



Resonant Local Systems on Complements of Discriminantal Arrangements and \mathfrak{sl}_2 Representations

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Abstract. We calculate the skew-symmetric cohomology of the complement of a discriminantal hyperplane arrangement with coefficients in local systems arising in the context of the representation theory of the Lie algebra \mathfrak{sl}_2 . For a discriminantal arrangement in \mathbb{C}^k , the skew-symmetric cohomology is nontrivial in dimension $k - 1$ precisely when the ‘master function’ which defines the local system on the complement has nonisolated critical points. In symmetric coordinates, the critical set is a union of lines. Generically, the dimension of this nontrivial skew-symmetric cohomology group is equal to the number of critical lines.

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1. Introduction

Let $z = (z_1, \dots, z_n)$ be an n -tuple of distinct complex numbers, $z_i \neq z_j$ for $i \neq j$, and let $m = (m_1, \dots, m_n)$ be an n -tuple of nonnegative integers. The ‘master function’

$$\Phi_{k,n} = \Phi_{k,n}(t; z, m) = \prod_{i=1}^k \prod_{j=1}^n (t_i - z_j)^{-m_j} \prod_{1 \leq p < q \leq k} (t_p - t_q)^2 \quad (1.1)$$

corresponding to z and m arises in a number of contexts. For instance, hypergeometric integrals involving this function are used in [10] to construct solutions of the \mathfrak{sl}_2 KZ differential equations. Furthermore, the critical point equations of the master function coincide with the Bethe ansatz equations for the \mathfrak{sl}_2 Gaudin model, see [8, 12].

Let $\mathcal{A}_{k,n}$ be the discriminantal arrangement in \mathbb{C}^k consisting of the hyperplanes $H_i^j = \{t_i = z_j\}$ and $H_{p,q} = \{t_p = t_q\}$ defined by the linear polynomials occurring in $\Phi_{k,n}$. Denote the complement of $\mathcal{A}_{k,n}$ by $X = X(\mathcal{A}_{k,n}) = \mathbb{C}^k \setminus \bigcup_{H \in \mathcal{A}_{k,n}} H$.

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For $\kappa \in \mathbb{C}^*$, the function $\Phi_{k,n}^{1/\kappa}$ defines a local system of coefficients \mathcal{L} on X , with monodromy $\exp(2\pi i m_j/\kappa)$ about the hyperplane H_i^j , and monodromy $\exp(-4\pi i/\kappa)$ about $H_{p,q}$. These local systems are often resonant, in the sense that the monodromy about intersections of hyperplanes of $\mathcal{A}_{k,n}$ may be trivial. For applications in conformal field theory, the case where κ is a positive integer (greater than 2) is of most interest. In this instance, the aforementioned hypergeometric integrals are integrals of algebraic functions, providing motivation for the study of these local systems.

The symmetric group Σ_k acts on \mathbb{C}^k by permuting coordinates. This action preserves the complement X of $\mathcal{A}_{k,n}$, and also the local system \mathcal{L} , as the monodromy about $H_{\sigma(i)}^j$ (resp., $H_{\sigma(p),\sigma(q)}$) is the same as that about H_i^j (resp., $H_{p,q}$) for all $\sigma \in \Sigma_k$. Consequently, Σ_k acts on the local system cohomology $H^*(X; \mathcal{L})$. Let $H_-^*(X; \mathcal{L})$ denote the subspace of all skew-symmetric cohomology classes, those classes η for which $\sigma(\eta) = (-1)^{|\sigma|}\eta$ for all $\sigma \in \Sigma_k$. The purpose of this note is to determine the dimensions of these skew-symmetric cohomology groups.

This determination is given in terms of the representation theory of the Lie algebra \mathfrak{sl}_2 . Let e, f , and h be the standard generators of \mathfrak{sl}_2 , satisfying $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let L_a be the irreducible \mathfrak{sl}_2 -module with highest weight $a \in \mathbb{C}$. The module L_a is generated by its singular vector v_a , which satisfies $ev_a = 0$, $hv_a = av_a$. The vectors v_a, fv_a, f^2v_a, \dots form a basis for L_a . If a is a nonnegative integer, then $\dim L_a = a + 1$; otherwise L_a is infinite-dimensional.

For nonnegative integers m_1, \dots, m_n as above, the tensor product $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$ is a direct sum of irreducible representations with highest weights $|m| - 2k$, where

$$|m| = m_1 + \dots + m_n$$

and k is a nonnegative integer. Let $w(m, k)$ denote the multiplicity of $L_{|m|-2k}$ in $L^{\otimes m}$. Then

$$w(m, k) \geq 0 \quad \text{if} \quad |m| - 2k \geq 0 \quad \text{and} \quad w(m, k) = 0 \quad \text{if} \quad |m| - 2k < 0.$$

THEOREM 1.1. *Let m_1, \dots, m_n be nonnegative integers, and let \mathcal{L} be the local system on the complement X of the discriminantal arrangement $\mathcal{A}_{k,n}$ induced by $(\Phi_{k,n}(t; z, m))^{1/\kappa}$ for generic κ .*

- (1) *If $0 \leq |m| - k + 1 < k$, then $H_-^q(X; \mathcal{L}) = 0$ for $q < k - 1$, $\dim H_-^{k-1}(X; \mathcal{L}) = w(m, |m| - k + 1)$, and $\dim H_-^k(X; \mathcal{L}) = w(m, |m| - k + 1) + \binom{n+k-2}{k}$.*
- (2) *Otherwise, $H_-^q(X; \mathcal{L}) = 0$ for $q \neq k$ and $\dim H_-^k(X; \mathcal{L}) = \binom{n+k-2}{k}$.*

In [11], I. Scherbak and the second author relate the critical set of the master function $\Phi_{k,n}(t; z, m)$ of (1.1) and the representation theory of the Lie algebra \mathfrak{sl}_2 in the context of Fuchsian differential equations. These results prompted us to investigate the behavior of local systems on the complement of the underlying discriminantal arrangement $\mathcal{A}_{k,n}$ induced by powers of the master function, our aim being to

determine if the aforementioned relationship is reflected in the dimensions of the (skew-symmetric) local system cohomology groups. For the sake of comparison, we briefly state results from [11, Theorem 1] concerning the critical set of the function $\Phi_{k,n}(t; z, m)$.

