightarrow H_*\big(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z}\big) \tag{1.7}$$

is isomorphic to $E_0^*(G)_k$;

(b) the Hurewicz homomorphism induces isomorphisms of graded Lie algebras

$$\pi_*\big(\Omega M\big(\mathcal{A}^{k+1}\big)\big)/\,\text{Torsion} \longrightarrow \text{Prim}\, H_*\big(\Omega M\big(\mathcal{A}^{k+1}\big);\mathbb{Z}\big) \cong E_0^*(G)_k, \tag{1.8}$$

where the Lie algebra structure of the source is induced by the Samelson product. □

This paper is organized as follows. Given a hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^\ell$, an associated arrangement $\mathcal{A}^k$ of the codimension k subspaces in $\mathbb{C}^{k\ell}$ is constructed in Section 2. The combinatorics and topology of the subspace arrangement $\mathcal{A}^k$ are studied in this section. In Section 3, the topology of the subspace arrangement $\mathcal{A}^k$, in the instance where the underlying hyperplane arrangement $\mathcal{A}$ is fiber-type, is studied further. In Section 4, the (known) structure of the Lie algebra $E_0^*(G)$ associated to the descending central series of the fundamental group $G = \pi_1(M(\mathcal{A}))$ of the complement of a fiber-type hyperplane arrangement $\mathcal{A}$ is analyzed. In Section 5, the structure of the Lie algebra of primitive elements in the homology of the loop space of the complement of the subspace arrangement $\mathcal{A}^k$ is analyzed, and the isomorphisms of graded Lie algebras asserted in Theorem 1.3 are established. In Section 6, the Poisson algebra structure on the homology of an iterated loop space of the complement of the subspace arrangement $\mathcal{A}^k$ is briefly analyzed. The above results admit generalizations in several directions. Some of these are discussed in Section 7.

2 Redundant arrangements

Let H be an affine hyperplane in $\mathbb{C}^\ell$, an affine subspace of codimension one. For each positive integer k, there is an affine subspace H^k of codimension k in $\mathbb{C}^{k\ell}$ obtained from H in the following manner. Choose coordinates $\mathbf{x} = (x_1, \dots, x_\ell)$ on $\mathbb{C}^\ell$ and $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ on $\mathbb{C}^{k\ell} = \mathbb{C}^\ell \times \cdots \times \mathbb{C}^\ell$, where for each i, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,\ell}) \in \mathbb{C}^\ell$. Then, if the hyperplane H in $\mathbb{C}^\ell$ is given by $H = \{\mathbf{x} \in \mathbb{C}^\ell \mid \sum_{j=1}^{\ell} a_j x_j = b\}$, define a codimension k affine subspace H^k in $\mathbb{C}^{k\ell}$ by $H^k = \{(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \mathbb{C}^{k\ell} \mid \sum_{j=1}^{\ell} a_j x_{i,j} = b,\ 1 \le i \le k\}$.

Now, let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, a finite collection of (affine) hyperplanes. Via the above process, there is an arrangement $\mathcal{A}^k = \{H^k \mid H \in \mathcal{A}\}$ of codimension k affine subspaces in $\mathbb{C}^{k\ell}$ obtained from $\mathcal{A}$. For evident reasons, call the subspace arrangement $\mathcal{A}^k$ *redundant*. A brief description of the relationship between the combinatorics and topology of the hyperplane arrangement $\mathcal{A} = \mathcal{A}^1$ and the redundant subspace arrangement $\mathcal{A}^k$ is given in this section.

Example 2.1. Let $\mathcal{A}_n$ be the braid arrangement in $\mathbb{C}^n$, consisting of the hyperplanes $H_{i,j} = \{x \in \mathbb{C}^n \mid x_i = x_j\}$. As is well known, the complement $M(\mathcal{A}_n) = F(\mathbb{C}, n)$ is the configuration space of n ordered points in $\mathbb{C}$.

For each positive integer k, the associated redundant arrangement $\mathcal{A}_n^k$ consists of subspaces $H_{i,j}^k = \{(x_1, \dots, x_k) \in (\mathbb{C}^n)^k \mid x_{p,i} = x_{p,j},\ 1 \le p \le k\}$. These subspaces may be realized as $H_{i,j}^k = \{(y_1, \dots, y_n) \in (\mathbb{C}^k)^n \mid y_i = y_j\}$. Thus, the complement $M(\mathcal{A}_n^k) = F(\mathbb{C}^k, n)$ is the configuration space of n ordered points in $\mathbb{C}^k$.

For an arbitrary hyperplane arrangement $\mathcal{A}$ and for each k, let $\mathbf{L}(\mathcal{A}^k)$ be the intersection poset of the arrangement $\mathcal{A}^k$, the partially ordered set of nonempty multi-intersections of elements of $\mathcal{A}^k$. Order $\mathbf{L}(\mathcal{A}^k)$ by reverse inclusion and include the ambient space $\mathbb{C}^{k\ell}$ in $\mathbf{L}(\mathcal{A}^k)$ as the minimal element corresponding to the intersection of no elements of $\mathcal{A}^k$. For the hyperplane arrangement $\mathcal{A} = \mathcal{A}^1$, it is known that $\mathbf{L}(\mathcal{A})$ is a geometric poset, see [23, Section 2.3]. This need not be the case for an arbitrary subspace arrangement. However, for redundant arrangements, the following proposition holds.

Proposition 2.2. If $\mathcal{A}$ is a hyperplane arrangement, then $\mathbf{L}(\mathcal{A}) \cong \mathbf{L}(\mathcal{A}^k)$ for all k. □

Proof. It suffices to show that the bijection between $\mathcal{A}$ and $\mathcal{A}^k$ given by $H \leftrightarrow H^k$ induces an isomorphism of posets $\mathbf{L}(\mathcal{A}) \cong \mathbf{L}(\mathcal{A}^k)$. For this, it is enough to show that to each codimension r flat $X \in \mathbf{L}(\mathcal{A})$, there corresponds a codimension kr flat $X^k \in \mathbf{L}(\mathcal{A}^k)$.

Write $\mathcal{A} = \{H_1, \dots, H_n\}$, where $H_i = \{x \in \mathbb{C}^\ell \mid \sum_{j=1}^{\ell} a_{i,j}x_j = b_i\}$, and let $X = H_1 \cap \cdots \cap H_m$. The flat X may be realized as the set of solutions of the system of equations $Ax = \mathbf{b}$, where $A = (a_{i,j})$ is $m \times \ell$ and $\mathbf{b} = (b_1, \dots, b_m)$. Then, X has codimension r in $\mathbb{C}^\ell$ if and only if $\operatorname{rank}[A \mid \mathbf{b}] = \operatorname{rank} A = \ell - r$ if and only if

$$\operatorname{rank}\begin{bmatrix} A & & & \mid & \mathbf{b} \\ & A & & \mid & \mathbf{b} \\ & & & \mid & \\ & & A & \mid & \mathbf{b} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & & \\ & A & \\ & & \\ & & A \end{bmatrix} = k(\ell - r) \tag{2.1}$$

if and only if $X^k = H_1^k \cap \cdots \cap H_m^k$ has codimension kr in $\mathbb{C}^{k\ell}$. ■

For each k, let $M(\mathcal{A}^k) = \mathbb{C}^{k\ell} \setminus \bigcup_{H^k \in \mathcal{A}^k} H^k$ denote the complement of the (subspace) arrangement $\mathcal{A}^k$. In the case where $k = 1$, the cohomology of the hyperplane complement $M(\mathcal{A}) = M(\mathcal{A}^1)$ is well known. It is isomorphic to the Orlik-Solomon algebra $\mathbf{A}(\mathcal{A})$, see [23, Sections 3.1 and 3.2]. A family of algebras which includes $\mathbf{A}(\mathcal{A})$ is defined next.

For each positive integer k, let $E_{2k-1}[k] = \bigoplus_{H \in \mathcal{A}} \mathbb{Z} e_H^k$ be a free $\mathbb{Z}$-module generated by degree $2k-1$ elements e_H^k in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let $E[k] = \bigwedge E_{2k-1}[k]$ be the exterior algebra of $E_{2k-1}[k]$, and denote by $\mathbf{I}[k]$ the ideal of $E[k]$ generated by the homogeneous elements

$$\begin{aligned} &\sum_{p=1}^{q} (-1)^{p-1} e_{H_1}^k \wedge \cdots \widehat{e_{H_p}^k} \cdots \wedge e_{H_q}^k, \quad \text{if } 0 \leq \operatorname{codim} H_1 \cap \cdots \cap H_q < q, \\ &e_{H_1}^k \wedge \cdots \wedge e_{H_q}^k, \quad \text{if } H_1 \cap \cdots \cap H_q = \varnothing. \end{aligned} \tag{2.2}$$

Let $\mathbf{A}[k] = E[k]/\mathbf{I}[k]$. The Orlik-Solomon algebra is then given by $\mathbf{A}(\mathcal{A}) = \mathbf{A}[1]$.

Proposition 2.2 may be used to determine the cohomology of $M(\mathcal{A}^k)$ for $k > 1$ in terms of that of $M(\mathcal{A})$. Let $P(\mathcal{A}^k, t) = \sum_{q \geq 0} b_q(M(\mathcal{A}^k)) \cdot t^q$ be the Poincaré polynomial of $M(\mathcal{A}^k)$, where $b_q(X)$ is the qth Betti number of X. Results of Goresky and MacPherson [14], and Yuzvinsky [30] (see also Feichtner and Ziegler [13]), together with Proposition 2.2, yield the following corollary.

Corollary 2.3. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$.

(1) For each k, the integral (co)homology of $M(\mathcal{A}^k)$ is torsion free, and $P(\mathcal{A}^k, t) = P(\mathcal{A}, t^{2k-1})$.

(2) For each k, the cohomology algebra of $M(\mathcal{A}^k)$ is isomorphic to the algebra $\mathbf{A}[k]$, $H^*(M(\mathcal{A}^k); \mathbb{Z}) \cong \mathbf{A}[k]$. □

An explicit basis for the first nonzero (reduced) homology group $H_{2k-1}(M(\mathcal{A}^k); \mathbb{Z})$ of the complement of the subspace arrangement $\mathcal{A}^k$ is recorded next. Let $L \subset \mathbb{C}^\ell$ be a complex line that is transverse to the hyperplane arrangement $\mathcal{A}$. Write $L = \{t \cdot \mathbf{u} + \mathbf{v}\}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^\ell$ are fixed and $t \in \mathbb{C}$ varies. For each hyperplane H of $\mathcal{A}$, the intersection $L \cap H$ is a point, say $\mathbf{q}_H = \tau_H \cdot \mathbf{u} + \mathbf{v}$ for some $\tau_H \in \mathbb{C}$. The following lemma is immediate.

