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Cohomology and Intersection Cohomology of Complex Hyperplane Arrangements

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INTRODUCTION

Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbb{C}^d$. It is well known that the cohomology groups of the *complement*, $M := \mathbb{C}^d - \bigcup_{H \in \mathscr{A}} H$, are determined by combinatorial data, see [Br, OS]. In this paper, we analyze several generalizations of this type of problem. We consider

(1) the cohomology of the complement M with coefficients in a local system,

(2) the intersection cohomology of $\mathbb{C}^d$, stratified by $\mathscr{A}$, with coefficients in a local system, and

(3) more generally, the cohomology of a perverse sheaf on $\mathbb{C}^d$, which is constructible with respect to the stratification determined by $\mathscr{A}$.

Let $L = L(\mathscr{A})$ denote the partially ordered set of "flats" (i.e., of all multi-intersections of elements of $\mathscr{A}$), ordered by reverse inclusion:

$$v \leqslant w \Leftrightarrow w \subseteq v.$$

Thus the ambient space $\mathbb{C}^d$ is the minimal element (correponding to the empty intersection). Let L_q denote the subset of L consisting of all flats of codimension q. The arrangement $\mathscr{A}$ determines a Whitney stratification of $\mathbb{C}^d$, with one (connected) stratum of codimension q,

$$S(v) = v - \bigcup_{v < w} w,$$

for each flat $v \in L_q$. Let X denote $\mathbb{C}^d$ endowed with the stratification given by $\mathscr{A}$. We refer to L as the *lattice* of $\mathscr{A}$, and we do not distinguish between elements of L and the correponding strata of X.

We study each of the above cases by considering the cohomology of the appropriate perverse sheaf $\mathbf{P}^{\bullet}$ on X. In each case, we associate a group $A_w(\mathbf{P}^{\bullet})$ to each element w of the lattice of $\mathscr{A}$, and a homomorphism $\Phi_{v,w}: A_v(\mathbf{P}^{\bullet}) \to A_w(\mathbf{P}^{\bullet})$ whenever $v < w$ is a codimension one inclusion

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(i.e., $w \subset v$ and $\dim(v) = \dim(w) + 1$). Our principal result, Theorem 2.4, is that in each of these cases, the system of groups

$$\mathbf{K}^q(\mathbf{P}^\bullet) = \bigoplus_{w \in L_q} A_w(\mathbf{P}^\bullet),$$

and homomorphisms

$$\bigoplus_{w \in L_q} \sum_{\substack{v \in L_{q-1} \\ v < w}} \Phi_{v,w} : \mathbf{K}^{q-1}(\mathbf{P}^\bullet) \to \mathbf{K}^q(\mathbf{P}^\bullet),$$

is a differential complex, whose cohomology is isomorphic to $H^*(X; \mathbf{P}^\bullet)$, the cohomology of the sheaf $\mathbf{P}^\bullet$. The complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ is constructed using a *weakly self-indexing* Morse function (see Sections 1 and 2).

Given a local cofficient system of complex vector spaces $\mathbf{V}$ on the complement M of $\mathscr{A}$, using Theorem 2.4 we obtain a complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ whose cohomology is isomorphic to $H^*(M; \mathbf{V})$, the cohomology of the complement with coefficients in $\mathbf{V}$, by taking the perverse sheaf $\mathbf{P}^\bullet$ above to be the direct image, $Ri_*\mathbf{V}$, of the local system, where $i: M \to X$ denotes the natural inclusion. In Section 5, we give an explicit combinatorial description of the groups of the complex $\mathbf{K}^\bullet(Ri_*\mathbf{V})$ in terms of the *Möbius function* on the lattice of $\mathscr{A}$ (see Proposition 5.2). In the special case where the local system is trivial, the differentials of the complex $\mathbf{K}^\bullet(Ri_*\mathbf{V})$ all vanish, and we recover the formula for the Betti numbers of the complement due to Orlik and Solomon [OS] (see Remark 5.4).

If the perverse sheaf $\mathbf{P}^\bullet$ is taken to be $\mathbf{I}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{V})$, the complex of sheaves of intersection cochains with coefficients in the local system $\mathbf{V}$, Theorem 2.4 yields a complex $\mathbf{K}^\bullet(\mathbf{I}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{V}))$ whose cohomology is isomorphic to $I^{\bar{p}}H^*(X; \mathbf{V})$, the intersection cohomology of X with coefficients $\mathbf{V}$. The complex $\mathbf{K}^\bullet(\mathbf{I}^{\bar{m}}\mathbf{C}^\bullet(\mathbf{V}))$ for the "middle" intersection cohomology of X with coefficients in $\mathbf{V}$ is shown to be a subcomplex of the complex $\mathbf{K}^\bullet(Ri_*\mathbf{V})$ (see Theorem 6.3).

For other results on the cohomology of the complement of a complex hyperplane arrangement with a local coefficient system, most of which take the form of vanishing theorems, the reader is referred to [Ha, Ko, SP, SV, VGZ]. In Section 7, we obtain similar results for *general position* arrangements by giving explicit combinatorial descriptions of the differentials of the complex $\mathbf{K}^\bullet(Ri_*\mathbf{V})$, as well as the terms. This complete description of the complex $\mathbf{K}^\bullet(Ri_*\mathbf{V})$, together with the fact that $\mathbf{K}^\bullet(\mathbf{I}^{\bar{m}}\mathbf{C}^\bullet(\mathbf{V}))$ is a subcomplex of $\mathbf{K}^\bullet(Ri_*\mathbf{V})$, yields an algorithm for computing $I^{\bar{m}}H^*(X; \mathbf{V})$, the middle intersection cohomology of X with coefficients in the local system $\mathbf{V}$, where X denotes $\mathbb{C}^d$ stratified by the general position arrangement $\mathscr{A} = \{H_1, H_2, ..., H_n\}$. In the special case where the arrangement consists of the coordinate hyperplanes in $\mathbb{C}^d$ (i.e., $|\mathscr{A}| = n = d$), we show that the complex $\mathbf{K}^\bullet(\mathbf{I}^{\bar{m}}\mathbf{C}^\bullet(\mathbf{V}))$ is isomorphic to the complex $\mathbf{B}^\bullet$ constructed in [CKS] (see Example 7.8 and Proposition 7.9).