THEOREM 1.2. *Let m_1, \dots, m_n be nonnegative integers.*

- (1) *If $0 \leq |m| - k + 1 < k$, then for generic z , the function $\Phi_{k,n}(t; z, m)$ has only non-isolated critical points. In symmetric coordinates $\lambda_1 = \sum t_i$, $\lambda_2 = \sum t_i t_j, \dots$, $\lambda_k = t_1 \cdots t_k$, the critical set consists of $w(m, |m| - k + 1)$ lines.*
- (2) *If $|m| - k + 1 > k$, then for generic z , all critical points of $\Phi_{k,n}(t; z, m)$ are nondegenerate and the critical set consists of $w(m, k)$ Σ_k -orbits.*
- (3) *If $|m| - k + 1 = k$ or $|m| - k + 1 < 0$, then for any z , the function $\Phi_{k,n}(t; z, m)$ has no critical points in X .*

Thus, for generic z , the skew-symmetric cohomology group $H_-^{k-1}(X; \mathcal{L})$ is nontrivial if and only if the master function $\Phi_{k,n}(t; z, m)$ has critical lines. Moreover, the dimension of $H_-^{k-1}(X; \mathcal{L})$ is equal to the number of critical lines. Note also that the behavior of the critical set of $\Phi_{k,n}(t; z, m)$ differs in cases 2 (isolated critical points) and 3 (no critical points) of Theorem 1.2, while the skew-symmetric cohomology $H_-^q(X; \mathcal{L})$ vanishes in all dimensions except $q = k$ in both cases. It would be interesting to determine if these cases are reflected in cohomological properties of the local system \mathcal{L} .

For an arbitrary arrangement \mathcal{A} of N hyperplanes in \mathbb{C}^k , the set of complex rank one local systems on the complement $X(\mathcal{A})$ may be realized as the complex torus $(\mathbb{C}^*)^N \cong H^1(X(\mathcal{A}); \mathbb{C}^*)$. The correspondence is given by $\mathcal{L} \leftrightarrow \tau = (\dots \tau_H \dots) \in (\mathbb{C}^*)^N$, where τ_H is the monodromy of $\mathcal{L} = \mathcal{L}(\tau)$ about the hyperplane H of \mathcal{A} . Generically, the local system cohomology vanishes, except possibly in the top dimension, $H^q(X(\mathcal{A}); \mathcal{L}) = 0$ if $q \neq k$. Call these local systems nonresonant. The local systems for which the cohomology does not vanish (for $q \neq k$) are called resonant, and correspond to elements of the cohomology jumping loci, $V_p^q(\mathcal{A}) = \{\tau \in (\mathbb{C}^*)^N \mid \dim H^q(X(\mathcal{A}); \mathcal{L}(\tau)) \geq p\}$, which are known to be unions of torsion-translated subtori of $(\mathbb{C}^*)^N$.

The first cohomology jumping loci, $V_p^1(\mathcal{A})$, have been the subject of a great deal of recent attention, and are to some extent understood. There are, for instance, combinatorial algorithms for determining the components passing through the identity in $(\mathbb{C}^*)^N$, see [2, 5]. Less is known about the higher jumping loci $V_p^q(\mathcal{A})$ for $1 < q \leq k$. In the case where $\mathcal{A} = \mathcal{A}_{k,n}$ is a discriminantal arrangement, Theorem 1.1.1 provides new examples of resonant local systems, that is, nontrivial elements of the varieties $V_p^q(\mathcal{A}_{k,n})$ for $q = k - 1, k, 1 \leq p \leq w(m, |m| - k + 1)$, and arbitrary k .

This note is organized as follows. Some results on the local system cohomology of the complement of an arrangement, including a strengthening of a particular case of Theorem 1.1.2, are given in Section 2. See [6, 7] as general references in this context. In Section 3, we discuss the relationship between the skew-symmetric local system

cohomology of the complement of a discriminantal arrangement and the representation theory of \mathfrak{sl}_2 , and reformulate Theorem 1.1 in terms of the latter in Theorem 3.1. After a number of preliminary results are established in Section 4, the proof of Theorem 3.1 is given in Section 5.

2. Local System Cohomology

Choose coordinates $\mathbf{t} = (t_1, \dots, t_k)$ on \mathbb{C}^k , and let \mathcal{A} be an arbitrary arrangement of N hyperplanes in \mathbb{C}^k , with complement $X = X(\mathcal{A}) = \mathbb{C}^k \setminus \bigcup_{H \in \mathcal{A}} H$. Assume that \mathcal{A} contains k linearly independent hyperplanes. For each hyperplane H of \mathcal{A} , let f_H be a linear polynomial with $H = \{\mathbf{t} \in \mathbb{C}^k \mid f_H(\mathbf{t}) = 0\}$, and let $\omega_H = d \log f_H$ denote the corresponding logarithmic one-form. Let $\lambda = (\dots \lambda_H \dots) \in \mathbb{C}^N$ be a weight vector, where $\lambda_H \in \mathbb{C}$ corresponds to $H \in \mathcal{A}$. Associated to λ , we have

- (1) a flat connection on the trivial line bundle over X , with connection form $\nabla = d + \omega_\lambda \wedge: \Omega^0 \rightarrow \Omega^1$, where d is the exterior differential operator with respect to the coordinates \mathbf{t} , $\omega_\lambda = \sum_{H \in \mathcal{A}} \lambda_H \omega_H$, and Ω^q is the sheaf of germs of holomorphic differential forms of degree q on X ;
- (2) a rank one representation $\rho: \pi_1(X) \rightarrow \mathbb{C}^*$, given by $\rho(\gamma_H) = \tau_H = \exp(-2\pi i \lambda_H)$, where γ_H is any meridian loop about the hyperplane H of \mathcal{A} ; and
- (3) a rank one local system $\mathcal{L} = \mathcal{L}_\lambda$ on X associated to the flat connection ∇ (resp., the representation ρ).