Lemma 2.4. The subspace $L^k = \{(t_1 \cdot \mathbf{u} + \mathbf{v}, \ldots, t_k \cdot \mathbf{u} + \mathbf{v}) \mid t_1, \ldots, t_k \in \mathbb{C}\}$ of $\mathbb{C}^{k\ell}$ is transverse to the subspace arrangement $\mathcal{A}^k \subset \mathbb{C}^{k\ell}$. For each subspace H^k of $\mathcal{A}^k$, the intersection $L^k \cap H^k$ is the point $(\mathbf{q}_H, \ldots, \mathbf{q}_H) = (\tau_H \cdot \mathbf{u} + \mathbf{v}, \ldots, \tau_H \cdot \mathbf{u} + \mathbf{v})$. □

Let S^{2k-1} be the unit sphere in $\mathbb{C}^k$. For $\epsilon > 0$ sufficiently small, the point

$$\big((\tau_H + \epsilon' z_1) \cdot \mathbf{u} + \mathbf{v}, \ldots, (\tau_H + \epsilon' z_k) \cdot \mathbf{u} + \mathbf{v}\big) \in L^k \tag{2.3}$$

lies in the intersection $L^k \cap M(\mathcal{A}^k)$ for all ϵ', $0 < \epsilon' \leq \epsilon$. Fix such an ϵ, and define a map $c_H^k : S^{2k-1} \to L^k \cap M(\mathcal{A}^k)$ using the above formula,

$$c_H^k(\mathbf{z}) = c_H^k(z_1, \ldots, z_k) = \big((\tau_H + \epsilon z_1) \cdot \mathbf{u} + \mathbf{v}, \ldots, (\tau_H + \epsilon z_k) \cdot \mathbf{u} + \mathbf{v}\big). \tag{2.4}$$

Let ι_{2k-1} be the fundamental class of $H_{2k-1}(S^{2k-1};\mathbb{Z})$, and denote by $C^k_H \in H_{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$ the image of $(c^k_H)_*(\iota_{2k-1}) \in H_{2k-1}(L^k \cap M(\mathcal{A}^k);\mathbb{Z})$ under the map induced by the natural inclusion $L^k \cap M(\mathcal{A}^k) \hookrightarrow M(\mathcal{A}^k)$.

Proposition 2.5. The classes $\{C^k_H \mid H \in \mathcal{A}\}$ form a basis for $H_{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$. □

Proof. For $H \in \mathcal{A}$, define $p^k_H : L^k \cap M(\mathcal{A}^k) \to S^{2k-1}$ by

$$p^k_H(t_1 \cdot \boldsymbol{u} + \boldsymbol{v}, \ldots, t_k \cdot \boldsymbol{u} + \boldsymbol{v}) = \frac{\mathbf{t} - \tau_H \cdot \boldsymbol{e}}{\|\mathbf{t} - \tau_H \cdot \boldsymbol{e}\|}, \tag{2.5}$$

where $\mathbf{t} = (t_1, \ldots, t_k)$ and $\boldsymbol{e} = (1, \ldots, 1)$ are in $\mathbb{C}^k$. It is then readily checked that $p^k_H \circ c^k_H = \mathrm{id} : S^{2k-1} \to S^{2k-1}$ is the identity map. Furthermore, if $H' \neq H$ is another hyperplane of $\mathcal{A}$, the composition $p^k_H \circ c^k_{H'}$ is given by

$$p^k_H \circ c^k_{H'}(\boldsymbol{z}) = \frac{\boldsymbol{z} + \frac{1}{\epsilon}(\tau_{H'} - \tau_H) \cdot \boldsymbol{e}}{\left\|\boldsymbol{z} + \frac{1}{\epsilon}(\tau_{H'} - \tau_H) \cdot \boldsymbol{e}\right\|}, \tag{2.6}$$

so is null homotopic. Consequently, the classes $(c^k_H)_*(\iota_{2k-1}) \in H_{2k-1}(L^k \cap M(\mathcal{A}^k);\mathbb{Z})$ form a basis. Finally, stratified Morse theory may be used to show that the relative homology group $H_i(M(\mathcal{A}^k), L^k \cap M(\mathcal{A}^k);\mathbb{Z})$ vanishes for $i < 4k-2$, see [**14**, Parts II, III]. It follows that the inclusion $L^k \cap M(\mathcal{A}^k) \hookrightarrow M(\mathcal{A}^k)$ induces an isomorphism $H_{2k-1}(L^k \cap M(\mathcal{A}^k);\mathbb{Z}) \xrightarrow{\sim} H_{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$. So the classes C^k_H form a basis for $H_{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$ as asserted. ■

Remark 2.6. The cohomology classes $(C^k_H)^* \in H^{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$ dual to the classes $C^k_H \in H_{2k-1}(M(\mathcal{A}^k);\mathbb{Z})$ generate the cohomology algebra $H^*(M(\mathcal{A}^k);\mathbb{Z})$. Let $a^k_H \in \mathbf{A}[k]$ denote the image of $e^k_H \in \mathbf{E}[k]$ under the natural projection. Then, the map $H^{2k-1}(M(\mathcal{A}^k);\mathbb{Z}) \to \mathbf{A}_{2k-1}[k]$, $(C^k_H)^* \mapsto a^k_H$, induces an isomorphism of algebras $H^*(M(\mathcal{A}^k);\mathbb{Z}) \xrightarrow{\sim} \mathbf{A}[k]$, see Corollary 2.3.

3 Linearly fibered arrangements

In this section, the topology of those redundant arrangements arising from strictly linearly fibered and fiber-type hyperplane arrangements is studied further. Recall the definition of arrangements of the former type from [11, 23].

Definition 3.1. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^{\ell+1}$ is *strictly linearly fibered* if there is a choice of coordinates $(x, z) = (x_1, \ldots, x_\ell, z)$ on $\mathbb{C}^{\ell+1}$ so that the restriction p of the projection $\mathbb{C}^{\ell+1} \to \mathbb{C}^\ell$, $(x, z) \mapsto x$, to the complement $M(\mathcal{A})$ is a fiber bundle projection, with base $p(M(\mathcal{A})) = M(\mathcal{B})$, the complement of an arrangement $\mathcal{B}$ in $\mathbb{C}^\ell$, and fiber the complement of finitely many points in $\mathbb{C}$. Refer to the hyperplane arrangement $\mathcal{A}$ as strictly linearly fibered over $\mathcal{B}$.

The complements of hyperplane arrangements of this type are closely related to configuration spaces as we now illustrate. For each hyperplane H of $\mathcal{A}$, let f_H be a linear polynomial with $H = \{(x, z) \in \mathbb{C}^{\ell+1} \mid f_H(x, z) = 0\}$. Then $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H$ is a defining polynomial for $\mathcal{A}$. From the definition, if $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and $|\mathcal{A}| = |\mathcal{B}| + n$, there is a choice of coordinates for which a defining polynomial for $\mathcal{A}$ factors as

$$Q(\mathcal{A}) = Q(\mathcal{B}) \cdot \phi(x, z), \tag{3.1}$$

where $Q(\mathcal{B}) = Q(\mathcal{B})(x)$ is a defining polynomial for $\mathcal{B}$, and $\phi(x, z)$ is a product

$$\phi(x, z) = \big(z - g_1(x)\big)\big(z - g_2(x)\big)\cdots\big(z - g_n(x)\big), \tag{3.2}$$

with $g_j(x)$ linear. Define $g : \mathbb{C}^\ell \to \mathbb{C}^n$ by

$$g(x) = \big(g_1(x), g_2(x), \ldots, g_n(x)\big). \tag{3.3}$$

Since $\phi(x, z)$ necessarily has distinct roots for any $x \in M(\mathcal{B})$, the restriction of g to $M(\mathcal{B})$ takes values in the configuration space $F(\mathbb{C}, n)$. The following result was proven by the first author, see [3, Theorem 1.1.5 and Corollary 1.1.6].

Theorem 3.2. Let $\mathcal{B}$ be an arrangement of m hyperplanes and let $\mathcal{A}$ be an arrangement of $m + n$ hyperplanes, which is strictly linearly fibered over $\mathcal{B}$. Then, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of the bundle of the configuration spaces $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ along the map g. Consequently, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ admits a cross section and has trivial local coefficients in homology. □

Since it is relevant to the subsequent discussion, a proof is included.

Proof. Denote points in $F(\mathbb{C}, n + 1)$ by (y, z), where $y = (y_1, \ldots, y_n) \in F(\mathbb{C}, n)$ and $z \in \mathbb{C}$ satisfies $z \neq y_j$ for each j. Similarly, denote points in $M(\mathcal{A})$ by (x, z), where $x \in M(\mathcal{B})$ and $\phi(x, z) \neq 0$. In this notation, we have $p_{n+1}(y, z) = y$ and $p(x, z) = x$. Let $E = \{(x, (y, z)) \in M(\mathcal{B}) \times F(\mathbb{C}, n + 1) \mid g(x) = y\}$ be the total space of the pullback of $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ along the map g. It is then readily checked that the map $h : M(\mathcal{A}) \to E$, defined by $h(x, z) = (x, (g(x), z))$, is an equivalence of bundles.

Since the bundle $p_{n+1} : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ admits a cross section, so does the pullback $p : M(\mathcal{A}) \to M(\mathcal{B})$. Furthermore, the structure group of the latter is the pure braid group P_n. Consequently, the action of the fundamental group of the base $M(\mathcal{B})$ on that of the fiber $\mathbb{C} \setminus \{n \text{ points}\}$ is by pure braid automorphisms. As such, this action is by conjugation (see, e.g., [2, 15]), hence is trivial in homology. ■

It is now shown that redundant strictly linearly fibered arrangements admit (linear) fibrations, just as their codimension one progenitors do.

Theorem 3.3. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^{\ell+1}$ which is strictly linearly fibered over $\mathcal{B}$ with projection $p : M(\mathcal{A}) \to M(\mathcal{B})$ induced by the map $\mathbb{C}^{\ell+1} \to \mathbb{C}^{\ell}$ given by $(x_1, \dots, x_\ell, z) \mapsto (x_1, \dots, x_\ell)$. Then for each k, the map $\mathbb{C}^{k(\ell+1)} \to \mathbb{C}^{k\ell}$ given by $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_\ell, \boldsymbol{z}) \mapsto (\boldsymbol{x}_1, \dots, \boldsymbol{x}_\ell)$ induces a fiber bundle projection $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$. Furthermore, the bundle $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$ admits a cross section. □

Proof. By Theorem 3.2, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of $p_{n+1} : F(\mathbb{C}, n+1) \to F(\mathbb{C}, n)$ along the map g of (3.3). An analogous result for the complements of the subspace arrangements $\mathcal{A}^k$ and $\mathcal{B}^k$ is established next. For $k \geq 2$, view $\mathbb{C}^{k\ell}$ as $(\mathbb{C}^\ell)^k$ and $\mathbb{C}^{kn}$ as $(\mathbb{C}^k)^n$. Denote points in the configuration space $F(\mathbb{C}^k, n+1)$ by $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{z})$, where $(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n) \in F(\mathbb{C}^k, n)$ and $\boldsymbol{z} \neq \boldsymbol{y}_j$ for each j. Define $g^k : \mathbb{C}^{k\ell} \to \mathbb{C}^{kn}$ by

$$g^k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = \big((g_1(\boldsymbol{x}_1), \dots, g_1(\boldsymbol{x}_k)), \dots \dots, (g_n(\boldsymbol{x}_1), \dots, g_n(\boldsymbol{x}_k))\big), \tag{3.4}$$

where $(g_i(\boldsymbol{x}_1), \dots, g_i(\boldsymbol{x}_k)) \in \mathbb{C}^k$ for each i. It is readily checked that the restriction of g^k to $M(\mathcal{B}^k)$ takes values in the configuration space $F(\mathbb{C}^k, n)$. Let $\pi^k : E^k \to M(\mathcal{B}^k)$ be the pullback of the bundle $p^k_{n+1} : F(\mathbb{C}^k, n+1) \to F(\mathbb{C}^k, n)$ along this restriction with total space E^k consisting of all points

$$\big((\boldsymbol{x}_1, \dots, \boldsymbol{x}_k), (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{z})\big) \in M(\mathcal{B}^k) \times F(\mathbb{C}^k, n+1) \tag{3.5}$$

for which $g^k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = p^k_{n+1}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n, \boldsymbol{z}) = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$.