Using the complex $\mathbf{K}^\bullet(\mathbf{I}^{\bar{m}}\mathbf{C}^\bullet(\mathbf{V}))$, we obtain the following generalizations of an observation of Lusztig [Lu, 1.6]: If $\mathbf{V}$ is a nontrivial, rank one local system on the complement of the general position arrangement $\mathscr{A}$ in $\mathbb{C}^d$, then $I^{\bar{m}}H^i(X; \mathbf{V}) = 0$ for $i \neq d$, and

$$\dim IH^d(X; \mathbf{V}) = \begin{cases} \binom{n^*-1}{d} & \text{if } n^* > d \\ 0 & \text{if } n^* \leqslant d, \end{cases}$$

where n^* denotes the number of nontrivial "monodromy" transformations of the local system $\mathbf{V}$ (see Proposition 7.5). If $\mathbf{V}$ is a local system (of arbitrary rank) with the property that one of the monodromy transformation, say T_n, satisfies the condition $\det(Id - T_n) \neq 0$, then $I^{\bar{m}}H^i(X; \mathbf{V}) = 0$ for $i \neq d$, and

$$\dim I^{\bar{m}}H^d(X; \mathbf{V}) = \sum_{\substack{|J| = d \\ n \notin J}} \dim(Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_d}) V,$$

where $J = \{j_1, j_2, ..., j_d\} \subset \{1, 2, ..., n\}$ (see Proposition 7.7).

It has recently been shown [Or1] that the homotopy type of the complement of an arrangement $\mathscr{A}$ is determined by combinatorial data (see also [BZ]). It follows that the cohomology groups of the complement with coefficients in a local system are determined by abstract combinatorial data, which may be quite complicated. In this paper, given an arrangement $\mathscr{A}$ and a local system $\mathbf{V}$ on the complement M of $\mathscr{A}$, we construct a differential functor $\mathscr{F}$ from the lattice L of $\mathscr{A}$ to the category of groups, with the property that $H^*(\mathscr{F}(L))$ is isomorphic to $H^*(M; \mathbf{V})$. The values of $\mathscr{F}$ are explicitly computed using the Möbius function on L. For a general position arrangement, the differentials of the complex $\mathscr{F}(L)$ are given in terms of the local monodromy of the local system $\mathbf{V}$. For an arbitrary arrangement, if the local system is trivial, the functor $\mathscr{F}$ naturally reduces to the "Möbius functor" of Orlik and Solomon [OS].

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1. Weakly Self-Indexing Morse Functions and Arrangements

In this section, we construct a *Morse function* [GM3, I.2] well-suited for the study of arrangements.

Definition 1.1. Let Z be a Whitney stratified subset of Euclidean space. A Morse function $f: Z \to \mathbb{R}$ is said to be *weakly self-indexing* with respect to the stratification $\{S_\alpha\}$ of Z if for each q, $0 \leqslant q \leqslant \dim Z$, we have

$$\max_{\operatorname{codim} S_\alpha = q} \{\text{critical values of } f \mid S_\alpha\}$$

$$< \min_{\operatorname{codim} S_\beta = q+1} \{\text{critical values of } f \mid S_\beta\}.$$

Any arrangement of affine subspaces (of arbitrary dimension) in $\mathbb{R}^d$ gives rise to a Whitney stratification of $\mathbb{R}^d$, with one stratum for each multi-intersection of elements of the arrangement. A complex hyperplane arrangement may be thought of as a real subspace arrangement (with even-dimensional strata). Given an arbitrary subspace arrangement $\mathscr{A}$ in $\mathbb{R}^d$, we now show that there is a weakly self-indexing Morse function on $\mathbb{R}^d$ stratified by $\mathscr{A}$, which has a single minimum on each flat, and no other critical points. We have recently been informed that MacPherson has constructed a similar function in a more general setting [Mac2].

Proposition 1.2. *Let $\mathscr{A}$ be an arrangement of subspaces in $\mathbb{R}^d$. Then there is a positive definite quadratic form $f: \mathbb{R}^d \to \mathbb{R}$, which is a weakly self-indexing Morse function with respect to the Whitney stratification $\{S_\alpha\}$ of $\mathbb{R}^d$ given by $\mathscr{A}$, whose critical points consist of a unique minimum on each stratum.*

Proof. By induction on d, we show that there are positive constants $\omega_1, \omega_2, ..., \omega_d$ such that the positive definite quadratic form

$$f(x_1, x_2, ..., x_d) = \sum_{i=1}^{d} \omega_i x_i^2$$

is a weakly self-indexing Morse function. Such a function will clearly have a unique minimum on each flat (and no other critical points). It follows immediately that the critical points of such a quadratic form are nondegenerate.

The case $d = 1$ may be verified as follows: A subspace arrangement in $\mathbb{R}^1$ is a finite collection of points. For a generic choice of coordinates, the function $f_1: \mathbb{R}^1 \to \mathbb{R}$ defined by $f_1(x_1) = x_1^2$ is a weakly self-indexing Morse function with a unique critical point on each stratum of $\mathbb{R}^1$, since each singular stratum of $\mathbb{R}^1$ is zero-dimensional, hence is a critical point of f_1.

Now inductively assume that the proposition holds for any subspace arrangement in $\mathbb{R}^k$, where $k < d$, and let $\mathscr{A}$ be an arrangement of subspaces in $\mathbb{R}^d$. Choose a hyperplane H that is in *general position* with respect to the arrangement $\mathscr{A}$, i.e.,

$$\dim H \cap S_\alpha = d - 1 - \operatorname{codim} S_\alpha$$

for each stratum S_α, where a negative dimension indicates an empty intersection. Codimension d strata in $\mathbb{R}^d$ are points, say $y^1, ..., y^r$, which by construction, lie in the complement of H. For a generic choice of such a hyperplane, we have

$$\text{distance}(H, y^i) \neq \text{distance}(H, y^j) \qquad \text{if} \quad i \neq j.$$

Choose coordinates $\{x_1, x_2, ..., x_d\}$ so that $H = \text{span}\{x_1, x_2, ..., x_{d-1}\}$.