Let $A(\mathcal{A})$ be the Orlik–Solomon algebra of \mathcal{A} , generated by one-dimensional classes a_H corresponding to the hyperplanes of \mathcal{A} . It is the quotient of the exterior algebra generated by these classes by a homogeneous ideal, hence a finite dimensional graded \mathbb{C} -algebra. The weight vector λ determines an element $a_\lambda = \sum_{H \in \mathcal{A}} \lambda_H a_H$ in $A^1(\mathcal{A})$. Since $a_\lambda \wedge a_\lambda = 0$, multiplication by a_λ defines a differential on $A(\mathcal{A})$. The resulting complex $(A^\bullet(\mathcal{A}), a_\lambda \wedge)$ may be identified with a subcomplex of the twisted de Rham complex of X with coefficients in \mathcal{L} via $a_H \mapsto \omega_H$.

An edge of an arrangement \mathcal{A} is a nonempty intersection of hyperplanes in \mathcal{A} . Associated to each flag F of edges of \mathcal{A} and the weight vector λ , there is an element of the Orlik–Solomon algebra and a corresponding logarithmic flag form Ω_F . If $F = (F_q \subset F_{q-1} \subset \dots \subset F_2 \subset F_1)$ is a flag of edges of \mathcal{A} with $\text{codim } F_j = j$ for each j , then

$$\Omega_F = \lambda_{F_1} \omega_{F_1} \wedge \sum_{F_2 \subset H} \lambda_H \omega_H \wedge \dots \wedge \sum_{F_q \subset H} \lambda_H \omega_H. \tag{2.1}$$

An edge is dense if the subarrangement of hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that, after a change of coordinates, hyperplanes in different sets are in different coordinates. Let $\mathcal{A}_\infty = \mathcal{A} \cup H_\infty$ be the projective closure of \mathcal{A} , the union of \mathcal{A} and the hyperplane at infinity in $\mathbb{C}P^k$. Set $\lambda_{H_\infty} = -\sum_{H \in \mathcal{A}} \lambda_H$. For an edge L of \mathcal{A}_∞ , let $\lambda_L = \sum_{L \subset H} \lambda_H$, where the sum is over hyperplanes of \mathcal{A}_∞ .

THEOREM 2.1 ([9]). *Let \mathcal{L} be the rank one local system on the complement X of \mathcal{A} corresponding to the weight vector λ .*

- (1) *If $\lambda_L \notin \mathbb{Z}_{>0}$ for every dense edge L of \mathcal{A}_∞ , then $H^*(X; \mathcal{L}) \cong H^*(A^\bullet(\mathcal{A}), a_\lambda \wedge)$.*
- (2) *If $\lambda_L \notin \mathbb{Z}_{\geq 0}$ for every dense edge L of \mathcal{A}_∞ , then $H^q(X; \mathcal{L}) = 0$ for $q \neq k$ and $\dim H^k(X; \mathcal{L}) = |\chi(X)|$, where $\chi(X)$ is the Euler characteristic of X .*

Remark 2.2. Call two weight vectors λ and μ equivalent, and write $\lambda \equiv \mu$, if $\lambda - \mu$ is an integer vector. Note that if $\lambda \equiv \mu$ then $\exp(-2\pi i \lambda_j) = \exp(-2\pi i \mu_j)$ for each j , so λ and μ give rise to the same rank one local system \mathcal{L} on X . Consequently, if \mathcal{L} is the local system corresponding to λ , and λ is equivalent to a weight vector which satisfies the conditions of Theorem 2.1.2 above, then $H^q(X; \mathcal{L}) = 0$ for $q \neq k$.

Now let $\mathcal{A} = \mathcal{A}_{k,n}$ be a discriminantal arrangement, and λ the weight vector corresponding to the master function $\Phi_{k,n}(t; z, m)$ of (1.1) and $\kappa \in \mathbb{C}^*$. Explicitly, the weight of the hyperplane H_i^j is $\lambda_i^j = -m_j/\kappa$, and the weight of $H_{i,j}$ is $\lambda_{i,j} = 2/\kappa$. Denote the corresponding local system by \mathcal{L}_κ to indicate the dependence on κ . The following is an immediate consequence of Theorem 2.1.1.

PROPOSITION 2.3. *For generic κ , we have $H^*(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) \cong H^*(A^\bullet(\mathcal{A}_{k,n}), a_\lambda \wedge)$.*

Remark 2.4. For an arbitrary arrangement \mathcal{A} of N hyperplanes in \mathbb{C}^k , the resonance varieties of \mathcal{A} are the cohomology jumping loci, $R_p^q(\mathcal{A}) = \{\lambda \in \mathbb{C}^N \mid \dim H^q(A^\bullet(\mathcal{A}), a_\lambda \wedge) \geq p\}$, of the Orlik–Solomon complex, see [3]. The variety $R_p^q(\mathcal{A})$ coincides with the tangent cone of the cohomology jumping locus $V_p^q(\mathcal{A})$ at the identity in $(\mathbb{C}^*)^N$, and is consequently a union of linear subspaces of \mathbb{C}^N , see, for instance, [1].

Explicit combinatorial descriptions of the first resonance varieties $R_p^1(\mathcal{A})$ may be found in [3, 5]. Less is known about the higher resonance varieties $R_p^q(\mathcal{A})$ for $1 < q \leq k$. In the case where $\mathcal{A} = \mathcal{A}_{k,n}$ is a discriminantal arrangement, and the weights λ correspond to the local system \mathcal{L}_κ for generic κ , the local system cohomology is isomorphic to that of the Orlik–Solomon complex by Proposition 2.3. Hence, Theorem 1.1.1 provides nontrivial elements of the varieties $R_p^q(\mathcal{A}_{k,n})$ for $q = k - 1, k, 1 \leq p \leq w(m, |m| - k + 1)$, and arbitrary k .

The next result strengthens Theorem 1.1.2 in the case $|m| - k + 1 \geq k$.

THEOREM 2.5. *If $|m| - k + 1 \geq k$ and κ is generic, then $H^q(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) = 0$ for $q \neq k$ and $\dim H^k(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) = (n + k - 2)!/(n - 2)!$*

Proof. Note that $|\chi(X(\mathcal{A}_{k,n}))| = (n + k - 2)!/(n - 2)!$.