Since the hyperplane arrangement $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$, the complement of the subspace arrangement $\mathcal{A}^k$ may be realized as

$$M(\mathcal{A}^k) = \big\{(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k, \boldsymbol{z}) \in M(\mathcal{B}^k) \times \mathbb{C}^k \mid \boldsymbol{z} \neq (g_i(\boldsymbol{x}_1), \dots, g_i(\boldsymbol{x}_k)) \text{ for } 1 \leq i \leq n\big\}. \tag{3.6}$$

Define $h^k : M(\mathcal{A}^k) \to E^k$ by $h^k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k, \boldsymbol{z}) = ((\boldsymbol{x}_1, \dots, \boldsymbol{x}_k), (g^k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k), \boldsymbol{z}))$.

The map h^k is a homeomorphism. Moreover, the following diagram commutes

$$\begin{array}{ccc} M(\mathcal{A}^k) & \xrightarrow{h^k} & E^k \\ \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle \pi^k} \\ M(\mathcal{B}^k) & \xrightarrow{\text{id}} & M(\mathcal{B}^k). \end{array} \tag{3.7}$$

It follows that $p^k : M(\mathcal{A}^k) \to M(\mathcal{B}^k)$ is a bundle which is equivalent to the pullback of the bundle of configuration spaces $p^k_{n+1} : F(\mathbb{C}^k, n+1) \to F(\mathbb{C}^k, n)$ along the map $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k, n)$, and therefore has a cross section. ■

An analysis of the map in homology induced by the map $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k, n)$ defined in (3.4) is given next. For $1 \le i < j \le n$, define $p_{i,j} : F(\mathbb{C}^k, n) \to S^{2k-1}$ by $p_{i,j}(\mathbf{y}_1, \ldots, \mathbf{y}_n) = (\mathbf{y}_j - \mathbf{y}_i)/\|\mathbf{y}_j - \mathbf{y}_i\|$. Recall that $\iota_{2k-1} \in H_{2k-1}(S^{2k-1}; \mathbb{Z})$ denotes the fundamental class. The classes $p^*_{i,j}(\iota_{2k-1})$ form a basis for $H^{2k-1}(F(\mathbb{C}^k, n))$, and generate the cohomology algebra $H^*(F(\mathbb{C}^k, n))$, see [6, 7]. Denote the dual classes in $H_{2k-1}(F(\mathbb{C}^k, n))$ by $A_{i,j}$, $1 \le i < j \le n$. Note that the classes $A_{i,j}$ may be represented by spheres linking the subspaces $H^k_{i,j} = \{\mathbf{y}_i = \mathbf{y}_j\}$ in $\mathbb{C}^{kn}$, as in (2.4).

As in Section 2, let $L = \{t \cdot \mathbf{u} + \mathbf{v}\} \subset \mathbb{C}^\ell$ be a line transverse to the hyperplane arrangement $\mathcal{B}$, and L^k the corresponding codimension k subspace of $\mathbb{C}^{k\ell}$ transverse to the subspace arrangement $\mathcal{B}^k$. Recall the maps $c^k_H : S^{2k-1} \to L^k \cap M(\mathcal{B}^k)$ from (2.4) and the resulting basis $\{C^k_H \mid H \in \mathcal{B}\}$ for $H_{2k-1}(M(\mathcal{B}^k))$ exhibited in Proposition 2.5.

Proposition 3.4. Let $\mathcal{B} \subset \mathbb{C}^\ell$ be an arrangement of complex hyperplanes and let $g : \mathbb{C}^\ell \to \mathbb{C}^n$ be an affine transformation whose restriction $g : M(\mathcal{B}) \to F(\mathbb{C}, n)$ to the complement of $\mathcal{B}$ takes values in the configuration space $F(\mathbb{C}, n)$. Then, for every $k \ge 1$, the induced map $(g^k)_* : H_{2k-1}(M(\mathcal{B}^k); \mathbb{Z}) \to H_{2k-1}(F(\mathbb{C}^k, n); \mathbb{Z})$ is given by $(g^k)_*(C^k_H) = \sum A_{i,j}$ for each hyperplane H of $\mathcal{B}$, where the sum is over all distinct i and j for which $g(H)$ is contained in the hyperplane $H_{i,j} = \{y_i = y_j\}$ in $\mathbb{C}^n$. □

Proof. For each hyperplane H of $\mathcal{B}$, let $\tilde{c}^k_H : S^{2k-1} \to M(\mathcal{B}^k)$ denote the composition of $c^k_H : S^{2k-1} \to L^k \cap M(\mathcal{B}^k)$ and the natural inclusion $L^k \cap M(\mathcal{B}^k) \hookrightarrow M(\mathcal{B}^k)$. It will be shown that the composition $p_{i,j} \circ g^k \circ \tilde{c}^k_H : S^{2k-1} \to S^{2k-1}$ induces the identity in homology if $g(H) \subset H_{i,j}$, and induces the trivial homomorphism if $g(H) \not\subset H_{i,j}$, thereby establishing the result.

For $\mathbf{x} \in \mathbb{C}^\ell$, write $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ as in (3.3). Then, $g^k : \mathbb{C}^{k\ell} \to \mathbb{C}^{kn}$ is given by $g^k(\mathbf{x}_1, \ldots, \mathbf{x}_k) = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$, where $\mathbf{y}_i = (g_i(\mathbf{x}_1), \ldots, g_i(\mathbf{x}_k))$, see (3.4). Since the restriction of g to $M(\mathcal{B})$ takes values in $F(\mathbb{C}, n)$, the restriction of g^k to $M(\mathcal{B}^k)$ takes values in $F(\mathbb{C}^k, n)$.

From (2.4), the map $\tilde{c}_H^k : S^{2k-1} \to M(\mathcal{B}^k)$ is given by $\tilde{c}_H^k(\boldsymbol{z}) = (\boldsymbol{w}_1, \dots, \boldsymbol{w}_k)$, where $\boldsymbol{w}_j = (\tau_H + \epsilon z_j) \cdot \boldsymbol{u} + \boldsymbol{v}$, and $L \cap H$ is the point $\boldsymbol{q}_H = \tau_H \cdot \boldsymbol{u} + \boldsymbol{v}$. Let $\alpha_i = g_i(\boldsymbol{q}_H)$ and define β_i by

$$g_i(\boldsymbol{w}_j) = g_i\big((\tau_H + \epsilon z_j) \cdot \boldsymbol{u} + \boldsymbol{v}\big) = g_i\big(\boldsymbol{q}_H + \epsilon z_j \cdot \boldsymbol{u}\big) = \alpha_i + \epsilon \beta_i z_j. \tag{3.8}$$

Then, a calculation yields $g^k \circ \tilde{c}_H^k(\boldsymbol{z}) = (\alpha_1 \cdot \boldsymbol{e} + \epsilon\beta_1 \cdot \boldsymbol{z}, \dots, \alpha_n \cdot \boldsymbol{e} + \epsilon\beta_n \cdot \boldsymbol{z})$ and

$$p_{i,j} \circ g^k \circ \tilde{c}_H^k(\boldsymbol{z}) = \frac{\epsilon(\beta_j - \beta_i)\boldsymbol{z} + (\alpha_j - \alpha_i)\boldsymbol{e}}{\|\epsilon(\beta_j - \beta_i)\boldsymbol{z} + (\alpha_j - \alpha_i)\boldsymbol{e}\|}, \tag{3.9}$$

where, as before, $\boldsymbol{e} = (1, \dots, 1)$. Recall that $\epsilon > 0$ was chosen sufficiently small so as to insure that the point $(\boldsymbol{w}'_1, \dots, \boldsymbol{w}'_k)$, where $\boldsymbol{w}'_j = (\tau_H + \epsilon' z_j) \cdot \boldsymbol{u} + \boldsymbol{v}$, lies in $L^k \cap M(\mathcal{B}^k)$ for all ϵ', $0 < \epsilon' \leq \epsilon$. Since $g^k : M(\mathcal{B}^k) \to F(\mathbb{C}^k, n)$, it follows that $g^k \circ \tilde{c}_H^k(\boldsymbol{z}) \in F(\mathbb{C}^k, n)$ for all $\boldsymbol{z} \in S^{2k-1}$. In other words, $\epsilon(\beta_j - \beta_i)\boldsymbol{z} + (\alpha_j - \alpha_i)\boldsymbol{e} \neq \boldsymbol{0}$ for all distinct i and j.

If $g(H) \not\subset H_{i,j}$, then $g(\boldsymbol{q}_H) \notin H_{i,j}$ since $\boldsymbol{q}_H = L \cap H$ is generic in H. Thus, $\alpha_i = g_i(\boldsymbol{q}_H) \neq g_j(\boldsymbol{q}_H) = \alpha_j$, and the point $(\alpha_i \boldsymbol{e} + \epsilon'\beta_i \boldsymbol{z}, \alpha_j \boldsymbol{e} + \epsilon'\beta_j \boldsymbol{z})$ lies in the configuration space $F(\mathbb{C}^k, 2)$ for all $\epsilon' \leq \epsilon$ including $\epsilon' = 0$. It follows that $p_{i,j} \circ g^k \circ c_H^k$ is trivial in homology in this instance.

If, on the other hand, $g(H) \subset H_{i,j}$, then $\alpha_i = \alpha_j$ and thus $\beta_j - \beta_i$ is necessarily nonzero. In this instance, $p_{i,j} \circ g^k \circ \tilde{c}_H^k(\boldsymbol{z}) = \lambda \cdot \boldsymbol{z}$, where $\lambda \in S^1 \subset \mathbb{C}^*$ is given by $\lambda = (\beta_j - \beta_i)/|\beta_j - \beta_i|$ which clearly induces the identity in homology. ■

These results extend immediately to fiber-type arrangements which are defined next.

Definition 3.5. An arrangement $\mathcal{A} = \mathcal{A}_1$ of finitely many points in $\mathbb{C}^1$ is *fiber-type*. An arrangement $\mathcal{A} = \mathcal{A}_\ell$ of hyperplanes in $\mathbb{C}^\ell$ is *fiber-type* if $\mathcal{A}$ is strictly linearly fibered over a fiber-type hyperplane arrangement $\mathcal{A}_{\ell-1}$ in $\mathbb{C}^{\ell-1}$.

Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$. Then there is a choice of coordinates $(x_1, \dots, x_\ell)$ on $\mathbb{C}^\ell$ so that a defining polynomial for $\mathcal{A}$ factors as

$$Q(\mathcal{A}) = \prod_{j=1}^{\ell} Q_j(x_1, \dots, x_j), \tag{3.10}$$

see (3.1). Write $Q_j = \prod_{i=1}^{d_j}(x_j - g_{i,j}(x_1, \dots, x_{j-1}))$, where d_j is the degree of Q_j and each $g_{i,j}$ is linear. The nonnegative integers $\{d_1, \dots, d_\ell\}$ are called the exponents of $\mathcal{A}$. For each $j \leq \ell$, the polynomial $\prod_{i=1}^{j} Q_i$ defines a fiber-type arrangement $\mathcal{A}_j$ in $\mathbb{C}^j$ with exponents $\{d_1, \dots, d_j\}$, and $\mathcal{A}_j$ is strictly linearly fibered over $\mathcal{A}_{j-1}$. Furthermore, the map $g_j = (g_{1,j}, \dots, g_{d_j,j}) : \mathbb{C}^{j-1} \to \mathbb{C}^{d_j}$ gives rise to maps $g_j^k : M(\mathcal{A}_{j-1}^k) \to F(\mathbb{C}^k, d_j)$ for each k. Theorems 3.2 and 3.3 yield the following theorem.