The hyperplane H inherits a stratification $H \cap \mathscr{A}$ from the original arrangement. By induction, there are constants $\omega_1, \omega_2, ..., \omega_{d-1}$ so that the function $f_{d-1}: H = \mathbb{R}^{d-1} \to \mathbb{R}$, defined by

$$f_{d-1}(x_1, x_2, ..., x_{d-1}) = \sum_{i=1}^{d-1} \omega_i x_i^2$$

is a weakly self-indexing Morse function. Denote by a the maximum of the critical values of f_{d-1}, and let

$$b = \min_{1 \leqslant i \leqslant r} \{1, (y_d^i)^2\}, \qquad \text{where} \quad y^i = (y_1^i, ..., y_d^i).$$

Then choose $\omega_d > a/b$, and define $f = f_d: \mathbb{R}^d \to \mathbb{R}$ by

$$f(x_1, x_2, ..., x_d) = f_{d-1}(x_1, x_2, ..., x_{d-1}) + \omega_d x_d^2.$$

Check that the set $X_{\leqslant a} = \{x \in X \mid f(x) \leqslant a\}$ contains all critical points of f restricted to flats of codimension less than d, and that $y^i \notin X_{\leqslant a}$ for each i. By the choice of the hyperplane H, we have $y_d^i \neq y_d^j$ for $i \neq j$. Using this fact, it is an easy exercise to verify that for ω_d sufficiently large, the critical values $\{f(y^i)\}$ are distinct. In other words, f is a weakly self-indexing Morse function with respect to the codimension d strata. So it remains to show that there is a choice of $\omega_d > a/b$ such that f is weakly self-indexing with respect to the strata of codimension less than d, and has distinct critical values on these strata.

To do this, fix a stratum S of say codimension q. It is sufficient to show that the (unique) critical value of $f \mid S$ tends to the critical value of $f_{d-1} \mid S \cap H$ as ω_d tends to infinity, since f_{d-1} is a weakly self-indexing

Morse function by induction. Suppose that the flat S is defined by the q linear equations $l_1 = l_2 = \cdots = l_q = 0$, where

$$l_i(x_1, x_2, \ldots, x_d) = \beta_i + \sum_{j=1}^{d} \alpha_{i,j} x_j.$$

Using the technique of Lagrange multipliers, we observe that the critical point $x = (x_1, x_2, \ldots, x_d)$ of f restricted to S is given by the first d components of the solution of the system of equations

$$\begin{bmatrix} 2W_d & -A_d^T \\ -A_d & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

where

$$W_k = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_k \end{bmatrix}, \qquad A_k = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{q,1} & \alpha_{q,2} & \cdots & \alpha_{q,k} \end{bmatrix},$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_q \end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{bmatrix}.$$

Similarly, the critical point $y = (y_1, y_2, \ldots, y_{d-1})$ of f_{d-1} restricted to $S \cap H$ may be found by solving the system

$$\begin{bmatrix} 2W_{d-1} & -A_{d-1}^T \\ -A_{d-1} & 0 \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}.$$

We find explicit expressions for the components of the points x and y via Cramer's rule. From these computations, it follows immediately that

$$\lim_{\omega_d \to \infty} (x_1, x_2, \ldots, x_d) = (y_1, y_2, \ldots, y_{d-1}, 0),$$

and consequently that

$$\lim_{\omega_d \to \infty} f(x_1, x_2, \ldots, x_d) = f_{d-1}(y_1, y_2, \ldots, y_{d-1}). \quad \blacksquare$$

Remark 1.3. For any subspace arrangement $\mathscr{A}$ in $\mathbb{R}^d$, the above proof gives an inductive algorithm for the construction of a complete flag in $\mathbb{R}^d$ that is in general position with respect to the arrangement, and a positive definite quadratic form "about" the flag, which is a weakly self-indexing Morse function with respect to the stratification of $\mathbb{R}^d$ determined by the arrangement $\mathscr{A}$.

2. Construction of the Complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ and the Main Theorem

Let $\mathbf{P}^\bullet$ be a complex of sheaves (of complex vector spaces) on $\mathbb{C}^d$ which is constructible with respect to the stratification of $\mathbb{C}^d$ determined by the arrangement $\mathscr{A}$, so that each cohomology sheaf $\mathscr{H}^i(\mathbf{P}^\bullet)$ is locally constant on each stratum $w \in L$. Assume that $\mathbf{P}^\bullet$ has been chosen to be a *perverse* sheaf on $X = \mathbb{C}^d$ stratified by $\mathscr{A}$. Here we use the definition of perverse which has non-zero cohomology only in non-negative dimensions, as in [Mac1].

Fix a weakly self-indexing Morse function $f: X \to \mathbb{R}$, and denote by $X_{\leqslant a}$ the subspace of all points x in X such that $f(x) \leqslant a$. By construction, the Morse function f has a single minimum, x_w, on each flat $w \in L$, and no other critical points. Let $\eta_w = f(x_w)$ denote the critical value of f corresponding to the critical point x_w. For $\varepsilon > 0$ sufficiently small, the level set $X_{\leqslant \eta_w + \varepsilon}$ is obtained from the set $X_{\leqslant \eta_w - \varepsilon}$ by attaching the "Morse data" associated to the stratum w and the Morse function f, which is the product of the tangential and normal Morse data [GM3, I.3]. Since f has a minimum at $x_w \in w$, the tangential Morse data is topologically trivial. The Morse group of the stratum w is defined to be the cohomology of the (normal) Morse data associated to w (see [GM3, II.6.A]).

Definition 2.1. The *Morse group*, $A_w(\mathbf{P}^\bullet)$, associated to the codimension q stratum $w \in L$ and the Morse function f is the relative cohomology group

$$A_w(\mathbf{P}^\bullet) = H^q(X_{\leqslant \eta_w + \varepsilon}, X_{\leqslant \eta_w - \varepsilon}; \mathbf{P}^\bullet).$$

Remarks. 2.1.1. Since the sheaf $\mathbf{P}^\bullet$ is perverse, the groups $H^i(X_{\leqslant \eta_w + \varepsilon}, X_{\leqslant \eta_w - \varepsilon}; \mathbf{P}^\bullet)$ vanish for $i \neq q$.

2.1.2. The Morse group $A_w(\mathbf{P}^\bullet)$ is independent of the number ε used in its definition [GM3, II.6.A], and may be described in terms of the local geometry of X (see Section 4).

2.1.3. If $g: X \to \mathbb{R}$ is another Morse function, then the choice of a path between f and g in the space of all Morse functions on X determines an isomorphism $A_w(\mathbf{P}^\bullet) \approx \tilde{A}_w(\mathbf{P}^\bullet)$, where $\tilde{A}_w(\mathbf{P}^\bullet)$ is the Morse group of w constructed using the function g (compare [GM3, II.6]).

2.1.4. Let $i: M \to X$ denote the natural inclusion. If the perverse sheaf $\mathbf{P}^\bullet$ on X is the direct image, $Ri_*\mathbb{C}^\bullet$, of the constant sheaf on M, then the

Morse group $A_w(\mathbf{P}^\bullet)$ is isomorphic to $\mathbb{C}^{|\mu(w)|}$, where $\mu: L \to \mathbb{Z}$ denotes the Möbius function on the lattice of $\mathscr{A}$.