If $k = 1$, then $\mathcal{A}_{1,n}$ is an arrangement of n points in \mathbb{C} . In this case, the condition $|m| - k + 1 \geq k$ insures that $m_j \neq 0$ for some $j, 1 \leq j \leq n$. For any κ for which m_j/κ is not an integer, the local system \mathcal{L}_κ on $X(\mathcal{A}_{1,n})$ is nontrivial, and we have $\dim H^1(X(\mathcal{A}_{1,n}); \mathcal{L}_\kappa) = n - 1$.

For $k \geq 2$, by Theorem 2.1.2 and Remark 2.2, it suffices to show that there is a weight vector μ equivalent to λ for which $\mu_L \notin \mathbb{Z}_{\geq 0}$ for every dense edge L of \mathcal{A}_∞ , where \mathcal{A}_∞ is the projective closure of the discriminantal arrangement $\mathcal{A}_{k,n}$. We will show that there are integers a_1, \dots, a_n , so that the weight vector μ given by $\mu_i^j = a_j + \lambda_i^j = a_j - m_j/\kappa$ and $\mu_{i,j} = \lambda_{i,j} = 2/\kappa$ satisfies these conditions.

Denote the hyperplanes of \mathcal{A}_∞ by $H_i^j, H_{i,j}$, and H_∞ . The dense edges of \mathcal{A}_∞ may be described as follows. For $I \subseteq [k]$ and $j \in [n]$, let $L_I = \bigcap_{p,q \in I} H_{p,q}$ and $L_I^j = \bigcap_{i \in I} H_i^j$. If $|I| = 1$, set $L_I = \mathbb{C}^k$. For $\emptyset \neq J \subseteq [n]$, set $L_I^J = \bigcap_{j \in J} L_I^j$. One can check that the dense edges of \mathcal{A}_∞ , and their weights with respect to μ , are then given by

dense edge L	weight μ_L
(a) $L_I, I \subseteq [k], I = l, 2 \leq l \leq k$	$l(l-1)/\kappa$
(b) $L_I^j \cap L_I, j \in [n], I \subseteq [k], I = l$	$la_j + l(l-m_j-1)/\kappa$
(c) H_∞	$k(m - k + 1)/\kappa - k a $
(d) $H_\infty \cap L_i^{[n]}, i \in [k]$	$(k-1)(m - k)/\kappa - (k-1) a $
(e) $H_\infty \cap L_I^J \cap L_I, J \subseteq [n], J \geq 2$ $I \subseteq [k], I = l, 2 \leq l < k$	$(k-l)(m - k - l + 1)/\kappa +$ $+l m^J /\kappa - k a + l a_J $

In (e) above, we use the notation $|m^J| = |m| - \sum_{j \in J} m_j$ and $|a_J| = \sum_{j \in J} a_j$.

Now assume that $|m| - k + 1 \geq k$ and that κ is generic. These conditions imply that the weights in (a), (c), (d), and (e) above are not integers, for any choice of $a = (a_1, \dots, a_n)$. Choosing $a_j = -1$ whenever $1 \leq m_j \leq k - 1$ insures that the weights in (b) are not in $\mathbb{Z}_{\geq 0}$. The result follows. \square

3. Skew-Symmetric Cohomology and \mathfrak{sl}_2 Representations

We now turn to the skew-symmetric cohomology groups $H^q(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$, and their relation to representations of \mathfrak{sl}_2 .

For $a \in \mathbb{C}$, let M_a be the corresponding Verma module, the infinite-dimensional \mathfrak{sl}_2 -module generated by the vector v_a , where $ev_a = 0$ and $hv_a = av_a$. The vectors $\{f^k v_a \mid k \geq 0\}$ form a basis for M_a , and the \mathfrak{sl}_2 action is given by

$$e: f^k v_a \mapsto k(a - k + 1)f^{k-1} v_a, \quad f: f^k v_a \mapsto f^{k+1} v_a, \quad h: f^k v_a \mapsto (a - 2k)f^k v_a.$$

Recall [4] that the Shapovalov form is the symmetric bilinear form S_a on M_a defined by $S_a(v_a, v_a) = 1$, $S_a(hx, y) = S_a(x, hy)$ and $S_a(fx, y) = S_a(x, ey)$ for all $x, y \in M_a$, and

$$S_a - (f^k v_a, f^l v_a) = \begin{cases} k! a(a-1) \cdots (a-k+1) & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Given $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, let $M^{\otimes m} = M_{m_1} \otimes \cdots \otimes M_{m_n}$ denote the corresponding tensor product of Verma modules. A basis for $M^{\otimes m}$ is given by

$$F_J v := f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n}, \quad \text{where } J = (j_1, \dots, j_n) \text{ with } j_i \geq 0 \text{ for each } i.$$

The action of \mathfrak{sl}_2 on $M^{\otimes m}$ is given by

$$\begin{aligned} e: F_J v &\mapsto \sum_{i=1}^n j_i(m_i - j_i + 1)F_{J-1_i}v, & f: F_J v &\mapsto \sum_{i=1}^n F_{J+1_i}v, \\ h: F_J v &\mapsto (|m| - 2|J|)F_J v, \end{aligned} \quad (3.1)$$

where $J \pm 1_i = (j_1, \dots, j_i \pm 1, \dots, j_n)$. Denote the Shapovalov form $S_{m_1} \otimes \dots \otimes S_{m_n}$ on $M^{\otimes m}$ by S .

For an \mathfrak{sl}_2 -module V and $\lambda \in \mathbb{C}$, let $V[\lambda] = \{x \in V \mid hx = \lambda x\}$ be the weight subspace of weight λ . For the tensor product $M^{\otimes m}$ and an integer k such that $|m| - 2k \geq 0$, the weight subspace $M^{\otimes m}[|m| - 2k]$ has basis

$$F_J v := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n},$$

where J runs through all multi-indices such that $|J| = j_1 + \dots + j_n = k$ with nonnegative j_i . The dual space $(M^{\otimes m}[|m| - 2k])^*$ has the dual basis, denoted by

$$(F_J v)^* := (f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n})^*.$$

Note that $\dim(M^{\otimes m}[|m| - 2k])^* = \binom{n+k-1}{k}$.

