MORSE INEQUALITIES FOR ARRANGEMENTS

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Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of complex hyperplanes, and let $\mathcal{L}$ be a local system of coefficients on the complement M of $\mathcal{A}$. The cohomology of M with coefficients in $\mathcal{L}$ arises in a number of contexts—representations of braid groups, generalized hypergeometric functions, Knizhnik-Zamolodchikov equations, etc.—and has been the subject of considerable recent interest, see for instance [Ko], [SV], [AK], [Va], [CS], and see [OT] as a general reference for arrangements. In this brief note, we settle a question raised by Aomoto and Kita concerning the ranks of the cohomology groups $H^k(M;\mathcal{L})$ in the case where $\mathcal{L}$ is a complex local system of rank one.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ be a collection of "weights." Associated to α, we have a representation $\rho = \rho_\alpha : \pi_1(M) \to \mathbb{C}^*$ given by $\rho(g_j) = \exp(-2\pi \mathrm{i}\alpha_\mathrm{j})$ for any meridian g_j about the hyperplane H_j of $\mathcal{A}$, and a local system of coefficients $\mathcal{L} = \mathcal{L}_\alpha$ on M. If α satisfies certain genericity conditions, the cohomology, $H^*(M;\mathcal{L})$, of M with coefficients in $\mathcal{L}$ may be computed using the Orlik-Solomon algebra of $\mathcal{A}$, see [ESV], [STV]. This leads to results such as the following.

Proposition (Aomoto and Kita [AK], Prop. 2.13.2)**.** *For almost all weights α, we have*

$$\operatorname{rank} H^k(M;\mathcal{L}) \leq \operatorname{rank} H^k(M;\mathbb{C}). \tag{1}$$

Aomoto and Kita subsequently remark that it is not known if the above inequality holds for ALL α.

Our purpose is to point out that the inequality (1) does indeed hold for any local system on the complement of any arrangement. Let $\mathcal{L}$ be an arbitrary complex rank one local system on the complement M of an (essential) arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, and let $b_k = \operatorname{rank} H^k(M;\mathbb{C})$ and $\beta_k = \operatorname{rank} H^k(M;\mathcal{L})$ denote the betti numbers of M with trivial $\mathbb{C}$-coefficients and local coefficients $\mathcal{L}$ respectively.

Theorem. *For $0 \leq k \leq \ell$, we have*

$$\beta_k \leq b_k, \tag{2}$$

and

$$\beta_k - \beta_{k-1} + \cdots \pm \beta_0 \leq b_k - b_{k-1} + \cdots \pm b_0. \tag{3}$$

This result is a straightforward consequence of those of [Co] (see also [Ma]). A sketch of proof is as follows.

Let $\emptyset = F_{-1} \subset F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = \mathbb{C}^\ell$ be a flag in $\mathbb{C}^\ell$ which is transverse to the arrangement $\mathcal{A}$. An algorithm for constructing such a flag may be found in [Co], Section 1. Let $M_k = M \cap F_k$. We then have:

Proposition. *For each k, $0 \leq k \leq \ell$*

(i) $H^i(M_k, M_{k-1};\mathcal{L}) = 0$ *if $i \neq k$; and* (ii) $\operatorname{rank} H^k(M_k, M_{k-1};\mathcal{L}) = b_k$.

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These results may be proved using stratified Morse theory [GM]. For further details, see [Co], Sections 2 and 3 for (i), and Section 5 for (ii).

By the above result, the only nonzero terms in the cohomology exact sequence of the triple (M_k, M_{k-1}, M_{k-2}) occur in degrees $k-1$ and k. It is readily checked that the boundary homomorphisms, $\Delta^k : H^{k-1}(M_{k-1}, M_{k-2}; \mathcal{L}) \to H^k(M_k, M_{k-1}; \mathcal{L})$, of these triples satisfy $\Delta^k \circ \Delta^{k-1} = 0$. Consequently, by splicing together these sequences, we obtain a complex $S^\bullet$ with terms $S^k = H^k(M_k, M_{k-1}; \mathcal{L})$ of rank b_k, and boundary maps Δ^k.

A standard argument shows that the cohomology of the complex $S^\bullet$ is naturally isomorphic to the cohomology of M with coefficients in the local system $\mathcal{L}$. Thus, the inequalities (2) and (3) of the theorem are simply the weak and strong Morse inequalities (see [Mi], Section 5) arising from the complex $S^\bullet$ since we have $\operatorname{rank} S^k = b_k$. Note that for $k = \ell$, the inequality (3) is an equality, with both sides yielding the Euler characteristic of M.

Remark. The above theorem generalizes to local systems of rank greater than one. If $\mathcal{L}$ is a complex local system of rank r, then $\operatorname{rank} H^k(M; \mathcal{L}) \leq \operatorname{rank} H^k(M; \mathbb{C}^r) = r \cdot b_k$. Inequality (3) generalizes analogously.

Remark. An analogue of the above theorem also holds for (middle perversity) intersection cohomology. The arrangement $\mathcal{A}$ determines a Whitney stratification of $X = \mathbb{C}^\ell$, and the complement of $\mathcal{A}$ (on which the local system is defined) is the smooth stratum of X. In [Co], Section 6, we show that there is a subcomplex $IS^\bullet$ of the complex $S^\bullet$ constructed above whose cohomology is naturally isomorphic to $IH^*(X; \mathcal{L})$, the intersection cohomology of X with coefficients in $\mathcal{L}$. So we have $\operatorname{rank} IH^k(X; \mathcal{L}) \leq r \cdot b_k$ if $\mathcal{L}$ is of rank r.

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References

[AK] K. Aomoto, M. Kita, *Hypergeometric Functions*, Springer-Verlag, 1994 (in Japanese).

[Co] D. Cohen, *Cohomology and intersection cohomology of complex hyperplane arrangements*, Advances in Math. **97** (1993), 231–266.

[CS] D. Cohen, A. Suciu, *Homology of iterated semidirect products of free groups*, J. Pure Appl. Algebra **126** (1998), 87–120.

[ESV] H. Esnault, V. Schechtman, V. Viehweg, *Cohomology of local systems on the complement of hyperplanes*, Invent. Math. **109** (1992), 557–561. Erratum, ibid. **112** (1993), 447.

[GM] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Ergeb. Math. Grenzgeb., Vol. 14, Springer-Verlag, 1988.

[Ko] T. Kohno, *One-parameter family of linear representations of Artin's braid groups*, In: Galois Representations and Arithmetic Algebraic Geometry. Advanced Studies in Pure Math., Vol. 12, North-Holland, 1987, pp. 189–200.

[Ma] D. Massey, *Numerical invariants of perverse sheaves*, Duke Math. J. **73** (1994), 307–369.

[Mi] J. Milnor, *Morse Theory*, Annals of Math. Studies **51**, Princeton University Press, 1962.

[OT] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren Math. Wiss., Vol. 300, Springer-Verlag, 1992.

[STV] V. Schechtman, H. Terao, A. Varchenko, *Cohomology of local systems and the Kac-Kazhdan condition for singular vectors*, J. Pure Appl. Algebra **100** (1995), 93–102.

[SV] V. Schechtman, A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139–194.

[Va] A. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, Advanced Series in Mathematical Physics, Vol. 21, World Scientific, 1995.

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