2.1.5. More generally, if the perverse sheaf $\mathbf{P}^\bullet$ on X is the direct image, $Ri_*\mathbf{V}$, of a local system $\mathbf{V}$ on M, then the Morse group $A_w(\mathbf{P}^\bullet)$ is isomorphic to $V^{|\mu(w)|}$, where V is the stalk of the local system (see Section 5).

2.1.6. We associate the Morse group $A_w(\mathbf{P}^\bullet)$ to the element w of the lattice of $\mathscr{A}$.

For each flat w of positive codimension q in the lattice of $\mathscr{A}$, since the Morse function f is weakly self-indexing, the level set $X_{\leqslant \eta_w - \varepsilon}$ intersects all strata v of X of codimension less than q. Fix real numbers α and β so that

$$\max_{u \in L_{q-2}} \{\eta_u\} < \alpha < \min_{v \in L_{q-1}} \{\eta_v\} \qquad \text{and} \qquad \max_{v \in L_{q-1}} \{\eta_v\} < \beta < \min_{w \in L_q} \{\eta_w\},$$

and consider the level sets $X_{\leqslant \alpha}$ and $X_{\leqslant \beta}$. Since f has precisely one critical point on each flat, the level set $X_{\leqslant \beta}$ is obtained from $X_{\leqslant \alpha}$ by attaching the *disjoint union* of the Morse data corresponding to each stratum of codimension $q-1$, i.e., all $v \in L_{q-1}$. As noted above, the tangential Morse data at each critical point x_v is topologically trivial, so by excision we have

$$H^i(X_{\leqslant \beta}, X_{\leqslant \alpha}; \mathbf{P}^\bullet) = \begin{cases} \bigoplus_{v \in L_{q-1}} A_v(\mathbf{P}^\bullet) & \text{if } i = q-1 \\ 0 & \text{if } i \neq q-1. \end{cases}$$

For each flat $v \in L_{q-1}$, let $j_v^* : A_v(\mathbf{P}^\bullet) \to H^{q-1}(X_{\leqslant \beta}, X_{\leqslant \alpha}; \mathbf{P}^\bullet)$ denote the natural inclusion.

Let X_w denote the space obtained from $X_{\leqslant \beta}$ by attaching the Morse data associated to the codimension q stratum w, and consider the exact sequence of the triple $(X_w, X_{\leqslant \beta}, X_{\leqslant \alpha})$. Using excision, we canonically identify the Morse group of w with the group $H^q(X_w, X_{\leqslant \beta}, \mathbf{P}^\bullet)$, i.e.,

$$A_w(\mathbf{P}^\bullet) = H^q(X_{\leqslant \eta_w + \varepsilon}, X_{\leqslant \eta_w - \varepsilon}; \mathbf{P}^\bullet) = H^q(X_w, X_{\leqslant \beta}; \mathbf{P}^\bullet).$$

Thus the boundary map, Δ_w, of the triple $(X_w, X_{\leqslant \beta}, X_{\leqslant \alpha})$ in degree q is a homomorphism from the direct sum of the Morse groups associated to flats $v \in L_{q-1}$ to the Morse group of w:

$$\bigoplus_{v \in L_{q-1}} A_v(\mathbf{P}^\bullet) = H^{q-1}(X_{\leqslant \beta}, X_{\leqslant \alpha}; \mathbf{P}^\bullet) \xrightarrow{\Delta_w^q} H^q(X_w, X_{\leqslant \beta}; \mathbf{P}^\bullet) = A_w(\mathbf{P}^\bullet).$$

If $v \in L_{q-1}$ satisfies $v \nless w$ (i.e., the flat v does not contain w), then by excising the Morse data associated to v from the sets X_w and $X_{\leqslant \beta}$, we observe that the composition $\Delta_w^q \circ j_v^*$ is trivial.

DEFINITION 2.2. With notation as above, for each stratum $w \in L_q$, and each codimension one inclusion $v < w$, let

$$\Phi_{v,w}: A_v(\mathbf{P}^\bullet) \to A_w(\mathbf{P}^\bullet)$$

denote the composition $\Phi_{v,w} := \Delta_w^q \circ j_v^*$.

Remarks. 2.2.1. The homomorphism Δ_w^q is the only boundary map of the triple $(X_w, X_{\leqslant\beta}, X_{\leqslant\alpha})$ which is not necessarily trivial, since $H^i(X_{\leqslant\beta}, X_{\leqslant\alpha}; \mathbf{P}^\bullet) = 0$ for $i \neq q-1$ and $H^i(X_w, X_{\leqslant\beta}; \mathbf{P}^\bullet) = 0$ for $i \neq q$.

2.2.2. If the perverse sheaf $\mathbf{P}^\bullet$ on X is the direct image of the constant sheaf (or any other trivial local system) on M, then each of the homomorphisms $\Phi_{v,w}$ is trivial (see Section 5).

2.2.3. For general position arrangements, if the preverse sheaf $\mathbf{P}^\bullet$ on X is the direct image of a local system on M, then the homomorphisms $\Phi_{v,w}$ can be explicitly expressed in terms of the local monodromy of the local system (see Section 7).

2.2.4. We associate the homomorphism $\Phi_{v,w}$ to the codimension one inclusion $v < w$ in the lattice of $\mathscr{A}$.

DEFINITION 2.3. Let $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ denote the system of groups

$$\mathbf{K}^q(\mathbf{P}^\bullet) = \bigoplus_{w \in L_q} A_w(\mathbf{P}^\bullet),$$

and homomorphisms

$$\Phi^q = \bigoplus_{w \in L_q} \sum_{\substack{v \in L_{q-1} \\ v < w}} \Phi_{v,w} : \mathbf{K}^{q-1}(\mathbf{P}^\bullet) \to \mathbf{K}^q(\mathbf{P}^\bullet).$$

Our main result is:

THEOREM 2.4. *Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbb{C}^d$, let $\mathbf{P}^\bullet$ be a perverse sheaf on $\mathbb{C}^d$ which is constructible with respect to the stratification determined by $\mathscr{A}$, and let X denote $\mathbb{C}^d$ endowed with this stratification. Then*

(i) *The system of groups and homomorphisms $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ is a complex (i.e., $\Phi \circ \Phi = 0$), and*

(ii) *the cohomology of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ is canonically isomorphic to $H^*(X; \mathbf{P}^\bullet)$, the cohomology of the sheaf $\mathbf{P}^\bullet$.*

Remark 2.5. To prove this result, we use the weakly self-indexing Morse function f to construct a filtration $\{X_i\}$ of X. We then use the homomorphisms in cohomology induced by the inclusions of the level sets of this filtration to construct explicit isomorphisms between the cohomology groups, $H^q(X; \mathbf{P}^\bullet)$, of the sheaf $\mathbf{P}^\bullet$ and the cohomology groups, $H^q(\mathbf{K}^\bullet(\mathbf{P}^\bullet))$, of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$.