Let $(M^{\otimes m})^* = \bigoplus_k (M^{\otimes m}[|m| - 2k])^*$ denote the restricted dual of the \mathfrak{sl}_2 -module $M^{\otimes m}$, with basis $(F_J v)^*$ as above, for all relevant J . The contragredient action of \mathfrak{sl}_2 on $(M^{\otimes m})^*$ is given by

$$\begin{aligned} f: (F_J v)^* &\mapsto \sum_{i=1}^n (j_i + 1)(m_i - j_i)(F_{J+1_i} v)^*, \\ e: (F_J v)^* &\mapsto \sum_{j_i \neq 0} (F_{J-1_i} v)^*, \\ h: (F_J v)^* &\mapsto (|m| - 2|J|)(F_J v)^*. \end{aligned} \quad (3.2)$$

The Shapovalov form gives rise to a homomorphism of \mathfrak{sl}_2 -modules $S: M^{\otimes m} \rightarrow (M^{\otimes m})^*$ defined by

$$S(F_J v) = c_J (F_J v)^*, \quad \text{where} \quad c_J = \prod_{i=1}^n j_i! m_i (m_i - 1) \dots (m_i - j_i + 1). \quad (3.3)$$

By Proposition 2.3, the skew-symmetric cohomology $H_-^*(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$ is isomorphic to the skew-symmetric part of the cohomology of the Orlik–Solomon complex $(A^\bullet(\mathcal{A}_{k,n}), a_\lambda \wedge)$. By results of [10], this in turn is isomorphic to the cohomology of the complex

$$0 \rightarrow (M^{\otimes m}[|m| - 2k + 2])^* \xrightarrow{f} (M^{\otimes m}[|m| - 2k])^* \rightarrow 0. \quad (3.4)$$

Recall that the Orlik–Solomon complex may be realized as a subcomplex of the twisted de Rham complex of $X(\mathcal{A}_{k,n})$ with coefficients in \mathcal{L}_κ by identifying generators with the corresponding logarithmic forms. Recall also that the hyperplanes of $\mathcal{A}_{k,n}$ include $H_i^j = \{t_i - z_j = 0\}$.

Associated to each monomial $(F_J v)^* = (f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n})^*$ in $(M^{\otimes m})^*$, there is a skew-symmetric logarithmic form ω_J defined as follows. Given $(F_J v)^*$, let

$\ell_i(J) = j_1 + \dots + j_i$ for $1 \leq i \leq n$. Note that $\ell_n(J) = |J|$, and set $\ell_0(J) = 0$. Define $\eta_J = \alpha_J \eta_{J,1} \wedge \eta_{J,2} \wedge \dots \wedge \eta_{J,n}$, where

$$\alpha_J = \frac{1}{j_1! j_2! \dots j_n!},$$

$$\eta_{J,i} = \frac{d(t_{\ell_{i-1}(J)+1} - z_i)}{t_{\ell_{i-1}(J)+1} - z_i} \wedge \dots \wedge \frac{d(t_{\ell_i(J)-1} - z_i)}{t_{\ell_i(J)-1} - z_i} \wedge \frac{d(t_{\ell_i(J)} - z_i)}{t_{\ell_i(J)} - z_i} \quad \text{if } j_i > 0,$$

and $\eta_{J,i} = 1$ if $j_i = 0$. Observe that η_J is a $|J|$ -form. Let

$$\omega_J = \sum_{\sigma \in \Sigma_k} (-1)^{\ell(\sigma)} \eta_J(t_{\sigma(1)}, \dots, t_{\sigma(k)}, z_1, \dots, z_n)$$

be the skew-symmetrization of η_J , where $\ell(\sigma)$ is the length of the permutation σ . If $|J| \neq k - 1, k$, one can check that the skew-symmetric form ω_J vanishes. If $|J| = k - 1, k$, identifying ω_J with the corresponding element of $A(\mathcal{A}_{k,n})$ defines an injective chain map from the complex (3.4) to the Orlik–Solomon complex. This map, given by

$$(F_J v)^* \mapsto \omega_J, \tag{3.5}$$

induces an isomorphism between the cohomology of the complex (3.4) and the skew-symmetric cohomology of $(A^\bullet(\mathcal{A}_{k,n}), a_\lambda \wedge)$, see [10, Theorem 6.16.2].

Since the forms ω_J are nontrivial only for $|J| = k - 1, k$, the complex (3.4) is located in dimensions $k - 1$ and k . Consequently, $H_-^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) \cong \ker f$, $H_-^k(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) \cong \operatorname{coker} f$, and the skew-symmetric cohomology groups vanish in other dimensions, $H_-^q(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) = 0$ for $q \neq k - 1, k$. The differential f of the complex (3.4) is given by the action of $f \in \mathfrak{sl}_2$ on $(M^{\otimes m})^*$ recorded in (3.2) above. Recall that $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$ is the tensor product of the irreducible \mathfrak{sl}_2 -modules L_{m_i} , $1 \leq i \leq n$, and that $w(m, j)$ denotes the multiplicity of $L_{|m|-2j}$ in $L^{\otimes m}$.

THEOREM 3.1. *Let $m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}$, and*

$$f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*.$$

- (1) *If $0 \leq |m| - k + 1 < k$, then the dimension of the kernel of f is $\dim \ker f = w(m, |m| - k + 1)$, and $\dim \operatorname{coker} f = w(m, |m| - k + 1) + \binom{n+k-2}{k}$.*
- (2) *Otherwise, $\ker f = 0$ and $\dim \operatorname{coker} f = \binom{n+k-2}{k}$.*

Remark 3.2. Since the skew-symmetric cohomology $H_-^*(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$ is isomorphic to the cohomology of the complex (3.4), Theorem 1.1 from the Introduction is a consequence of Theorem 3.1. Consequently, the next two sections are devoted to establishing this latter result.

Remark 3.3. If $m_i \geq k$ for some i , $1 \leq i \leq n$, then $\ker f = 0$ in either case of Theorem 3.1 above, see Theorem 5.1.1. Accordingly, in the case $0 \leq |m| - k + 1 < k$ and $m_i \geq k$ for some i , we have $w(m, |m| - k + 1) = 0$, see Lemma 4.5.