Theorem 3.6. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with defining polynomial $Q(\mathcal{A}) = \prod_{j=1}^{\ell} Q_j$. Then, for each j, $2 \le j \le \ell$, and each $k \ge 1$, the projection $\mathbb{C}^j \to \mathbb{C}^{j-1}$, $(x_1, \dots, x_{j-1}, x_j) \mapsto (x_1, \dots, x_{j-1})$, gives rise to a bundle map $p_j^k : M(\mathcal{A}_j^k) \to M(\mathcal{A}_{j-1}^k)$. This bundle is equivalent to the pullback of the bundle of configuration spaces $F(\mathbb{C}^k, d_j + 1) \to F(\mathbb{C}^k, d_j)$ along the map $g_j^k : M(\mathcal{A}_{j-1}^k) \to F(\mathbb{C}^k, d_j)$. Consequently, the bundle $p_j^k : M(\mathcal{A}_j^k) \to M(\mathcal{A}_{j-1}^k)$ admits a cross section, has trivial local coefficients in homology, and the fiber is the complement of d_j points in $\mathbb{C}^k$. □

Proposition 3.4 also extends to fiber-type arrangements. The specific statement is omitted.

4 The descending central series

In this section, the structure of the Lie algebra $E_0^*(G)$ associated to the descending central series of the fundamental group G of the complement of a fiber-type arrangement is analyzed. Additively, this structure is given by well-known results of Falk and Randell [11, 12] stated below. Moreover, as shown by Jambu and Papadima [16], this Lie algebra is isomorphic to the (integral) holonomy Lie algebra of the arrangement $\mathcal{A}$ defined by Kohno [18]. An alternate description of $E_0^*(G)$, which will facilitate the comparison with the Lie algebra of primitives in the homology of the loop space of the complement of the subspace arrangement $\mathcal{A}^k$ in Section 5, is given here.

Example 4.1. Let P_n be the Artin pure braid group, the fundamental group of the configuration space $F(\mathbb{C}, n)$. The structure of the Lie algebra $E_0^*(P_n)$ was first determined rationally by Kohno [19]. As observed by many authors, the following description holds over the integers as well. For each $n \ge 2$, let $L[n]$ be the free Lie algebra generated by elements $A_{1,n+1}, \dots, A_{n,n+1}$. Then the Lie algebra $E_0^*(P_n)$ is additively isomorphic to the direct sum $\bigoplus_{j=1}^{n-1} L[j]$, and the Lie bracket relations in $E_0^*(P_n)$ are the infinitesimal pure braid relations, given by

$$\begin{aligned} &[A_{i,j} + A_{i,k} + A_{j,k}, A_{m,k}] = 0, \quad \text{for } m = i \text{ or } m = j, \\ &[A_{i,j}, A_{k,l}] = 0, \quad \text{for } \{i,j\} \cap \{k,l\} = \varnothing. \end{aligned} \tag{4.1}$$

Note that this description realizes the Lie algebra $E_0^*(P_{n+1})$ as the semidirect product of $E_0^*(P_n)$ by $L[n]$ determined by the Lie homomorphism $\theta_n : E_0^*(P_n) \to \mathrm{Der}(L[n])$ given by $\theta_n(A_{i,j}) = \mathrm{ad}(A_{i,j})$. The infinitesimal pure braid relations imply that

$$\mathrm{ad}\,(A_{i,j})(A_{m,n+1}) = \begin{cases} [A_{m,n+1}, A_{i,n+1} + A_{j,n+1}], & \text{if } m = i \text{ or } m = j, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

This extension of Lie algebras arises topologically from the bundle of configuration spaces $F(\mathbb{C}, n+1) \to F(\mathbb{C}, n)$.

Remark 4.2. The universal Yang-Baxter Lie algebra and the infinitesimal pure braid relations arise in a number of contexts. These include the classification of pure braids by Vassiliev invariants, see Kohno [20], and the Knizhnik-Zamolodchikov differential equations from the conformal field theory, where the relations appear as integrability conditions on the associated Gauss-Manin connection, see Varchenko [28]. Moreover, any finite-dimensional representation of the Lie algebra $\mathcal{L}(n)$ induces a representation of the pure braid group P_n on the same vector space, see Kapovich and Millson [17].

The additive structure of the universal Yang-Baxter Lie algebra may be obtained from the following result of Falk and Randell [11, 12].

Theorem 4.3. Let $1 \to H \to G \to K \to 1$ be a split extension of groups such that the conjugation action of K is trivial on H_1/H_2. Then, there is a short exact sequence of the Lie algebras $0 \to E_0^*(H) \to E_0^*(G) \to E_0^*(K) \to 0$, which is split as a sequence of abelian groups. Furthermore, if the descending central series quotients of H and K are free abelian, then so are those of G. □

Now, let $\mathcal{A} = \mathcal{A}_\ell$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$. The complement of $\mathcal{A}_\ell$ sits atop a tower of fiber bundles

$$M(\mathcal{A}_\ell) \xrightarrow{p_\ell} M(\mathcal{A}_{\ell-1}) \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} M(\mathcal{A}_1) = \mathbb{C} \setminus \{d_1 \text{ points}\}, \tag{4.3}$$

where the fiber of p_j is homeomorphic to the complement of d_j points in $\mathbb{C}$. For simplicity, write $\mathcal{B} = \mathcal{A}_{\ell-1}$ and $n = d_\ell$. Then, $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$, and by Theorem 3.2, the bundle $p : M(\mathcal{A}) \to M(\mathcal{B})$ is equivalent to the pullback of the configuration space bundle $p_{n+1} : F(\mathbb{C}, n+1) \to F(\mathbb{C}, n)$ along the map g of (3.3).

Application of the homotopy exact sequence of a bundle (and induction) shows that $M(\mathcal{A})$ is a $K(G,1)$-space, where $G = G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$. In the light of Theorem 3.2, there is also a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & G(\mathcal{A}) & \longrightarrow & G(\mathcal{B}) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle g_\#} & & \\
1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & 1,
\end{array} \tag{4.4}$$

where $g_\# : G(\mathcal{B}) \to P_n$ is induced by $g : M(\mathcal{B}) \to F(\mathbb{C}, n)$, and the fundamental group of the fiber $\mathbb{C} \setminus \{n \text{ points}\}$ is identified with the free group $\mathbb{F}_n$ on n generators. Since the underlying bundles admit cross sections, the rows in the diagram above are split exact.

Theorem 4.4. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement. If the exponents of $\mathcal{A}$ are $\{d_1, \ldots, d_\ell\}$, then $E_0^*(G(\mathcal{A})) \cong L[d_1] \oplus \cdots \oplus L[d_\ell]$ as abelian groups. □

Proof. The proof is by induction on ℓ.

In the case where $\ell = 1$, $\mathcal{A}$ is an arrangement of $d = d_1$ points in $\mathbb{C}$, the fundamental group of the complement is $\mathbb{F}_d$, the free group on d generators, and it is well known that $E_0^*(\mathbb{F}_d)$ is isomorphic to the free Lie algebra $L[d]$ (see, e.g., [26, Chapter IV]).

We added the highlighted parentheses. Please check.

In general, assume that the fiber-type arrangement $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and that $d_\ell = n$ as above. Then, there is a split, short exact sequence of groups $1 \to \mathbb{F}_n \to G(\mathcal{A}) \to G(\mathcal{B}) \to 1$, and by Theorem 3.2, the action of $G(\mathcal{B})$ on $\mathbb{F}_n$ is by pure braid automorphisms. As such, this action is by conjugation, hence is trivial on $H_*(\mathbb{F}_n;\mathbb{Z})$. By Theorem 4.3, the descending central series quotients of $G(\mathcal{A})$ are free abelian, and there is a short exact sequence of Lie algebras

$$0 \longrightarrow E_0^*(\mathbb{F}_n) \longrightarrow E_0^*(G(\mathcal{A})) \longrightarrow E_0^*(G(\mathcal{B})) \longrightarrow 0, \tag{4.5}$$

which splits as a sequence of abelian groups. The result follows by induction. ■

The additive decomposition provided by this result does not, in general, preserve the underlying Lie algebra structure. An inductive description of the Lie algebra structure of $E_0^*(G(\mathcal{A}))$ is given next.

Theorem 4.5. Let $\mathcal{A}$ and $\mathcal{B}$ be fiber-type hyperplane arrangements with $|\mathcal{A}| = |\mathcal{B}| + n$, and suppose that $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$. Then, the Lie algebra $E_0^*(G(\mathcal{A}))$ is isomorphic to the semidirect product of $E_0^*(G(\mathcal{B}))$ by $L[n]$ determined by the Lie homomorphism $\Theta = \theta_n \circ g_* : E_0^*(G(\mathcal{B})) \to \mathrm{Der}(L[n])$, where $g_* : E_0^*(G(\mathcal{B})) \to E_0^*(P_n)$ is induced by the map $g : M(\mathcal{B}) \to F(\mathbb{C}, n)$, and $\theta_n : E_0^*(P_n) \to \mathrm{Der}(L[n])$ is given by $\theta_n(A_{i,j}) = \mathrm{ad}(A_{i,j})$. □

Proof. From the exact sequence of Lie algebras (4.5), it follows that $E_0^*(G(\mathcal{A}))$ is isomorphic to the semidirect product of $E_0^*(G(\mathcal{B}))$ by $L[n]$ determined by the Lie homomorphism $\Theta : E_0^*(G(\mathcal{B})) \to \mathrm{Der}(L[n])$ given by $\Theta(b) = \mathrm{ad}_{L[n]}(b)$ for $b \in E_0^*(G(\mathcal{B}))$. It suffices to show that the homomorphism Θ factors as asserted.

From diagram (4.4) and the results of Falk and Randell stated in Theorem 4.3, there is a commutative diagram of Lie algebras with split exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L[n] & \longrightarrow & E_0^*(G(\mathcal{A})) & \longrightarrow & E_0^*(G(\mathcal{B})) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle g_*} & & \\ 0 & \longrightarrow & L[n] & \longrightarrow & E_0^*(P_{n+1}) & \longrightarrow & E_0^*(P_n) & \longrightarrow & 0. \end{array} \tag{4.6}$$

Via the splittings, view $E_0^*(G(\mathcal{B}))$ and $E_0^*(P_n)$ as Lie subalgebras of $E_0^*(G(\mathcal{A}))$ and $E_0^*(P_{n+1})$, respectively. Then, for $a \in L[n]$ and $b \in E_0^*(G(\mathcal{B}))$, we have $[b, a] = [g_*(b), a]$ in $L[n]$. Thus, $\mathrm{ad}_{L[n]}(b) = \mathrm{ad}_{L[n]}(g_*(b))$ in $\mathrm{Der}(L[n])$ and $\Theta = \theta_n \circ g_*$. ■

This result, together with **Proposition 3.4**, provides an inductive description of the Lie bracket structure of $E_0^*(G(\mathcal{A}))$. Recall the basis $\{C_H^1 \mid H \in \mathcal{B}\}$ for $H_1(M(\mathcal{B}); \mathbb{Z}) = E_0^1(G(\mathcal{B}))$ exhibited in **Proposition 2.5**, and recall that the free Lie algebra $L[n]$ is generated by $A_{1,n+1}, \ldots, A_{n,n+1}$.

