genus $g > 1$ and $\xi \neq 0$. As $\text{cat}(M_g, \xi) \leq 1$ by Theorem 1 this gives a first example where $\text{cat}(X, \xi) \neq \text{cat}^1(X, \xi)$.

For direct products of spaces there exist similar estimates as for the classical Lusternik-Schnirelmann category and so we obtain the following theorem.

Theorem 2. *Let $M^{2k} = \Sigma_1 \times \cdots \times \Sigma_k$, where each Σ_i is a closed orientable surface of genus $g_i > 1$. Given $\xi_i \in H^1(\Sigma_i; \mathbb{R})$, one has*

$$\begin{aligned} \text{cat}(M^{2k}, \xi) &= 1 + 2r \\ \text{cat}^1(M^{2k}, \xi) &= 1 + r + k \end{aligned}$$

where $\xi = p_1^*\xi_1 + \cdots + p_k^*\xi_k$ and $r \leq k$ is the number of indices i such that $\xi_i = 0$.

An immediate consequence is

Corollary 1. *The difference*

$$\text{cat}^1(X, \xi) - \text{cat}(X, \xi)$$

can be arbitrary large.

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Topological complexity of almost-direct products of free groups

Daniel C. Cohen

Let X be a path-connected topological space, and let PX be the space of all continuous paths $\gamma\colon [0,1] \to X$, equipped with the compact-open topology. The map $\pi\colon PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$, defined by sending a path to its endpoints is a fibration.

Definition. The *topological complexity* of X, $\mathsf{TC}(X)$, is the minimal k for which $X \times X = U_1 \cup \cdots \cup U_k$, where U_i is open and there exists a continuous section $s_i\colon U_i \to PX$, $\pi \circ s_i = \text{id}_{U_i}$, for each i, $1 \leq i \leq k$. In other words, the topological complexity of X is the sectional category (or Schwarz genus) of the path space fibration, $\mathsf{TC}(X) = \text{secat}(\pi\colon PX \to X \times X)$.

This notion, introduced by Farber, provides a topological approach to the motion planning problem from robotics, see the survey [6] and the references therein.

Assume that X is a finite-dimensional cell complex. We shall make use of the following properties of topological complexity, which may be found in [6].

(1) $\mathsf{TC}(X)$ is an invariant of the homotopy type of X;
(2) $\mathsf{TC}(X) = 1 \iff X$ is contractible;
(3) $\mathsf{TC}(X) \leq 2\dim(X) + 1$;
(4) $\mathsf{TC}(X) > \mathsf{zcl}(H^*(X)) :=$ cup length$\big(\ker\big(H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)\big)\big)$.

For the *zero-divisor cup length* $\mathsf{zcl}(H^*(X))$, use cohomology with field coefficients.

Problem (Farber [6])**.** For a discrete group G, define the topological complexity of G to be the topological complexity of an Eilenberg-Mac Lane space of type $K(G,1)$, $\mathsf{TC}(G) := \mathsf{TC}(K(G,1))$. Determine the topological complexity of G in terms of other invariants of G.

Definition. An *almost-direct product of free groups* is an iterated semi-direct product $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ of finitely generated free groups F_{n_i}, $n_i < \infty$, such that the action of F_{n_i} on the homology $H^*(F_{n_j};\mathbb{Z})$ is trivial for $1 \leq i < j \leq \ell$.

We pursue the topological complexity of groups of this type. This is motivated by the following results.

Theorem (Farber-Yuzvinsky [8])**.** Let P_ℓ be the Artin pure braid group, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C}$. Then $\mathsf{TC}(P_\ell) = 2\ell - 2$.

Theorem (Farber-Grant-Yuzvinsky [7])**.** Let $P_{\ell,k} = \ker(P_\ell \to P_k)$ denote the kernel of the map which forgets the last $\ell - k$ strands of an ℓ-strand pure braid, the fundamental group of the configuration space of ℓ ordered points in $\mathbb{C} \setminus \{k \text{ points}\}$. If $k \geq 2$, then $\mathsf{TC}(P_{\ell,k}) = 2(\ell - k) + 1$.