For the tensor product $L^{\otimes m} = L_{m_1} \otimes \cdots \otimes L_{m_n}$, a basis for the weight subspace $L^{\otimes m}[|m| - 2k]$ is given by monomials $F_j v = f^{j_1} v_1 \otimes \cdots \otimes f^{j_n} v_n$, where j_1, \dots, j_n are integers satisfying $j_1 + \cdots + j_n = k$ and $0 \leq j_i \leq m_i$ for each i . The map (3.3) induced by the Shapovalov form gives rise to an injective map of complexes

$$\begin{array}{ccc} 0 \longrightarrow L^{\otimes m}[|m| - 2k + 2] & \xrightarrow{f} & L^{\otimes m}[|m| - 2k] \longrightarrow 0 \\ \downarrow s & & \downarrow s \\ 0 \longrightarrow (M^{\otimes m}[|m| - 2k + 2])^* & \xrightarrow{f} & (M^{\otimes m}[|m| - 2k])^* \longrightarrow 0 \end{array}$$

where the action of f is given by (3.1) on $L^{\otimes m}[|m| - 2k + 2]$, and by (3.2) on $(M^{\otimes m}[|m| - 2k + 2])^*$.

As noted in the Introduction, the tensor product $L^{\otimes m}$ is a direct sum of irreducible \mathfrak{sl}_2 -modules of the form L_a with highest weight $a = |m| - 2j$. If $j = |m| - k + 1$ and the condition $0 \leq |m| - k + 1 < k$ does not hold, then $L_{|m| - 2j}$ does not occur in this direct sum. For each irreducible L_a which does occur in the direct sum, the vector $f^a v_a$ has the lowest weight $-a$ and lies in the kernel of f . This observation yields the following.

PROPOSITION 3.4. *If $0 \leq |m| - k + 1 < k$, the dimension of the kernel the map $f: L^{\otimes m}[|m| - 2k + 2] \rightarrow L^{\otimes m}[|m| - 2k]$ is equal to $w(m, |m| + 1 - k)$.*

Remark 3.5. Via the embedding $S: L^{\otimes m}[|m| - 2k + 2] \rightarrow (M^{\otimes m}[|m| - 2k + 2])^*$ induced by the Shapovalov form, this result yields a subspace of

$$\ker(f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*)$$

of dimension $w(m, |m| - k + 1)$, with basis corresponding to lowest weight vectors in the case $0 \leq |m| - k + 1 < k$. These, in fact, form a basis for

$$\ker(f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*),$$

as asserted in Theorem 3.1.1, and shown in Section 5.

Identifying elements of $(M^{\otimes m})^*$ with skew-symmetric logarithmic forms using (3.5), the image of the embedding $S: L^{\otimes m}[|m| - 2k + 2] \rightarrow (M^{\otimes m}[|m| - 2k + 2])^*$ is realized by skew-symmetric flag forms, see (2.1). Thus, upon establishing Theorem 3.1.1, the above considerations yield a basis of the skew-symmetric cohomology group $H_-^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$ consisting of flag forms in the case $0 \leq |m| - k + 1 < k$.

4. Preliminary Results

The purpose of this section is to establish a number of results which will be of use in the proof of Theorem 3.1. First, we describe the matrix of the endomorphism

$$f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*.$$

Order the bases of $(M^{\otimes m}[|m| - 2k + 2])^*$ and $(M^{\otimes m}[|m| - 2k])^*$ lexicographically: $(F_j v)^* > (F_i v)^*$ if the left-most nonzero entry in $J - I$ is positive, where $J =$

(j_1, \dots, j_n) and $I = (i_1, \dots, i_n)$. Let $A_k(m)$ denote the matrix of the map $f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*$ with respect to these ordered bases, and acting on row vectors. If $m = (m_1, m_2, \dots, m_n)$, let $m^1 = (m_2, \dots, m_n)$.

PROPOSITION 4.1. *The matrix of $f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*$ is given by*

$$A_k(m) = \begin{pmatrix} D_{1,k}(m) & A_1(m^1) & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & D_{2,k}(m) & A_2(m^1) & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & D_{k-1,k}(m) & A_{k-1}(m^1) & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & D_{k,k}(m) & A_k(m^1) \end{pmatrix},$$

where $A_q(m^1)$ is the matrix of $f: (M^{\otimes m^1}[|m^1| - 2q + 2])^* \rightarrow (M^{\otimes m^1}[|m^1| - 2q])^*$, $D_{q,k}(m)$ is the diagonal matrix $(k - q + 1)(m_1 - k + q)\mathbf{I}$, and $\mathbf{0}$ and \mathbf{I} denote the zero matrix and identity matrix of the appropriate sizes.

Proof. Suppose the ordered basis of $(M^{\otimes m}[|m| - 2k + 2])^*$ corresponds to the n -tuples

$$J_{1,1}, \dots, J_{1,p_1}, J_{2,1}, \dots, J_{2,p_2}, \dots, J_{n,1}, \dots, J_{n,p_n},$$

where $J_{i,1}, \dots, J_{i,p_i}$ correspond to those basis elements for which $j_1 = \dots = j_{p-1} = 0$ and $j_p \neq 0$. Then the ordered basis for $(M^{\otimes m}[|m| - 2k])^*$ corresponds to the n -tuples

$$\begin{array}{ccccccc} J_{1,1} + 1_1, \dots, J_{1,p_1} + 1_1, & J_{2,1} + 1_1, \dots, J_{2,p_2} + 1_1, & \dots, & J_{n,1} + 1_1, \dots, J_{n,p_n} + 1_1, \\ & J_{2,1} + 1_2, \dots, J_{2,p_2} + 1_2, & \dots, & J_{n,1} + 1_2, \dots, J_{n,p_n} + 1_2, \\ & & \ddots & \vdots \\ & & & J_{n,1} + 1_n, \dots, J_{n,p_n} + 1_n. \end{array}$$

So if row i of $A_k(m)$ corresponds to the basis element $(F_J v)^*$ of $(M^{\otimes m}[|m| - 2k + 2])^*$, then column i corresponds to the basis element $(F_{J+1} v)^*$ of $(M^{\otimes m}[|m| - 2k])^*$. Hence, the diagonal entries are $(A_k(m))_{i,i} = (j_1 + 1)(m_1 - j_1)$. Furthermore, since $J + 1_1 > J + 1_2 > \dots > J + 1_n$ in the lexicographic ordering, the entries below the diagonal are $(A_{k,n}(m))_{i,j} = 0$ for $i > j$.