Corollary 4.6. The generators C_H^1 of $E_0^1(G(\mathcal{B}))$ and $A_{m,n+1}$ of $L[n]$ satisfy

$$\Theta(C_H^1)(A_{m,n+1}) = \sum_{g(H) \subset H_{i,j}} [A_{i,j}, A_{m,n+1}]. \tag{4.7}$$

□

Proof. **Proposition 3.4** implies that $g_*(C_H^1) = \sum A_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. The result follows. ■

This corollary can be used to explicitly record the Lie bracket relations in $E_0^*(G(\mathcal{A}))$, and to show that these relations are combinatorial, that is, completely determined by the intersection poset $\mathbf{L}(\mathcal{A})$. The Lie algebra $E_0^*(G(\mathcal{A}))$ is generated by $\{C_H^1 \mid H \in \mathcal{A}\}$. For a flat $X \in \mathbf{L}(\mathcal{A})$, write $C_X^1 = \sum_{X \subset H} C_H^1$. The following was proven by Jambu and Papadima [16], see also [3].

Theorem 4.7. If $\mathcal{A}$ is a fiber-type hyperplane arrangement with exponents $\{d_1, \ldots, d_\ell\}$, then the Lie bracket relations in $E_0^*(G(\mathcal{A}))$ are given by

$$[C_X^1, C_H^1] = 0 \tag{4.8}$$

for codimension two flats $X \in \mathbf{L}(\mathcal{A})$ and hyperplanes $H \in \mathcal{A}$ containing X. □

Proof. The proof is by induction on ℓ.

In the case where $\ell = 1$, there is nothing to show since $G(\mathcal{A})$ is a free group on $d = d_1$ generators, $E_0^*(G(\mathcal{A}))$ is isomorphic to the free Lie algebra $L[d]$, and there are no codimension two flats in $\mathbf{L}(\mathcal{A})$.

In general, assume that $\mathcal{A}$ is strictly linearly fibered over $\mathcal{B}$ and that $d_\ell = n$ as before. Then $\mathcal{A}$ has a defining polynomial of the form $Q(\mathcal{A}) = Q(\mathcal{B}) \cdot \prod_{j=1}^n (z - g_j(\mathbf{x}))$, see (3.1). View $\mathcal{B}$ as a subarrangement of $\mathcal{A} = \{H \mid H \in \mathcal{B}\} \cup \{H_j \mid 1 \le j \le n\}$, where $H_j = \{(\mathbf{x}, z) \mid z = g_j(\mathbf{x})\}$. Then, the set $\{C_H^1 \mid H \in \mathcal{B}\} \cup \{C_{H_j}^1 \mid 1 \le j \le n\}$ generates $E_0^*(G(\mathcal{A}))$, where the generators $C_{H_j}^1$ correspond to the hyperplanes H_j of $\mathcal{A} \setminus \mathcal{B}$, and to the generators $A_{j,n+1}$ of the free Lie algebra $L[n]$ under the additive isomorphism $E_0^*(G(\mathcal{A})) \cong E_0^*(G(\mathcal{B})) \oplus L[n]$.

By Theorem 4.5, $E_0^*(G(\mathcal{A}))$ is isomorphic to an extension of $E_0^*(G(\mathcal{B}))$ by $L[n]$. Consequently, the Lie bracket relations in $E_0^*(G(\mathcal{A}))$ consist of those of $E_0^*(G(\mathcal{B}))$, and those arising from the extension. By induction, the Lie bracket relations in $E_0^*(G(\mathcal{B}))$ are given by $[C_X^1, C_H^1] = 0$ for the codimension two flats X contained only in hyperplanes $H \in \mathcal{B} \subset \mathcal{A}$. So, it remains to analyze those relations in $E_0^*(G(\mathcal{A}))$ arising from the extension. These are implicitly given in Corollary 4.6.

Recall from (4.1) that $[A_{i,j}, A_{m,n+1}] = [A_{m,n+1}, A_{i,n+1} + A_{j,n+1}]$ if $m \in \{i, j\}$, and equals zero otherwise. Thus, the results of Corollary 4.6 may be recorded as

$$\begin{aligned}\Theta(C_H^1)(A_{m,n+1}) &= [C_H^1, A_{m,n+1}] \\ &= \sum_{g(H)\subset H_{i,j}} [A_{m,n+1}, (\delta_{i,m} + \delta_{j,m})(A_{i,n+1} + A_{j,n+1})],\end{aligned} \tag{4.9}$$

where $C_H^1 \in E_0^*(G(\mathcal{B})) \subset E_0^*(G(\mathcal{A}))$, and $\delta_{i,m}$ is the Kronecker delta. Note that the expression on the right lies in $L[n]$. Under the above identifications, these relations take the form

$$[C_H^1, C_{H_m}^1] = \sum_{g(H)\subset H_{i,j}} [C_{H_m}^1, (\delta_{i,m} + \delta_{j,m})(C_{H_i}^1 + C_{H_j}^1)]. \tag{4.10}$$

It is already checked that $g(H) \subset H_{i,j}$ if and only if $H \cap H_i \cap H_j$ is a codimension two flat in $\mathbf{L}(\mathcal{A})$ if and only if $H_i \cap H_j \subset H$. Using this observation, relation (4.10) may be expressed as

$$[C_H^1, C_{H_m}^1] = \left[C_{H_m}^1, \sum_{H_m\cap H_j\subset H} C_{H_j}^1\right]. \tag{4.11}$$

A calculation then shows that this relation is equivalent to $[C_X^1, C_{H_m}^1] = 0$, where X is the codimension two flat in $\mathbf{L}(\mathcal{A})$ contained in H and H_m. Since this relation holds for all $H_m \in \mathcal{A} \setminus \mathcal{B}$, for which $X \subset H \cap H_m$, it follows that $[C_X^1, C_H^1] = 0$ as well. ■

5 Homology of the loop space

The structure of the Lie algebra of primitive elements in the homology of the loop space of the complement of a redundant subspace arrangement associated to a fiber-type hyperplane arrangement is analyzed in this section. In analogy with Section 4, begin by recalling this structure for the classical configuration spaces $F(\mathbb{C}^{k+1}, n)$ for $k \geq 1$.

Example 5.1. The integral homology of the loop space $\Omega F(\mathbb{C}^{k+1}, n)$ was calculated by Fadell and Husseini [10]. The structure of the Lie algebra $\operatorname{Prim} H_*(\Omega F(\mathbb{C}^{k+1}, n); \mathbb{Z})$ was subsequently determined by Cohen and Gitler [4]. For brevity, denote this Lie algebra by $\mathcal{L}(n)_k$. The structure of $\mathcal{L}(n)_k$ may be described as follows.

For each $n \geq 2$, let $L[n]_k$ denote the free Lie algebra generated by elements $B_{i,n+1}$, $1 \leq i \leq n$, of degree $2k$. Then, $\mathcal{L}(n)_k$ is additively isomorphic to the direct sum $\bigoplus_{j=1}^{n-1} L[j]_k$, and the Lie bracket relations in $\mathcal{L}(n)_k$ are given by the infinitesimal pure braid relations on the $B_{i,j}$, see (4.1). Thus, there is an isomorphism of graded Lie algebras $\mathcal{L}(n)_k \cong E_0^*(P_n)_k$, see Definition 1.1, Theorem 1.2, and Example 4.1.

Furthermore, as is the case for the descending central series of the pure braid group, the Lie algebra $\mathcal{L}(n+1)_k$ is isomorphic to the semidirect product of $\mathcal{L}(n)_k$ by $L[n]_k$ determined by the Lie homomorphism $\theta_n^k : \mathcal{L}(n)_k \to \operatorname{Der}(L[n]_k)$ given by $\theta_n^k(B_{i,j}) = \operatorname{ad}(B_{i,j})$. From the infinitesimal pure braid relations, there is a formula for $\operatorname{ad}(B_{i,j})$ analogous to that given in (4.2). As before, this extension of Lie algebras arises topologically from the bundle of configuration spaces $F(\mathbb{C}^{k+1}, n+1) \to F(\mathbb{C}^{k+1}, n)$.

Now, let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with the exponents $\{d_1, \dots, d_\ell\}$. Then, for each k, there is a tower of fiber bundles

$$M(\mathcal{A}_\ell^k) \xrightarrow{p_\ell^k} M(\mathcal{A}_{\ell-1}^k) \xrightarrow{p_{\ell-1}^k} \cdots \xrightarrow{p_2^k} M(\mathcal{A}_1^k) = \mathbb{C}^k \setminus \{d_1 \text{ points}\}, \tag{5.1}$$

where the fiber of p_j^k is homeomorphic to the complement of d_j points in $\mathbb{C}^k$, see Theorem 3.6. Furthermore, each of the fiber bundles $p_j^k : M(\mathcal{A}_j^k) \to M(\mathcal{A}_{j-1}^k)$ involving the complements of the redundant subspace arrangements $\mathcal{A}_j^k \subset \mathbb{C}^{jk}$ admits a cross section, and, as indicated above, $M(\mathcal{A}_1^k)$ is the complement of d_1 points in $\mathbb{C}^k$. By the work of the second and third authors [8, Theorem 1], the following theorem holds.

Theorem 5.2. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with the exponents $\{d_1, \dots, d_\ell\}$. Then, for each $k \geq 1$,

(a) there is a homotopy equivalence $\Omega M(\mathcal{A}^{k+1}) \to \prod_{j=1}^{\ell} \Omega(\mathbb{C}^{k+1} \setminus \{d_j \text{ points}\})$;

(b) the integral homology of $\Omega M(\mathcal{A}^{k+1})$ is torsion-free, and is isomorphic to the tensor product $\bigotimes_{j=1}^{\ell} H_*(\Omega(\mathbb{C}^{k+1} \setminus \{d_j \text{ points}\}); \mathbb{Z})$ as a coalgebra;

(c) the module of primitives in the integral homology of $\Omega M(\mathcal{A}^{k+1})$ is isomorphic to $\pi_*(\Omega M(\mathcal{A}^{k+1}))$ modulo torsion as a Lie algebra. □

Remark 5.3. The homotopy groups of a loop space admit a bilinear pairing, which satisfies the axioms for a graded Lie algebra in case there is no 2- or 3-torsion in the homotopy groups. The graded analogue of the symmetry law can fail if 2-torsion is present, while the graded analogue of the Jacobi identity can fail if 3-torsion is present. Thus, forming the quotient of the homotopy groups by the torsion gives a graded module which satisfies the axioms for a graded Lie algebra. Analogous properties of iterated loop spaces yield a graded Poisson algebra, see Section 6.