The groups $P_\ell = F_{\ell-1} \rtimes \cdots \rtimes F_1$ and $P_{\ell,k} = F_{\ell-1} \rtimes \cdots \rtimes F_k$ are almost direct products of free groups. For $i < j$, the action of F_i on F_j is given by (the restriction of) the Artin representation.

Let $P\Sigma_\ell$ be the group of basis-conjugating automorphisms of the free group $F_\ell = \langle x_1, \ldots, x_\ell \rangle$. McCool [10], found the following presentation for $P\Sigma_\ell$:

$$P\Sigma_\ell = \langle \beta_{i,j}, 1 \leq i \neq j \leq \ell \mid [\beta_{i,j}, \beta_{k,l}], [\beta_{i,k}, \beta_{j,k}], [\beta_{i,j}, (\beta_{i,k} \cdot \beta_{j,k})], \rangle,$$

where the indices in the relations are distinct, and the generators $\beta_{i,j}$ are the automorphisms defined by

$$\beta_{i,j}(x_k) = \begin{cases} x_k & \text{if } k \neq j, \\ x_i^{-1} x_j x_i & \text{if } k = j. \end{cases}$$

The subgroup $P\Sigma_\ell^+$ generated by $\beta_{i,j}$ for $i < j$ is an almost-direct product of free groups, see [4]. One has $P\Sigma_\ell^+ = F_{\ell-1} \rtimes P\Sigma_{\ell-1}^+ = F_{\ell-1} \rtimes \cdots \rtimes F_1$, where $F_\ell = \langle \beta_{1,\ell}, \ldots, \beta_{\ell-1,\ell} \rangle$, and the action of $P\Sigma_{\ell-1}^+$ on F_ℓ may be extracted from the above presentation. The *upper triangular McCool group* $P\Sigma_\ell^+$ is *not* isomorphic to the pure braid group P_ℓ.

Theorem (Cohen-Pruidze [2])**.** Let $P\Sigma_\ell^+$ be the upper triangular McCool group. Then $\mathsf{TC}(P\Sigma_\ell^+) = 2\ell - 2$.

Remark. The pure braid group P_ℓ and triangular McCool group $P\Sigma_\ell^+$ each have infinite cyclic center. Denoting the center of a group G by $Z(G)$, and writing $\overline{G} = G/Z(G)$, we have $P_\ell = \overline{P}_\ell \times \mathbb{Z}$ and $P\Sigma_\ell^+ = \overline{P\Sigma}_\ell^+ \times \mathbb{Z}$, where $\overline{P}_\ell$ and $\overline{P\Sigma}_\ell^+$ are almost-direct products of free groups, each of which has rank at least two.

The above results are unified by the following:

Theorem. Let $G = F_{n_\ell} \rtimes \cdots \rtimes F_{n_1}$ be an almost-direct product of free groups. If $n_j \geq 2$ for each j and m is a non-negative integer, then $\mathsf{TC}(G \times \mathbb{Z}^m) = 2\ell + m + 1$.

Problem. If $\mathbb{Z}^m$ acts nontrivially on G, what is $\mathsf{TC}(G \rtimes \mathbb{Z}^m)$?

For the sake of brevity, we focus on the case $m = 0$. Let $X = K(G, 1)$ be an Eilenberg-Mac Lane space of type $K(G, 1)$. The above result may be established using the bounds $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \leq 2\dim(X) + 1$ noted previously. First, it is not difficult to show that the cohomological dimension of G is equal to the geometric dimension of G, which in turn, is equal to ℓ,

$$\mathrm{cd}(G) = \mathrm{geom\ dim}(G) = \ell \implies \mathsf{TC}(G) \leq 2\ell + 1.$$

The lower bound $\mathsf{zcl}(H^*(G)) < \mathsf{TC}(G)$ may be established through analysis of the (integral) cohomology ring $H^*(G)$.