Since

$$\begin{aligned} & f((f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n})^*) \\ &= (j_1 + 1)(m_1 - j_1)(f^{j_1+1} v_{m_1} \otimes f^{j_2} v_{m_2} \otimes \dots \otimes f^{j_n} v_{m_n})^* + \\ &+ (f^{j_1} v_{m_1})^* \otimes f((f^{j_2} v_{m_2} \otimes \dots \otimes f^{j_n} v_{m_n})^*), \end{aligned}$$

the fact that the (nonzero) entries above the diagonal are as asserted also follows from the above considerations. □

LEMMA 4.5. *If $0 \leq |m| - k + 1 < k$ and there exists an i for which $m_i \geq k$, then $w(m, |m| - k + 1) = 0$.*

Note that if $|m| - k + 1 < k$, there can be at most one i for which $m_i \geq k$.

Proof. Let $m = (m_1, \dots, m_n)$. The proof is by induction on n , with the case $n = 1$ trivial. The case $n = 2$ is also known to hold, as noted in [11].

For general $n \geq 3$, assume that $m_1 \leq m_2 \leq \dots \leq m_{n-1} < k \leq m_n$. Take $j = |m| - k + 1$, $0 \leq j < k$ in each of the three cases in Lemma 4.4 above. Write $m_1 = k - p$, so $|m^1| = |m| - m_1 = |m| - k + p$.

If $0 \leq j < m_1$, then

$$|m^1| - p + 1 = j, \quad |m^1| - (p + 1) + 1 = j - 1, \dots, \quad |m^1| - (p + j) + 1 = 0.$$

Since $|m^1| - (p + j) + 1 = 0$ and $j < m_1 \leq r^1$, we have $|m^1| = p + j - 1 \geq 2r^1 \geq 2m_1 \geq 2j$. So $p - 1 \geq j$ and $p > j$. It follows that $0 \leq |m^1| - (p + i) + 1 < p + i$ for $i = 0, 1, \dots, j$. So by induction, $w(m^1, |m^1| - (p + i) + 1) = w(m^1, j - i) = 0$ for $i = 0, 1, \dots, j$. Thus $w(m, j) = w(m^1, j) + w(m^1, j - 1) + \dots + w(m^1, 1) + w(m^1, 0) = 0$ if $0 \leq j < m_1$.

If $m_1 \leq j \leq r^1$, then

$$\begin{aligned} |m^1| - p + 1 = j, \quad |m^1| - (p + 1) + 1 = j - 1, \dots, \\ |m^1| - (k - 1) + 1 = j - m_1 + 1, \quad |m^1| - k + 1 = j - m_1. \end{aligned}$$

are all nonnegative. Since $|m^1| - k + 1 = j - m_1$ and $j \leq r^1$, we have $|m^1| = j - m_1 + k - 1 \geq 2r^1 \geq 2j$. So $p - 1 \geq j$ and $p > j$. It follows that $0 \leq |m^1| - (p + i) + 1 < p + i$ for $i = 0, 1, \dots, k - p$. So by induction,

$$w(m^1, |m^1| - (p + i) + 1) = w(m^1, j - i) = 0 \quad \text{for } i = 0, 1, \dots, k - p.$$

Thus

$$w(m, j) = w(m^1, j) + w(m^1, j - 1) + \dots + w(m^1, j - m_1 + 1) + w(m^1, j - m_1) = 0$$

if $m_1 \leq j \leq r^1$.

If $r^1 < j \leq r$, then

$$\begin{aligned} |m^1| - k + 1 = j - m_1, \quad |m^1| - (k - 1) + 1 = j - m_1 + 1, \dots, \\ |m^1| - (k - |m| + 2j) + 1 = |m^1| - j. \end{aligned}$$

Since $|m| - k + 1 = j$, we have $k + j - |m| = 1$. The condition $r^1 < j$ implies that $2j > |m^1|$. It follows that $k - |m| + 2j - |m^1| + j > 0$, that is, $k - |m| + 2j > |m^1| - j$. From this, we obtain $0 \leq |m^1| - (k - i) + 1 = j - m_1 + i < k - i$ for $i = 0, 1, \dots, |m| - 2j$. So by induction, $w(m^1, |m^1| - (k - i) + 1) = w(m^1, j - m_1 + i) = 0$ for $i = 0, 1, \dots, |m| - 2j$. Thus $w(m, j) = w(m^1, j - m_1) + \dots + w(m^1, |m^1| - j - 1) + w(m^1, |m^1| - j) = 0$ if $r^1 < j \leq r$. \square

In case 3, where $|m| - k + 1 \geq k$, recall from Section 3 that the kernel of f is isomorphic to $H_-^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$, the skew-symmetric part of the $(k - 1)$ st cohomology of the complement of the discriminantal arrangement $\mathcal{A}_{k,n}$ with coefficients in the local system induced by $(\Phi_{k,n}(t; z, m))^{1/\kappa}$ for generic κ . If $|m| - k + 1 \geq k$, the local system cohomology $H_-^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa)$ vanishes by Theorem 2.5. Hence, in this instance, $\ker f \cong H_-^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) \subseteq H^{k-1}(X(\mathcal{A}_{k,n}); \mathcal{L}_\kappa) = 0$. \square

It remains to prove assertion 1 of Theorem 3.1.

Proof of Theorem 3.1.1. Since $\dim(M^{\otimes m}[|m| - 2j])^* = \binom{n+j-1}{j}$, we must show that

$$\begin{aligned} \dim \ker(f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*) \\ = \dim \ker A_k(m) = w(m, |m| - k + 1) \end{aligned}$$

if $0 \leq |m| - k + 1 < k$. By Proposition 3.4 and Remark 3.5, $\ker f = \ker A_k(m)$ contains a subspace of dimension $w(m, |m| - k + 1)$ generated by flag forms. Thus, $\dim \ker A_k(m) \geq w(m, |m| - k + 1)$, and to establish the result, it suffices to show that $\dim \ker A_k(m) \leq w(m, |m| - k + 1)$.