Proof of Theorem 5.2. Given a fibration $F \xrightarrow{i} E \to B$ with a section σ, there is a homotopy equivalence $\Omega E \simeq \Omega B \times \Omega F$ given by the composite

$$\Omega B \times \Omega F \xrightarrow{\Omega\sigma\times\Omega i} \Omega E \times \Omega E \xrightarrow{\mu} \Omega E, \tag{5.2}$$

where μ is the loop space multiplication and such that the inclusions of ΩB and ΩF into ΩE are maps of H-spaces. Moreover, if the spaces involved have torsion-free homology, then $H_*(\Omega E) \cong H_*(\Omega B) \otimes H_*(\Omega F)$. By a theorem of Milnor and Moore [22], we obtain

$$\operatorname{Prim} H_*(\Omega E) \cong \operatorname{Prim} H_*(\Omega B) \oplus \operatorname{Prim} H_*(\Omega F) \tag{5.3}$$

upon passing to the Lie algebra of primitives. This result is a topological analogue of Theorem 4.3 as the underlying Lie algebra structure need not be preserved.

Now, apply these considerations to the fiber bundle $p_j^{k+1} : M(\mathcal{A}_j^{k+1}) \to M(\mathcal{A}_{j-1}^{k+1})$. The fiber in this case is $F = \mathbb{C}^{k+1} \setminus \{d_j \text{ points}\} \simeq \bigvee_{d_j} S^{2k+1}$. Assertion (a) follows by induction, and then (b) by the Künneth theorem. By the Bott-Samelson theorem, $H_*(\Omega F)$ is isomorphic to $T[d_j]_k$, a tensor algebra on d_j generators of degree $2k$.

Thus, the module of primitive elements is generated as a Lie algebra by the primitive elements in degree $2k$ which are in the image of the Hurewicz map. Since the Hurewicz map takes values in the module of primitive elements, that module is generated as a Lie algebra by those spherical classes given by the homology classes of degree $2k$.

Next, notice that the homology groups here are torsion-free. Hence, the Hurewicz map factors through $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\text{Torsion}$. Furthermore, the homotopy groups of a loop space modulo torsion give a graded Lie algebra where the Lie bracket is induced by the classical Samelson product, and the Hurewicz map is a morphism of the graded Lie algebras. Thus, the induced map $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\text{Torsion} \to \operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z})$ is an epimorphism of Lie algebras.

Since all spaces are simply connected and are of finite type, the homotopy groups modulo torsion are finitely generated free abelian groups in any fixed degree. By a classical theorem of Milnor and Moore [22] concerning rational homotopy groups, the induced map $\pi_*\Omega(M(\mathcal{A}^{k+1}))/\text{Torsion} \to \operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z})$ is also a monomorphism. The result follows. ∎

By (5.3) above, there is an isomorphism of graded abelian groups

$$\operatorname{Prim} H_*\big(\Omega M\big(\mathcal{A}_j^{k+1}\big)\big) \cong \operatorname{Prim} H_*\big(\Omega M\big(\mathcal{A}_{j-1}^{k+1}\big)\big) \oplus L\big[d_j\big]_k. \tag{5.4}$$

Proceeding inductively, this implies that the Lie algebra $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ is isomorphic to $L[d_1]_k \oplus \cdots \oplus L[d_\ell]_k$ as a graded abelian group, where $\{d_1, \ldots, d_\ell\}$ are the exponents of $\mathcal{A}$. Thus, the Lie algebras $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ and $E_0^*(G(\mathcal{A}))_k$ are additively isomorphic, see Theorem 4.4. To show that they are isomorphic as Lie algebras, thereby completing the proof of Theorem 1.3, it remains to show that the Lie bracket structure of $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ coincides with that of $E_0^*(G(\mathcal{A}))_k$. This analysis parallels the determination of the Lie algebra structure of $E_0^*(G(\mathcal{A}))$ in Section 4.

The fiber-type hyperplane arrangement $\mathcal{A} = \mathcal{A}_\ell$ is strictly linearly fibered over $\mathcal{A}_{\ell-1}$ and $|\mathcal{A}| = |\mathcal{A}_{\ell-1}| + d_\ell$. As before, write $\mathcal{B} = \mathcal{A}_{\ell-1}$ and $n = d_\ell$. Recall the map $g^{k+1}: M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$ from (3.4). Recall also that the Lie algebra $\operatorname{Prim} H_*(\Omega F(\mathbb{C}^{k+1}, n);\mathbb{Z})$ is denoted by $\mathcal{L}(n)_k$. Analogously, denote the Lie algebra $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1});\mathbb{Z})$ by $\mathcal{L}(\mathcal{A})_k$.

Theorem 5.4. Let $\mathcal{A}$ and $\mathcal{B}$ be fiber-type hyperplane arrangements with $\mathcal{A}$ strictly linearly fibered over $\mathcal{B}$ and $|\mathcal{A}| = |\mathcal{B}| + n$. Then, the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is isomorphic to the semidirect product of $\mathcal{L}(\mathcal{B})_k$ by the free Lie algebra $L[n]_k$ determined by the Lie homomorphism $\Theta^k = \theta_n^k \circ \gamma_*^k : \mathcal{L}(\mathcal{B})_k \to \operatorname{Der}(L[n]_k)$, where $\gamma_*^k : \mathcal{L}(\mathcal{B})_k \to \mathcal{L}(n)_k$ is the map in a loop space homology induced by $g^{k+1}: M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$, and $\theta_n^k : \mathcal{L}(n)_k \to \operatorname{Der}(L[n]_k)$ is given by $\theta_n^k(B_{i,j}) = \operatorname{ad}(B_{i,j})$. □

Proof. The realization of the bundle $p^{k+1}: M(\mathcal{A}^{k+1}) \to M(\mathcal{B}^{k+1})$ as the pullback of the bundle of configuration spaces $p_{n+1}^{k+1}: F(\mathbb{C}^{k+1}, n+1) \to F(\mathbb{C}^{k+1}, n)$ along the map $g^{k+1}: M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$ from Theorem 3.3 yields a commutative diagram of the Hopf algebras

$$\begin{array}{ccccc}
H_*\big(\Omega\big(\mathbb{C}^{k+1} \setminus \{n \text{ points}\}\big)\big) & \longrightarrow & H_*\big(\Omega M\big(\mathcal{A}^{k+1}\big)\big) & \longrightarrow & H_*\big(\Omega M\big(\mathcal{B}^{k+1}\big)\big) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \gamma_*^k} \\
H_*\big(\Omega\big(\mathbb{C}^{k+1} \setminus \{n \text{ points}\}\big)\big) & \longrightarrow & H_*\big(\Omega F\big(\mathbb{C}^{k+1}, n+1\big)\big) & \longrightarrow & H_*\big(\Omega F\big(\mathbb{C}^{k+1}, n\big)\big)
\end{array} \tag{5.5}$$

with exact rows, and, on the level of primitives, a commutative diagram of Lie algebras

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L[n]_k & \longrightarrow & \mathcal{L}(\mathcal{A})_k & \longrightarrow & \mathcal{L}(\mathcal{B})_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \gamma_*^k} & & \\
0 & \longrightarrow & L[n]_k & \longrightarrow & \mathcal{L}(n+1)_k & \longrightarrow & \mathcal{L}(n)_k & \longrightarrow & 0,
\end{array} \tag{5.6}$$

where $\gamma_*^k : \mathcal{L}(\mathcal{B})_k \to \mathcal{L}(n)_k$ is induced by $g^{k+1} : M(\mathcal{B}^{k+1}) \to F(\mathbb{C}^{k+1}, n)$. Since the underlying bundles admit cross sections, the rows in the above diagrams are split exact. Via these splittings, view $\mathcal{L}(\mathcal{B})_k$ and $\mathcal{L}(n)_k$ as Lie subalgebras of $\mathcal{L}(\mathcal{A})_k$ and $\mathcal{L}(n+1)_k$, respectively.

From the above considerations, it follows that the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is isomorphic to the semidirect product of $\mathcal{L}(\mathcal{B})_k$ by $L[n]_k$ determined by the Lie homomorphism $\Theta^k : \mathcal{L}(\mathcal{B})_k \to \mathrm{Der}(L[n]_k)$ given by $\Theta^k(b) = \mathrm{ad}_{L[n]_k}(b)$ for $b \in \mathcal{L}(\mathcal{B})_k$. Moreover, for $a \in L[n]_k$, we have $[b, a] = [\gamma_*^k(b), a]$ in $L[n]_k$. Thus, $\mathrm{ad}_{L[n]_k}(b) = \mathrm{ad}_{L[n]_k}(\gamma_*^k(b))$ in $\mathrm{Der}(L[n]_k)$ and $\Theta^k = \theta_n^k \circ \gamma_*^k$. ■

This result, together with Proposition 3.4, provides an inductive description of the Lie bracket structure of $\mathcal{L}(\mathcal{A})_k$. The space $M(\mathcal{B}^{k+1})$ is 2k-connected, and the cohomology algebra $H^*(M(\mathcal{B}^{k+1}); \mathbb{Z})$ is generated by the classes a_H^{k+1} in one-to-one correspondence with the hyperplanes $H \in \mathcal{B}$, see Corollary 2.3. These classes are of degree $2k+1$, and are dual to the elements of the basis $\{C_H^{k+1} \mid H \in \mathcal{B}\}$ for $H_{2k+1}(M(\mathcal{B}^{k+1}); \mathbb{Z})$ exhibited in Proposition 2.5. See also Remark 2.6.

The above observations imply that a homology suspension induces an isomorphism

$$\sigma_* : H_{2k}\big(\Omega M(\mathcal{B}^{k+1}); \mathbb{Z}\big) \longrightarrow H_{2k+1}\big(M(\mathcal{B}^{k+1}); \mathbb{Z}\big). \tag{5.7}$$

Let $\beta_H^k \in H_{2k}(\Omega M(\mathcal{B}^{k+1}); \mathbb{Z})$ be the unique class satisfying $\sigma_*(\beta_H^k) = C_H^{k+1}$. Recall that the free Lie algebra $L[n]_k$ is generated by $B_{1,n+1}, \dots, B_{n,n+1}$.

Corollary 5.5. The generators β_H^k of $\mathcal{L}(\mathcal{B})_k$ and $B_{m,n+1}$ of $L[n]_k$ satisfy

$$\Theta^k(\beta_H^k)(B_{m,n+1}) = \sum_{g(H) \subset H_{i,j}} [B_{i,j}, B_{m,n+1}]. \tag{5.8}$$

□

Proof. Proposition 3.4 implies that $g_*^{k+1}(C_H^{k+1}) = \sum A_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. Since the homology suspension σ_* is an isomorphism and γ_*^k is the map in the loop space homology induced by g^{k+1}, it follows that $\gamma_*^k(\beta_H^k) = \sum B_{i,j}$, where the sum is over all i and j for which $g(H) \subset H_{i,j}$. The result follows. ■

Theorem 1.3 from the introduction is established next.

Proof of Theorem 1.3. The fact that the Hurewicz homomorphism induces an isomorphism of graded Lie algebras $\pi_*(\Omega M(\mathcal{A}^{k+1}))/\,\mathrm{Torsion} \to \mathrm{Prim}\, H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z})$ for each $k \geq 1$ is established in Theorem 5.2 and its proof. So it suffices to show that the Lie algebras $E_0^*(G(\mathcal{A}))_k$ and $\mathcal{L}(\mathcal{A})_k$ are isomorphic.