Theorem. The cohomology ring $H^*(G)$ is a quadratic algebra. That is, $H^*(G) \cong E/J$, where E is an exterior algebra generated in degree 1, and J is an ideal generated in degree 2.

The integral homology $H_*(G)$ is torsion-free and the Poincaré polynomial is given by $P(G, t) = \sum_{k=0}^{\ell} b_k(G) \cdot t^k = \prod_{j=1}^{\ell}(1 + n_j t)$, where $b_k(G)$ is the k-th Betti number of G, see [5]. A minimal, free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, which we denote by $C_\bullet(G) \xrightarrow{\epsilon} \mathbb{Z}$, is constructed in [3].

Let $N = b_1(G)$. The abelianization map $G \to \mathbb{Z}^N$ induces a chain map $\Phi_\bullet \colon C_\bullet \to K_\bullet$, where $C_\bullet = C_\bullet(G) \otimes_{\mathbb{Z}G} \mathbb{Z}\mathbb{Z}^N$ and $K_\bullet \to \mathbb{Z}$ is the standard $\mathbb{Z}\mathbb{Z}^N$-resolution of $\mathbb{Z}$. One can show that the induced map in cohomology $\Phi_2^* \colon H^2(\mathbb{Z}^N) \to H^2(G)$ is surjective, and that $H^*(G) \cong E/J$, where $E = H^*(\mathbb{Z}^N)$ and $J = \ker(\Phi_2^*)$.

The identification $H^*(G) \cong E/J$ may be used to show that $\mathsf{zcl}(H^*(G)) = 2\ell$. For each i, $1 \leq i \leq \ell$, let x_i and y_i be classes in $H^1(G)$ corresponding to distinct generators of the free group F_{n_i}. Then one can show that the product

$$\prod_{i=1}^{\ell}(x_i \otimes 1 - 1 \otimes x_i)(y_i \otimes 1 - 1 \otimes y_i)$$

is non-zero. So we have $2\ell = \mathsf{zcl}(H^*(G)) < \mathsf{TC}(G) \leq 2\dim(K(G, 1)) + 1 = 2\ell + 1$.

As another consequence of the calculation of the cohomology ring of an almost-direct product of free groups G, one can show that $H^*(G; \mathbb{Q})$ is, in fact, Koszul. If, additionally, the group G is 1-formal, then the space $K(G, 1)$ is formal. This is the case for the groups P_ℓ, $P_{\ell,k}$, and $P\Sigma_\ell^+$, see [9] and [1]. When G is 1-formal, the calculation of $\mathsf{TC}(G)$ above may be viewed as evidence in support of the conjecture that, for a formal space X, one has $\mathsf{TC}(X) = 1 + \mathsf{zcl}(H^*(X;\Bbbk))$ for some field $\Bbbk$.

The basis-conjugating automorphism group $P\Sigma_\ell$ is also known to be 1-formal, see [1]. Moreover, we have $\mathsf{TC}(P\Sigma_\ell) = 2\ell - 1 = 1 + \mathsf{zcl}(H^*(P\Sigma_\ell;\mathbb{Q}))$, see [2].

Problem.

(1) Presumably, $P\Sigma_\ell$ is not an almost-direct product of free groups. Prove (or disprove) this.
(2) Determine if the cohomology ring $H^*(P\Sigma_\ell;\mathbb{Q})$ is Koszul.

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Novikov homology and three-manifolds

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Novikov homology is a useful tool to study 1) singularities of closed one-forms and in particular the question of existence of a nonsingular one-form in a given cohomology class 2) finiteness properties in group theory: finite generation, finite presentation, FP_n. In dimension three, there is a close though as yet not fully understood relation with the Thurston norm on $H^1(M;\mathbb{R}) = H_2(M;\mathbb{R})$. This results from a reinterpretation of results of Stallings and Thurston, and is due to Bieri, Neumann and Strebel in 1987, and independently the author (same year, with some recent improvements). We first describe this relation, and then speculate how it could be interpreted thanks to some hypothetical noncommutative Alexander polynomial. Let M be a closed 3-manifold, with $H^1(M;\mathbb{R}) \neq 0$. We