Remark 2.6. The complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ depends on the weakly self-indexing Morse function f used in its construction. If $g: X \to \mathbb{R}$ is another such Morse function, and the construction of this section is carried out using g to obtain a complex $\tilde{\mathbf{K}}^\bullet(\mathbf{P}^\bullet)$, then the choice of a path between f and g in the space of all (weakly self-indexing) Morse functions on X determines an isomorphism of complexes $\mathbf{K}^\bullet(\mathbf{P}^\bullet) \approx \tilde{\mathbf{K}}^\bullet(\mathbf{P}^\bullet)$.

3. Proof of Theorem 2.4

Let $\mathscr{A}$ be a hyperplane arrangement in $\mathbb{C}^d$, and let $\mathbf{P}^\bullet$ be a perverse sheaf on $\mathbb{C}^d$ which is constructible with respect to the stratification determined by $\mathscr{A}$. Recall that X denotes $\mathbb{C}^d$ stratified by $\mathscr{A}$. By Proposition 1.2, there is a Morse function $f: X \to \mathbb{R}$, which has a unique minimum on each flat v in the lattice of $\mathscr{A}$, and is weakly self-indexing with respect to the Whitney stratification of X. Construct the system of groups and homomorphisms $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ using this Morse function as in Section 2.

It follows from the fact that f is weakly self-indexing that we can find positive real numbers $\zeta_0 < \zeta_1 < \cdots < \zeta_d$ so that

$$\max_{v \in L_q} \{\text{critical value of } f \mid v\} < \zeta_q < \min_{w \in L_{q+1}} \{\text{critical value of } f \mid w\}.$$

The level sets $X_q = \{x \in X \mid f(x) \leqslant \zeta_q\}$ form a filtration of X:

$$\varnothing \subset X_0 \subset X_1 \subset \cdots \subset X_d \subset X.$$

The set X_0 is contractible, and contains none of the singular strata of X, so $H^i(X_0; \mathbf{P}^\bullet) = 0$ for $i > 0$, since the sheaf $\mathbf{P}^\bullet$ is locally constant. Also, the set X_d is a stratum-preserving retraction of X, hence $H^*(X, X_d; \mathbf{P}^\bullet) = 0$.

As discussed in Section 2, since each flat contains precisely one critical point (a minimum) of f, one of the level sets X_q of the above filtration of X is obtained from the set X_{q-1} by attaching the disjoint union of the Morse data corresponding to each of the strata of codimension q, i.e., all $w \in L_q$. Since all critical points of f are minima, the tangential Morse data associated to each stratum are topologically trivial, so by excision we have

$$H^i(X_q, X_{q-1}; \mathbf{P}^\bullet) = \begin{cases} \bigoplus_{w \in L_q} A_w(\mathbf{P}^\bullet) & \text{if } i = q \\ 0 & \text{if } i \neq q. \end{cases}$$

Furthermore, since the Morse group $A_w(\mathbf{P}^\bullet)$ associated to the flat $w \in L$ is the cohomology of the Morse data of w (for the Morse function f), we canonically identify the groups $H^q(X_q, X_{q-1}; \mathbf{P}^\bullet)$ and $\mathbf{K}^q(\mathbf{P}^\bullet)$.

By the above observations, for each q the only nontrivial terms in the exact sequence of the triple (X_q, X_{q-1}, X_{q-2}) occur in degrees $q-1$ and q:

$$0 \longrightarrow H^{q-1}(X_q, X_{q-2}; \mathbf{P}^\bullet) \longrightarrow H^{q-1}(X_{q-1}, X_{q-2}; \mathbf{P}^\bullet)$$

$$\xrightarrow{\Delta^q} H^q(X_q, X_{q-1}; \mathbf{P}^\bullet) \longrightarrow H^q(X_q, X_{q-2}; \mathbf{P}^\bullet) \longrightarrow 0.$$

Checking that the boundary homomorphisms of these triples satisfy $\Delta^q \circ \Delta^{q-1} = 0$ for each q, by splicing these sequences together we obtain a complex $\mathbf{E}^\bullet$:

$$0 \longrightarrow H^0(X_0; \mathbf{P}^\bullet) \xrightarrow{\Delta^1} H^1(X_1, X_0; \mathbf{P}^\bullet) \xrightarrow{\Delta^2} H^2(X_2, X_1; \mathbf{P}^\bullet)$$

$$\xrightarrow{\Delta^3} \cdots \xrightarrow{\Delta^{d-1}} H^{d-1}(X_{d-1}, X_{d-2}; \mathbf{P}^\bullet) \xrightarrow{\Delta^d} H^d(X_d, X_{d-1}; \mathbf{P}^\bullet) \longrightarrow 0.$$

We now construct explicit isomorphisms between the cohomology groups $H^q(X; \mathbf{P}^\bullet)$ and $H^q(\mathbf{E}^\bullet)$.

Since $H^i(X_0; \mathbf{P}^\bullet) = 0$ for $i > 0$ and $H^i(X_q, X_{q-1}; \mathbf{P}^\bullet) = 0$ for $i \neq q$, an easy inductive argument shows that for each q, we have $H^i(X_q; \mathbf{P}^\bullet) = 0$ for $i > q$. Also, by successively considering the exact sequence of the triple $(X_{p+q+1}, X_{p+q}, X_q)$ for $p = 1, 2, ..., d-q-1$, we observe that $H^i(X, X_q; \mathbf{P}^\bullet) = 0$ for $i \leqslant q$.