The fiber-type hypeplane arrangement $\mathcal{A} = \mathcal{A}_\ell$ is strictly linearly fibered over $\mathcal{A}_{\ell-1}$ and $|\mathcal{A}| = |\mathcal{A}_{\ell-1}| + d_\ell$. Write $\mathcal{B} = \mathcal{A}_{\ell-1}$ and $n = d_\ell$ as before. Assume inductively that the Lie algebras $E_0^*(G(\mathcal{B}))_k$ and $\mathcal{L}(\mathcal{B})_k$ are isomorphic. By Theorem 4.5, the Lie algebra $E_0^*(G(\mathcal{A}))$ is the extension of $E_0^*(G(\mathcal{B}))$ by the free Lie algebra $L[n]$ (generated in degree one) determined by the Lie homomorphism $\Theta = \theta_n \circ g_*$. Thus, $E_0^*(G(\mathcal{A}))_k$ may be realized as the extension of $E_0^*(G(\mathcal{B}))_k$ by the free Lie algebra $L[n]_k$ (generated in degree $2k$) determined by Θ as specified in Definition 1.1. Similarly, by Theorem 5.4, the Lie algebra $\mathcal{L}(\mathcal{A})_k$ is the extension of $\mathcal{L}(\mathcal{B})_k$ by the free Lie algebra $L[n]_k$ determined by the Lie homomorphism $\Theta^k = \theta_n^k \circ \gamma_*^k$. A comparison of the results of Corollaries 4.6 and 5.5 reveals that these extensions coincide. Therefore, the Lie algebras $E_0^*(G(\mathcal{A}))_k$ and $\mathcal{L}(\mathcal{A})_k$ are isomorphic. ∎

Alternatively, Corollary 5.5 may be used to explicitly determine the Lie bracket structure in $\mathcal{L}(\mathcal{A})_k$. As the argument is completely analogous to that which established Theorem 4.7, the result is stated below without proof. The Lie algebra $\mathcal{L}(\mathcal{A})_k$ is generated by $\{\beta_H^k \mid H \in \mathcal{A}\}$. For a flat $X \in \mathbf{L}(\mathcal{A})$, write $\beta_X^k = \sum_{X \subset H} \beta_H^k$.

Theorem 5.6. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement. Then, for each $k \geq 1$, the Lie bracket relations in $\mathcal{L}(\mathcal{A})_k$ are given by

$$\left[\beta_X^k, \beta_H^k\right] = 0 \tag{5.9}$$

for codimension two flats $X \in \mathbf{L}(\mathcal{A})$ and hyperplanes $H \in \mathcal{A}$ containing X. □

6 Homology of iterated loop spaces

In this section, the Poisson algebra structure for the homology of an iterated loop space of the complement of a redundant subspace arrangement associated to a fiber-type hyperplane arrangement is analyzed. A graded associative algebra admits the structure of a graded Lie algebra with a Lie bracket given by the commutator

$$[a, b] = a \cdot b - (-1)^{|a| \cdot |b|} b \cdot a \tag{6.1}$$

for a of degree $|a|$ and b of degree $|b|$, any two elements in the graded associative algebra.

Remark 6.1. Let X be a topological space, and consider the q-fold loop space $\Omega^q X$. There is a bilinear map

$$\lambda_{q-1} : H_i\left(\Omega^q X\right) \otimes H_j\left(\Omega^q X\right) \longrightarrow H_{i+j+q-1}\left(\Omega^q X\right) \tag{6.2}$$

given by the Browder operation. In the case where $q = 1$, this is precisely the commutator for the underlying associative algebra $H_*(\Omega X)$. For $q > 1$, the homology of $\Omega^q X$ admits the structure of a graded Poisson algebra, see [6, pages 215–217], and the Browder operation satisfies the following axioms for a graded Poisson algebra (for which the explicit signs are omitted as they do not appear in what follows below):

(i) *Jacobi identity*: $\lambda_{q-1}[a, \lambda_{q-1}[b, c]] = (\pm 1)\lambda_{q-1}[\lambda_{q-1}[a, b], c] + (\pm 1)\lambda_{q-1}[\lambda_{q-1}[a, c], b]$;

(ii) *antisymmetry*: $\lambda_{q-1}[a, b] = (\pm 1)\lambda_{q-1}[b, a]$;

(iii) *product formula*: $\lambda_{q-1}[ax, b] = (\pm 1)\lambda_{q-1}[a, b] \cdot x + (\pm 1)\lambda_{q-1}[x, b] \cdot a$;

(iv) *commutation with homology suspension* σ_*: $\sigma_*(\lambda_{q-1}[a, b]) = \lambda_{q-2}[\sigma_*(a), \sigma_*(b)]$;

(v) *degree of the operation*: the degree of $\lambda_{q-1}[a, b]$ is $q - 1 + |a| + |b|$.

Please note that according to the style of IMRN all displayed equations must have numbers. Accordingly, it is not possible to make the mathematical expressions in (i)–(v) displayed without numbers.

In addition, it is known that this pairing is compatible with the Whitehead product structure for the classical Hurewicz homomorphism via the commutative diagram

$$\begin{array}{ccc}
\pi_{m+q}(X) \otimes \pi_{n+q}(X) & \xrightarrow{W_0} & \pi_{m+n+2q-1}(X) \\
\downarrow{\scriptstyle s_* \otimes s_*} & & \downarrow{\scriptstyle s_*} \\
\pi_m(\Omega^q X) \otimes \pi_n(\Omega^q X) & \xrightarrow{(\pm 1)W_q} & \pi_{m+n+q-1}(\Omega^q X) \\
\downarrow{\scriptstyle \phi \otimes \phi} & & \downarrow{\scriptstyle \phi} \\
H_m(\Omega^q X) \otimes H_n(\Omega^q X) & \xrightarrow{\lambda_{q-1}} & H_{m+n+q-1}(\Omega^q X),
\end{array} \tag{6.3}$$

where the map s_* is the natural isomorphism, the map ϕ is the classical Hurewicz homomorphism, and the map $(\pm 1)W_q$ is the adjoint of the classical Whitehead product $W_0 : \pi_{m+q}(X) \otimes \pi_{n+q}X \to \pi_{m+n+2q-1}(X)$ up to sign.

Proposition 6.2. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$. Then, for each $k \geq 1$, there is a map $\mathcal{E}^2 : M(\mathcal{A}^{k+1}) \to \Omega^2 M(\mathcal{A}^{k+2})$ such that the associated loop map $\Omega(\mathcal{E}^2) : \Omega M(\mathcal{A}^{k+1}) \to \Omega^3 M(\mathcal{A}^{k+2})$ induces an isomorphism on $H_{2k}(-; \mathbb{Z})$. Furthermore, the image of the map in homology induced by $\Omega(\mathcal{E}^2)$ is the subalgebra generated by the classes of degree $2k$. □

Proof. The complement of an arrangement $\mathcal{A} = \mathcal{A}^1$ of n hyperplanes in $\mathbb{C}^\ell$ may be realized as a slice of a complex torus $M(\mathcal{A}) = W \cap (\mathbb{C}^*)^n$, where $W \subset \mathbb{C}^n$ is an affine subspace of dimension ℓ, see for instance [1]. Let $h : M(\mathcal{A}) \to (S^1)^n$ denote the composition of the natural inclusion $M(\mathcal{A}) = W \cap (C^*)^n \hookrightarrow (\mathbb{C}^*)^n$ and the evident deformation retract $(\mathbb{C}^*)^n \to (S^1)^n$. It is known that the inclusion $M(\mathcal{A}) \hookrightarrow (\mathbb{C}^*)^n$, and hence h, induces a split epimorphism in cohomology. Consequently, after suspending once, the map h is split up to homotopy. Thus, the single suspension $\Sigma M(\mathcal{A})$ is homotopy equivalent to a finite bouquet of spheres of dimensions $2j$ for $1 \leq j \leq t$, where t is the largest dimension for which the cohomology of $M(\mathcal{A})$ does not vanish.

Now, let $\mathcal{A}^{k+1}$ be a redundant subspace arrangement associated to $\mathcal{A}$. A construction analogous to those given in Sections 2 and 3 yields a map $h^{k+1} : M(\mathcal{A}^{k+1}) \to (S^{2k+1})^n$, which induces a split epimorphism in cohomology. Thus, as above, the single suspension $\Sigma M(\mathcal{A}^{k+1})$ is homotopy equivalent to a finite bouquet of spheres of dimensions $2(k+1)j$ for $1 \leq j \leq t$.

Hence, for each k, there is a map $\eta^{k+1} : \Sigma^2 M(\mathcal{A}^{k+1}) \to \bigvee_L S^{2k+3}$ for some L, which induces a homology isomorphism in dimension $2k+3$. Let $\phi : \bigvee_L S^{2k+3} \to M(\mathcal{A}^{k+2})$ denote the natural map obtained from the Hurewicz homomorphism, which induces an isomorphism on the first nonvanishing homology group of $M(\mathcal{A}^{k+2})$. Consider the composite $\phi \circ \eta^{k+1} : \Sigma^2 M(\mathcal{A}^{k+1}) \to M(\mathcal{A}^{k+2})$. The adjoint $\mathcal{E}^2 : M(\mathcal{A}^{k+1}) \to \Omega^2 M(\mathcal{A}^{k+2})$ of this composite induces an isomorphism on the first nontrivial homology group.

Consequently, the associated loop map $\Omega(\mathcal{E}^2) : \Omega M(\mathcal{A}^{k+1}) \to \Omega^3 M(\mathcal{A}^{k+2})$ induces an isomorphism on $H_{2k}(-;\mathbb{Z})$. This, together with the fact that the homology of $\Omega^3 M(\mathcal{A}^{k+2})$ is abelian while the homology of $\Omega M(\mathcal{A}^{k+1})$ is generated by elements of degree $2k$, yields the result. ■

Remark 6.3. The map $\mathcal{E}^2 : M(\mathcal{A}^{k+1}) \to \Omega^2 M(\mathcal{A}^{k+2})$ constructed above may be viewed as an analogue of the classical Freudenthal suspension iterated twice (that is, the double suspension), where the spaces $M(\mathcal{A}^{k+i})$ are replaced by single odd-dimensional spheres. While the associated loop map induces an isomorphism on $H_{2k}(-;\mathbb{Z})$, it has nontrivial kernel in many other dimensions.

When $\mathcal{A} = \mathcal{A}_n$ is the braid arrangement, $q > 1$, and $k \geq 1$, the relations among the Poisson brackets in the homology of the q-fold loop space $\Omega^q M(\mathcal{A}_n^{k+1})$ are the "universal infinitesimal Poisson braid relations" given in Example 6.5. These may be viewed as Poisson analogues of the infinitesimal pure braid relations recorded in (4.1). For an arbitrary fiber-type hyperplane arrangement $\mathcal{A}$, relations among Poisson brackets in the homology of $\Omega^q M(\mathcal{A}^{k+1})$ are analogous to the relations in the Lie algebra $E_0^*(G(\mathcal{A}))$ associated to the descending central series of the fundamental group of $M(\mathcal{A})$.

Theorem 6.4. Let $\mathcal{A}$ be a fiber-type hyperplane arrangement in $\mathbb{C}^\ell$ with the exponents $\{d_1, \ldots, d_\ell\}$. Then, for each $k \geq 1$,

(a) there is a homotopy equivalence $\Omega^q M(\mathcal{A}^{k+1}) \to \prod_{j=1}^{\ell} \Omega^q(\mathbb{C}^{k+1} \setminus \{d_j \text{ points}\})$ for each $q \geq 1$;

(b) if $1 < q < 2k+1$, the rational homology of $\Omega^q M(\mathcal{A}^{k+1})$ is the free Poisson algebra generated by elements β_H of degree $2k+1-q$ for $H \in \mathcal{A}$ modulo the relations

$$\lambda_{q-1}\left[\beta_X, \beta_H\right] = 0, \tag{6.4}$$

for codimension two flats $X \in \mathbf{L}(\mathcal{A})$ and hyperplanes $H \in \mathcal{A}$ containing X, where $\beta_X = \sum_{X \subset H} \beta_H$. □

Proof. In the case where $q = 1$, part (a) was established in Theorem 5.2. The assertion for $q > 1$ follows immediately from this result.