The proof is by double induction on k and n . The result holds for $n = 1, 2$ and any k , and also for $k = 2$ and any n by direct calculation, see Proposition 4.1 and Example 4.2. (The condition $0 \leq |m| - k + 1 < k$ is vacuous for $k = 1$.) So assume that $k \geq 3$ and $n \geq 3$.

If $m_1 \geq k$, then $\ker f = 0$ by Theorem 5.1.1 and $w(m, |m| - k + 1) = 0$ by Lemma 4.5. So assume that $0 \leq m_1 < k$, and write $p = k - m_1$. Then the matrix $A_k(m)$ of $f: (M^{\otimes m}[|m| - 2k + 2])^* \rightarrow (M^{\otimes m}[|m| - 2k])^*$ is as given in the proof of Theorem 5.1.2, see (5.1). Since the diagonal matrix $D_{q,k} = D_{q,k}(m) = (k - q + 1)(k - q - m_1)\mathbf{I}$ is invertible for $q \neq p$, any nontrivial element of $\ker A_k(m)$ is necessarily of the form

$$v = (0 \cdots 0 v_p v_{p+1} \cdots v_q 0 \cdots 0), \tag{5.2}$$

where $v_q \in \ker A_q(m^1)$ for some $q \geq p$. It follows that

$$\dim \ker A_k(m) \leq \sum_{q=p}^k \dim \ker A_q(m^1). \tag{5.3}$$

In particular, if $m_1 = 0$, then all elements of $\ker A_k(m)$ are of the form $v = (0 \cdots 0 v_k)$, where $v_k \in \ker A_k(m^1)$. Thus,

$$\dim \ker A_k(m) = \dim \ker A_k(m^1) = w(m^1, |m^1| - k + 1),$$

the last equality by induction. Since $m_1 = 0$, we have $w(m, |m| - k + 1) = w(m^1, |m^1| - k + 1)$, which completes the proof in this instance.

For $0 < m_1 < k$, we consider the three cases specified in Lemma 4.4. Assume without loss that $0 \leq m_1 \leq m_2 \leq \cdots \leq m_n$. Let $r, r^1 \in \mathbb{Z}$ be maximal so that $|m| - 2r \geq 0$ and $|m^1| - 2r^1 \geq 0$. Since $n \geq 3$, we have $r^1 \geq m_1$. Write $|m| - k + 1 = j$. Since $p = k - m_1$, we have

$$|m^1| - p + 1 = j, \quad |m^1| - (p + 1) + 1 = j - 1, \dots,$$

$$|m^1| - (k - 1) + 1 = j - (m_1 - 1), \quad |m^1| - k + 1 = j - m_1.$$

First, consider the case $0 \leq j < m_1$. In this instance, we have $|m^1| - q + 1 < 0$ for $p + j < q \leq k$. Theorem 3.1.2 implies that $\ker A_q(m^1) = 0$ for these q . For $q = p + i$, $0 \leq i \leq p + j$, we have $|m^1| - (p + i) + 1 = j - i$. Since $|m^1| - (p + j) + 1 = 0$ and $j < m_1 \leq r^1$, we have $|m^1| = p + j - 1 \geq 2r^1 \geq 2m_1 > 2j$. So $p - 1 > j$ and $p > j$. Thus, $0 \leq |m^1| - (p + i) + 1 < p + i$ for $0 \leq i \leq j$. Hence, by induction,

$$\dim \ker A_{p+i}(m^1) = w(m^1, |m^1| - (p + i) + 1) = w(m^1, j - i)$$

for $0 \leq i \leq j$. Consequently, in this case, (5.3) yields

$$\dim \ker A_k(m) \leq \sum_{i=0}^j \dim \ker A_{p+i}(m^1) = \sum_{i=0}^j w(m^1, j - i) = w(m, j)$$

by Lemma 4.4. Hence, $\dim \ker A_k(m) = w(m, j) = w(m, |m| - k + 1)$. This completes the proof in the case $0 \leq j < m_1$.

Next, consider the case $m_1 \leq j \leq r^1$. In this instance, $|m^1| - (p + i) + 1$ is nonnegative for each i , $0 \leq i \leq m_1$. As above, the condition $j \leq r^1$ implies that $p > j$, and it follows that $0 \leq |m^1| - (p + i) + 1 < p + i$ for $i = 0, 1, \dots, k - p = m_1$. So, by induction, $\dim \ker A_{p+i}(m^1) = w(m^1, |m^1| - (p + i) + 1) = w(m^1, j - i)$ for $0 \leq i \leq m_1$. In this instance, (5.3) yields

$$\dim \ker A_k(m) \leq \sum_{i=0}^{m_1} \dim \ker A_{p+i}(m^1) = \sum_{i=0}^{m_1} w(m^1, j - i) = w(m, j)$$

by Lemma 4.4. Hence, $\dim \ker A_k(m) = w(m, j) = w(m, |m| - k + 1)$. This completes the proof in the case $m_1 \leq j \leq r^1$.

Finally, consider the case $r^1 < j \leq r$. Note that $2j - |m^1| > 0$, since $j \geq r^1$. This, and $j = |m| - k + 1$, implies that $|m^1| - (k - i) + 1 < k - i$ for $0 \leq i \leq |m| - 2j$, since

$$k - i - (|m^1| - (k - i) + 1) = k - j + m_1 - 2i \geq k - j + m_1 - 2|m| + 4j = k + j - |m| + 2j - |m^1| = 1 + 2j - |m^1|.$$

By induction, $\dim \ker A_{k-i}(m^1) = w(m^1, |m^1| - (k - i) + 1)$ for $0 \leq i \leq |m| - 2j$. Furthermore,

$$\sum_{i=0}^{|m|-2j} w(m^1, |m^1| - (k - i) + 1) = \sum_{i=0}^{|m|-2j} w(m^1, j - m_1 + i) = w(m, j),$$

by Lemma 4.4. So in this instance, (5.3) yields

$$\dim \ker A_k(m) \leq \sum_{i=0}^{|m|-2j} \dim \ker A_{k-i}(m^1) + \sum_{i=|m|-2j+1}^{m_1} \dim \ker A_{k-i}(m^1) = w(m, j) + d,$$

where $d = \sum_{i=|m|-2j+1}^{m_1} \dim \ker A_{k-i}(m^1)$.

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