Let $\imath_q : H^q(X; \mathbf{P}^\bullet) \to H^q(X_q; \mathbf{P}^\bullet)$ and $\jmath_q : H^q(X_q, X_{q-1}; \mathbf{P}^\bullet) \to H^q(X_q; \mathbf{P}^\bullet)$ denote the maps in cohomology induced by the inclusions of X_q into X and (X_q, X_{q-1}), respectively. Since $H^q(X_{q-1}; \mathbf{P}^\bullet) = 0$, the homomorphism $\jmath_q$ is surjective. Also, the map $\imath_q$ is injective since $H^q(X, X_q; \mathbf{P}^\bullet) = 0$. Using these results, together with the fact that the differential Δ^q of the complex $\mathbf{E}^\bullet$ (the boundary map of the triple (X_q, X_{q-1}, X_{q-2})) is equal to the composition

$$H^{q-1}(X_{q-1}, X_{q-2}; \mathbf{P}^\bullet) \xrightarrow{\jmath_{q-1}} H^{q-1}(X_{q-1}; \mathbf{P}^\bullet) \xrightarrow{\delta^q} H^q(X_q, X_{q-1}; \mathbf{P}^\bullet),$$

where δ^q denotes the boundary map of the pair (X_q, X_{q-1}), it is a straightforward exercise in homological algebra to show that

$$\jmath_q^{-1} \circ \imath_q : H^q(X : \mathbf{P}^\bullet) \to H^q(\mathbf{E}^\bullet)$$

is an isomorphism.

As noted above, the terms $\mathbf{E}^q$ of the complex $\mathbf{E}^\bullet$ are canonically identified with the groups $\mathbf{K}^q(\mathbf{P}^\bullet)$:

$$\mathbf{E}^q = H^q(X_q, X_{q-1}; \mathbf{P}^\bullet) = \bigoplus_{w \in L_q} A_w(\mathbf{P}^\bullet) = \mathbf{K}^q(\mathbf{P}^\bullet).$$

Hence we complete the proof of the theorem by showing that the differential of the complex $\mathbf{E}^{\bullet}$ decomposes as the direct sum

$$\Delta^q = \bigoplus_{w \in L_q} \sum_{\substack{v \in L_{q-1} \\ v < w}} \Phi_{v,w} = \Phi^q.$$

Let X_w denote the space obtained from the level set X_{q-1} by attatching the Morse data associated to the codimension q flat $w \in L$, and consider the exact sequence of the triple (X_q, X_w, X_{q-1}). The only nontrivial cohomology groups occur in degree q, and we have

$$0 \longrightarrow H^q(X_q, X_w; \mathbf{P}^{\bullet}) \longrightarrow H^q(X_q, X_{q-1}; \mathbf{P}^{\bullet}) \xrightarrow{i_w^*} H^q(X_w, X_{q-1}; \mathbf{P}^{\bullet}) \longrightarrow 0,$$

where i_w^* is induced by the inclusion $i_w: (X_w, X_{q-1}) \to (X_q, X_{q-1})$. Since

$$H^q(X_w, X_{q-1}; \mathbf{P}^{\bullet}) = A_w(\mathbf{P}^{\bullet}), \qquad H^q(X_q, X_{q-1}; \mathbf{P}^{\bullet}) = \mathbf{K}^q(\mathbf{P}^{\bullet}),$$

and

$$H^q(X_q, X_w; \mathbf{P}^{\bullet}) = \mathbf{K}^q(\mathbf{P}^{\bullet}) - A_w(\mathbf{P}^{\bullet}),$$

the homomorphism i_w^* is clearly the natural projection $\mathbf{K}^q(\mathbf{P}^{\bullet}) \twoheadrightarrow A_w(\mathbf{P}^{\bullet})$. Using these projections, the differential of the complex $\mathbf{E}^{\bullet}$ may be written as the sum

$$\Delta^q = \bigoplus_{w \in L_q} i_w^* \circ \Delta^q : \mathbf{K}^{q-1}(\mathbf{P}^{\bullet}) \to \mathbf{K}^q(\mathbf{P}^{\bullet}).$$

We identify each of the homomorphisms $i_w^* \circ \Delta^q$ with the boundary map of the triple (X_w, X_{q-1}, X_{q-2}).

For each codimension $q-1$ stratum v, let X^v denote the space obtained from the level set X_{q-2} by attatching the Morse data corresponding to all codimension $q-1$ flats *except* v. Then $H^{q-1}(X_{q-1}, X^v; \mathbf{P}^{\bullet}) = A_v(\mathbf{P}^{\bullet})$, and we have a canonical isomorphism

$$\sum_{v \in L_{q-1}} j_v^* : \bigoplus_{v \in L_{q-1}} H^{q-1}(X_{q-1}, X^v; \mathbf{P}^{\bullet}) \to H^{q-1}(X_{q-1}, X_{q-2}; \mathbf{P}^{\bullet}),$$

where each of the homomorphisms j_v^* (induced by the inclusion $j_v: (X_{q-1}, X_{q-2}) \to (X_{q-1}, X^v)$) is the natural inclusion of the Morse group $A_v(\mathbf{P}^{\bullet})$ into $\mathbf{K}^{q-1}(\mathbf{P}^{\bullet}) = H^{q-1}(X_{q-1}, X_{q-2}; \mathbf{P}^{\bullet})$.

Thus the differential Δ^q decomposes as the sum

$$\Delta^q = \bigoplus_{w \in L_q} \sum_{v \in L_{q-1}} i_w^* \circ \Delta^q \circ j_v^* : \bigoplus_{v \in L_{q-1}} A_v(\mathbf{P}^{\bullet}) \to \bigoplus_{w \in L_q} A_w(\mathbf{P}^{\bullet}).$$

Each of the summands $i_w^* \circ \Delta^q \circ j_v^*$ is the boundary map of the triple (X_w, X_{q-1}, X^v). If v is a codimension $q-1$ stratum such that $v \nless w$, then by excising the Morse data associated to v from X_w and X_{q-1}, we observe that the homomorphism $i_w^* \circ \Delta^q \circ j_v^*$ is trivial. If $v < w$ is a codimension one inclusion, then we have

$$i_w^* \circ \Delta^q \circ j_v^* = \Phi_{v,w} : A_v(\mathbf{P}^\bullet) \to A_w(\mathbf{P}^\bullet)$$

(see Definition 2.2). Hence the differential of the complex $\mathbf{E}^\bullet$ may be written as

$$\Delta^q = \bigoplus_{w \in L_q} \sum_{\substack{v \in L_{q-1} \\ v < w}} \Phi_{v,w} = \Phi^q,$$

and the system of groups and homomorphisms $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ is canonically identified with the complex $\mathbf{E}^\bullet$.