For part (b), first note that in the case where $q = 1$, Theorems 5.2 and **??** imply that the rational homology of $\Omega M(\mathcal{A}^{k+1})$ is isomorphic to the universal enveloping algebra of $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Q}) \cong E_0^*(G(\mathcal{A}))_k \otimes \mathbb{Q}$, which is isomorphic to the image of the rational Hurewicz homomorphism. Denote the generators of $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Q})$ by the same symbols as those of $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Z})$, namely, β_H^k. By Theorem 5.6, the Lie bracket relations in $\operatorname{Prim} H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Q})$ are given by $[\beta_X^k, \beta_H^k] = 0$ for all codimension two flats $X \in \mathbf{L}(\mathcal{A})$ and hyperplanes $H \in \mathcal{A}$ containing X.

Now, fix q, $1 < q < 2k+1$. Then, $M(\mathcal{A}^{k+1})$ is q-connected, and the rational homology of $\Omega^q M(\mathcal{A}^{k+1})$ is isomorphic to the graded symmetric algebra generated by the image of the rational Hurewicz homomorphism, as is well known from work of Milnor and Moore [22]. Since $M(\mathcal{A}^{k+1})$ is q-connected and $q > 1$, the homology suspension

$$\sigma_* : \operatorname{Prim} H_i\left(\Omega^q M\left(\mathcal{A}^{k+1}\right); \mathbb{Q}\right) \longrightarrow \operatorname{Prim} H_{i+1}\left(\Omega^{q-1} M\left(\mathcal{A}^{k+1}\right); \mathbb{Q}\right) \tag{6.5}$$

is an isomorphism. It follows that the composite

$$\sigma_*^{q-1} : \operatorname{Prim} H_i\left(\Omega^q M\left(\mathcal{A}^{k+1}\right); \mathbb{Q}\right) \longrightarrow \operatorname{Prim} H_{i+q-1}\left(\Omega M\left(\mathcal{A}^{k+1}\right); \mathbb{Q}\right) \tag{6.6}$$

is also an isomorphism. Let $\beta_H \in H_{2k+1-q}(\Omega^q M(\mathcal{A}^{k+1}); \mathbb{Q})$ be the unique class satisfying $\sigma_*^{q-1}(\beta_H) = \beta_H^k$. These elements generate the Poisson algebra $H_*(\Omega^q M(\mathcal{A}^{k+1}); \mathbb{Q})$.

Now, recall from Remark 6.1 that, for the single loop space $\Omega M(\mathcal{A}^{k+1})$, the Browder operation is the commutator for the associative algebra $H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Q})$ and that, in general, the homology suspension σ_* satisfies $\sigma_*(\lambda_{q-1}[a, b]) = \lambda_{q-2}[\sigma_*(a), \sigma_*(b)]$. Since the relations $[\beta_X^k, \beta_H^k] = 0$ hold in $H_*(\Omega M(\mathcal{A}^{k+1}); \mathbb{Q})$ by Theorem 5.6 and $\sigma_*^{q-1}(\beta_H) = \beta_H^k$ by construction, the above considerations imply that $\lambda_{q-1}[\beta_X, \beta_H] = 0$. Thus, $H_*(\Omega^q M(\mathcal{A}^{k+1}); \mathbb{Q})$ is the free Poisson algebra generated by the elements β_H with defining relations given by those for a Poisson algebra, and the relations $\lambda_{q-1}[\beta_X, \beta_H] = 0$ for codimension two flats $X \in \mathbf{L}(\mathcal{A})$ and hyperplanes $H \in \mathcal{A}$ containing X. ■

Example 6.5. Let $\mathcal{A} = \mathcal{A}_n$ be the braid arrangement in $\mathbb{C}^n$. Then, $M(\mathcal{A}_n^{k+1}) = F(\mathbb{C}^{k+1}, n)$ for all k, as noted in Example 2.1. For the braid arrangement, the codimension two flats in $\mathbf{L}(\mathcal{A}_n)$ (the partition lattice) are of the forms

$$\begin{aligned} &H_{i,j} \cap H_{i,k} \cap H_{j,k}, \quad \text{for } 1 \le i < j < k \le n, \\ &H_{i,j} \cap H_{k,l}, \quad \text{for } \{i,j\} \cap \{k,l\} = \varnothing. \end{aligned} \tag{6.7}$$

Thus, by Theorem 6.4, for $1 < q < 2k+1$, the rational homology of $\Omega^q F(\mathbb{C}^{k+1}, n)$ is generated as a Poisson algebra by elements $B_{i,j} = \beta_{H_{i,j}}$ of degree $2k+1-q$ for $1 \le i < j \le n$. In addition to the relations recorded in Remark 6.1, these generators also satisfy the following universal infinitesimal Poisson braid relations:

$$\begin{aligned} &\lambda_{q-1}\left[B_{i,j} + B_{i,k} + B_{j,k}, B_{m,k}\right] = 0, \quad \text{for } m = i \text{ or } m = j, \\ &\lambda_{q-1}\left[B_{i,j}, B_{k,l}\right] = 0, \quad \text{for } \{i,j\} \cap \{k,l\} = \varnothing. \end{aligned} \tag{6.8}$$

These are precisely the infinitesimal pure braid relations in case $q = 1$, see [4], and Examples 4.1 and 5.1.

Remark 6.6. The Poisson algebra structure admitted by the homology of an iterated loop space has been used in homotopy theory, and to study the cohomology of certain algebraic groups. This algebra provides a crude measure of “nonstable” homotopy theoretic properties of a space. For instance, Lehrer and Segal use the Poisson bracket to give the “first” nonstable element outside the stable range in the homology of several natural algebraic groups in [21]. Here, the specific spaces in question are the generalizations of configuration space given by $F(\mathbb{C}^{k+1}, n) \times_{\Sigma_n} (\mathbb{CP}^\infty)^n$. The homology of these spaces was determined in [6].

7 Generalizations

The relationship between the descending central series Lie algebra and the Lie algebra of primitives in loop space homology established here for fiber-type arrangements is known to hold in several other situations. These, and some potential generalizations and applications, are discussed below.

First, a number of natural families of examples fit within the general framework described in the introduction. Recall that if M is a manifold and Γ a group which acts properly discontinuously on M, then the orbit configuration space $F_\Gamma(M, \ell)$ consists of all ℓ-tuples of points in M, no two of which lie in the same Γ-orbit. These spaces often satisfy conditions (1), (2), (3), and (4) recorded in the introduction.

For instance, consider orbit configuration spaces of the form $F_\Gamma(\mathbb{E} \times \mathbb{C}^n, \ell)$, where Γ operates diagonally on $\mathbb{E} \times \mathbb{C}^n$ and trivially on $\mathbb{C}^n$. Relevant examples include

(a) a parameterized lattice Γ acting on $\mathbb{E} = \mathbb{C}$ so that the orbit space is an elliptic curve;

(b) a discrete subgroup Γ of $PSL(2, \mathbb{R})$ acting properly discontinuously on the upper half-plane $\mathbb{E} = \mathbb{H}$ by fractional linear transformations so that the orbit space is a complex curve;

(c) a torsion-free subgroup of $\Gamma < Sp(2g, \mathbb{Z})$ acting properly discontinuously on the Siegel upper half-space $\mathbb{E} = \mathbb{H}^g$;

(d) a torsion-free subgroup Γ of the mapping class group for genus g surfaces, acting on Teichmuller space $\mathbb{E}$;

(e) a torsion-free subgroup Γ of $GL(n, \mathbb{Z})$ acting properly discontinuously on $\mathbb{R}^n$ so that the resulting orbit space $\mathbb{R}^n/\Gamma$ is a Bieberbach manifold.

The orbit configuration spaces associated to the action of the standard integral lattice $\Gamma = \mathbb{Z} + i\mathbb{Z}$ on $\mathbb{C}$ by translation were the subject of [8], where it is shown that the analogue of Theorem 1.3 holds for these spaces. The analogue of Theorem 1.3 in the case where Γ is a discrete subgroup of $PSL(2, \mathbb{R})$ is considered in [5]. It is not yet known if the analogue of Theorem 1.3 holds for many of the above examples.

When conditions (1), (2), (3), and (4) from the introduction hold, we obtain generalizations of the universal Yang-Baxter Lie algebra parameterized by the group Γ. This is the case for the family of orbit configuration spaces $F_\Gamma(M, \ell)$, where $M = \mathbb{C}^k \setminus \{0\}$ and $\Gamma = \mathbb{Z}/p\mathbb{Z}$. As noted by D. Matei (personal communication), the resulting generalized Yang-Baxter Lie algebra with cyclic symmetry is of use in constructing Vassiliev invariants of links in the lens space $L(p, 1)$. The Lie algebras arising from other families of orbit configuration spaces may be of similar use for other 3-manifolds, among other potential applications.

Finally, recent work of Papadima and Suciu [24] provides a generalization of Theorem 1.3 over the rationals. For an arbitrary hyperplane arrangement $\mathcal{A}$, it is well known that the complement $X = M(\mathcal{A})$ is a formal space in the sense of Sullivan. It follows from work of Yuzvinsky [30] that the complement $Y = M(\mathcal{A}^{k+1})$ of any redundant subspace arrangement $\mathcal{A}^{k+1}$ is also formal. If $\mathcal{A}$ is a fiber-type arrangement, then the rational cohomology $H^*(X; \mathbb{Q})$ of the complement is a Koszul algebra, see Shelton and Yuzvinsky [27]. Motivated in part by the results of this paper, Papadima and Suciu [24] have obtained analogous results in this generality. In particular, they show that if X and Y are formal, finite-type spaces whose cohomology rings are isomorphic up to regrading (as is the case for $X = M(\mathcal{A})$ and $Y = M(\mathcal{A}^{k+1})$, see Corollary 2.3), then $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong E_0^*(\pi_1(X)) \otimes \mathbb{Q}$ if and only if $H^*(X; \mathbb{Q})$ is Koszul.

The results of Papadima and Suciu [24] provide information concerning the Milnor-Moore group $\mathrm{Hom}^{\mathrm{coalg}}(H_*(\Omega S^2;\mathbb{Q}), H_*(\Omega Y;\mathbb{Q}))$ of degree 0 coalgebra maps, an analogue of the group $[\Omega S^2, \Omega Y]$ of pointed homotopy classes of based maps. In particular, they recover a result of Sato [25] which shows that when Y is a bouquet of odd-dimensional spheres, the Milnor-Moore group is isomorphic to the Malcev completion of a free group.

Acknowledgments

D. C. Cohen was partially supported by grants LEQSF(1996–99)-RD-A-04 and LEQSF(1999–2002)-RD-A-01 from the Louisiana Board of Regents, and by grant MDA904-00-1-0038 from the National Security Agency (NSA). F. R. Cohen was partially supported by the National Science Foundation (NSF). M. Xicoténcatl was partially supported by the CINVESTAV of the IPN and by the MPI für Mathematik.

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