4. Local Geometry of Arrangements

Throughout this section, let $w \in L$ denote a fixed flat of positive codimension q, and let $f : X \to \mathbb{R}$ be a weakly self-indexing Morse function. We recall [GM2, 3; GM3, II.2.2] the construction of the complex link and related spaces associated to the stratum w of X, and discuss the connection between these spaces and the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$.

Let p be the unique critical point of the restriction of f to w, and choose an affine subspace N of $\mathbb{C}^d$ meeting w transversely at p. (The dimension of the *normal slice* N will necessarily be $q = \operatorname{codim}(w)$, and N inherits a Whitney stratification $N \cap \mathscr{A}$ from X.) Let $\pi : N \to \mathbb{C}$ be the restriction of a linear projection such that $\pi(p) = 0$, $\operatorname{Re}(\pi)$ is a Morse function near p, and $d(\operatorname{Re}(\pi))(p) = df(p)$. Let $B_\delta(p)$ denote the closed ball of radius $\delta > 0$ about p in $\mathbb{C}^d$, and choose δ so small that for all $\delta' \leqslant \delta$, the boundary $\partial B_{\delta'}(p)$ is transverse to each stratum of N. Choose $\varepsilon > 0$ so small that for any $\zeta \in D_\varepsilon(0)$ in $\mathbb{C}$, $\pi^{-1}(\zeta)$ is transverse to each stratum of $N \cap B_\delta(p)$, with the single exception that $\pi^{-1}(0)$ fails to be transverse to the stratum p.

Definition 4.1. Associated to the flat $w \in L_q$, define the *complex link*, $\mathscr{L}$, and its boundary, $\partial\mathscr{L}$, by

$$\mathscr{L} = \pi^{-1}(\xi) \cap N \cap B_\delta(p) \qquad \text{and} \qquad \partial\mathscr{L} = \pi^{-1}(\xi) \cap N \cap \partial B_\delta(p),$$

where $\xi = \varepsilon + 0\sqrt{-1} \in \mathbb{C}$; the *cylindrical neighborhood* by $C = \pi^{-1}(D_\varepsilon(0)) \cap N \cap B_\delta(p)$; and the *cut off space* by $C_{<0} = \pi^{-1}\{\zeta \mid \operatorname{Re}(\zeta) < 0\} \cap C$.

Remarks. 4.1.1. The topological type of each of the above spaces is independent of choices of the normal slice N, and the numbers ε and δ [GM3, II.2].

4.1.2. If $\tilde{\pi}: N \to \mathbb{C}$ is the restriction of another linear projection such that $\tilde{\pi}(p) = 0$ and $\mathrm{Re}(\tilde{\pi})$ is a Morse function near p, but $d(\mathrm{Re}(\tilde{\pi}))(p) \neq df(p)$, then the choice of a path between π and $\tilde{\pi}$ in the space of all such projections determines stratum-preserving homeomorphisms $\mathscr{L} \cong \tilde{\mathscr{L}}$ and $C \cong \tilde{C}$, where $\tilde{\mathscr{L}}$ and $\tilde{C}$ are the complex link and cylindrical neighborhood defined using $\tilde{\pi}$.

4.1.3. By [GM2, 3], the Morse group $A_w(\mathbf{P}^{\bullet})$ is canonically identified with the group $H^q(C, C_{<0}; \mathbf{P}^{\bullet})$.

4.1.4. Since the sheaf $\mathbf{P}^{\bullet}$ is perverse, the groups $H^i(C, C_{<0}; \mathbf{P}^{\bullet})$ vanish for $i \neq q$, and the groups $H^i(C_{<0}; \mathbf{P}^{\bullet})$ vanish for $i \geqslant q$ (see Proposition 4.4(ii)).

4.1.5. There is a stratum-preserving homeomorphism between the interior of the cylindrical neighborhood C and $\mathbb{C}^q$ stratified by the central hyperplane arrangement $\{v \in L \mid v \leqslant w\}$. Similarly, there is a stratum-preserving homeomorphism between the interior of the complex link $\mathscr{L}$ and $\mathbb{C}^{q-1}$ stratified by the affine hyperplane arrangement $\{v \in L \mid v < w\}$.

DEFINITION 4.2. The *variation map* associated to the codimension q stratum $w \in L$ is the boundary homomorphism of the exact sequence of the pair $(C - \{p\}, C_{<0})$:

$$\mathrm{var}_w : H^{q-1}(C_{<0}; \mathbf{P}^{\bullet}) \to H^q(C - \{p\}, C_{<0}; \mathbf{P}^{\bullet}).$$

Remarks. 4.2.1. Closely related to the variation map is the boundary homomorphism of the pair $(C, C_{<0})$. By Remark 4.1.4 above, the only nontrivial boundary map occurs in degree q:

$$\delta_w := \delta^q : H^{q-1}(C_{<0}; \mathbf{P}^{\bullet}) \to H^q(C, C_{<0}; \mathbf{P}^{\bullet}).$$

The relation between this homomorphism and the variation is given by the following commutative diagram

$$\begin{array}{ccc} & H_q(C, C_{<0}; \mathbf{P}^{\bullet}) & \\ {}^{\delta_w}\nearrow & & \searrow^{j^*} \\ H^{q-1}(C_{<0}; \mathbf{P}^{\bullet}) & \xrightarrow{\mathrm{var}_w} & H^q(C - \{p\}, C_{<0}; \mathbf{P}^{\bullet}) \end{array}$$

in which the map j^* is induced by the inclusion $j: (C - \{p\}, C_{<0}) \to (C, C_{<0})$.

4.2.2. Since the boundary homomorphism of the pair $(C, C_{<0})$ vanishes for $i \neq q$, the variation map is the only boundary map of the pair $(C - \{p\}, C_{<0})$ which is not necessarily trivial. This fact and Remark 4.1.4 imply that the groups $H^i(C - \{p\}, C_{<0}; \mathbf{P}^\bullet)$ vanish for $i < q$.

DEFINITION 4.3. Let $\mathbf{K}^\bullet_{\leqslant w}(\mathbf{P}^\bullet)$ denote the system of groups

$$\mathbf{K}^i_{\leqslant w}(\mathbf{P}^\bullet) = \bigoplus_{\substack{v \in L_i \\ v \leqslant w}} A_v(\mathbf{P}^\bullet),$$

and homomorphisms

$$\Phi^i_w = \bigoplus_{\substack{v \in L_i \\ v \leqslant w}} \sum_{\substack{u \in L_{i-1} \\ u < v}} \Phi_{u,v} : \mathbf{K}^{i-1}_{\leqslant w}(\mathbf{P}^\bullet) \to \mathbf{K}^i_{\leqslant w}(\mathbf{P}^\bullet).$$

Similarly, let $\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)$ denote the system of groups $\mathbf{K}^i_{\leqslant w}(\mathbf{P}^\bullet)$ and homomorphisms Φ^i_w satisfying $i < q$.

We refer to the homomorphism Φ^i_w as the component of the differential of $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ associated to the stratum w. Notice that the groups $\mathbf{K}^i_{\leqslant w}(\mathbf{P}^\bullet)$ vanish for $i > q$, and that $\mathbf{K}^\bullet_{\leqslant w}(\mathbf{P}^\bullet)$ and $\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)$ are subcomplexes of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ by construction.

PROPOSITION 4.4. (i) *The cohomology of the complex* $\mathbf{K}^\bullet_{\leqslant w}(\mathbf{P}^\bullet)$ *is canonically isomorphic to* $H^*(C; \mathbf{P}^\bullet)$, *the cohomology of the restriction of the sheaf* $\mathbf{P}^\bullet$ *to the cylindrical neighborhood* C *of the flat* w.

(ii) *The cohomology of the complex* $\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)$ *is canonically isomorphic to* $H^*(C_{<0}; \mathbf{P}^\bullet)$, *the cohomology of the restriction of the sheaf* $\mathbf{P}^\bullet$ *to the cut off space* $C_{<0}$ *of the flat* w.

Proof. We prove (i). The argument for (ii) is analogous.

The cylindrical neighborhood C is a stratum-preserving retraction of the space $\pi^{-1}(D_\varepsilon(0)) \cap N$. As noted in Remark 4.1.5, there is a stratum-preserving homeomorphism between this space and $\mathbb{C}^q$ stratified by the (hyperplane) arrangement $\{v \in L \mid v \leqslant w\}$. Since the restriction of the weakly self-indexing Morse function f to the space $\pi^{-1}(D_\varepsilon(0)) \cap N$ is again a weakly self-indexing Morse function, by applying the techniques of Sections 2 and 3 to this last space, we obtain the desired result. ∎

COROLLARY 4.5. *The cohomology of the complex* $\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)$ *is isomorphic to* $H^*(\mathscr{L}; \mathbf{P}^\bullet)$, *the cohomology of the restriction of the sheaf* $\mathbf{P}^\bullet$ *to the complex link* $\mathscr{L}$ *of the stratum* w.

Proof. By Proposition 4.4(ii), the cohomology of $\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)$ is isomorphic to $H^*(C_{<0}; \mathbf{P}^\bullet)$. Using the techniques of [GM2, 3], for each i, we canonically identify the groups $H^i(C_{<0};\mathbf{P}^\bullet)$ and $H^i(\mathscr{L}^-;\mathbf{P}^\bullet)$, where $\mathscr{L}^- = \pi^{-1}(-\xi)\cap N\cap B_\delta(p)$. A choice of path from $-\xi = -\varepsilon+0\sqrt{-1}$ to $\xi=\varepsilon+0\sqrt{-1}$ in $\mathbb{C}-\{0\}$ determines a homeomorphism $\mathscr{L}^-\to\mathscr{L}$. Using the induced isomorphism to identify the groups $H^i(\mathscr{L}^-;\mathbf{P}^\bullet)$ and $H^i(\mathscr{L};\mathbf{P}^\bullet)$ for each i, we have

$$H^i(\mathscr{L};\mathbf{P}^\bullet)\approx H^i(\mathscr{L}^-;\mathbf{P}^\bullet)=H^i(C_{<0};\mathbf{P}^\bullet)=H^i(\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet)). \quad \blacksquare$$

Remark 4.6. Using Proposition 4.4(ii), for each i we canonically identify the group $H^i(C_{<0};\mathbf{P}^\bullet)$ with $H^i(\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet))$, and the boundary homomorphism δ_w of the pair $(C, C_{<0})$ as

$$\delta_w\colon H^{q-1}(\mathbf{K}^\bullet_{<w}(\mathbf{P}^\bullet))=\operatorname{coker}\Phi_w^{q-1}\to A_w(\mathbf{P}^\bullet)=H^q(C, C_{<0};\mathbf{P}^\bullet).$$

The relation between the homomorphsm δ_w and the component Φ_w^q of the differential of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ associated to the flat w is given by the following commutative diagram

$$\begin{array}{ccc} & \operatorname{coker}\Phi_w^{q-1} & \\ {}^{\pi_w}\nearrow & & \searrow^{\delta_w} \\ \bigoplus_{\substack{v\in L_{q-1}\\ v<w}} A_v(\mathbf{P}^\bullet)=\mathbf{K}^{q-1}_{<w}(\mathbf{P}^\bullet) & \xrightarrow{\Phi_w^q} & A_w(\mathbf{P}^\bullet) \end{array}$$

where $\pi_w\colon \mathbf{K}^{q-1}_{<w}(\mathbf{P}^\bullet)\twoheadrightarrow\operatorname{coker}\Phi_w^{q-1}$ denotes the natural projection.

It follows that the differential Φ^q of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ may be realized as the sum, over all codimension q flats w, of lifts Φ_w^q of the homomorphisms δ_w to the groups $\mathbf{K}^{q-1}_{<w}(\mathbf{P}^\bullet)$. Furthermore, each of these homomorphisms Φ_w^q is the direct sum, over all codimension one inclusions $v<w$, of the homomorphisms $\Phi_{v,w}\colon A_v(\mathbf{P}^\bullet)\to A_w(\mathbf{P}^\bullet)$ of Definition 2.2.

5. Consequences in Ordinary Cohomology

Recall that M denotes the complement of the arrangement $\mathscr{A}$ in $\mathbb{C}^d$, and that $i\colon M\to X=\mathbb{C}^d$ stratified by $\mathscr{A}$ denotes the natural inclusion. The manifold M is the nonsingular stratum of X. Given a basepoint x_0 in M, a complex representation

$$\rho\colon \pi_1(M, x_0)\to \operatorname{Aut}(V)$$

of the fundamental group of M gives rise to a local coefficient system of complex vector spaces on M, which we denote by $\mathbf{V}$. Since M is the complement of a hypersurface in the manifold $\mathbb{C}^d$, it follows from [LM, 4.6] and duality that the (derived) direct image of the local system $\mathbf{V}$ on M is a perverse sheaf, $\mathbf{P}^{\bullet} := Ri_{*}\mathbf{V}$, on X. The cohomology of M with coefficients in $\mathbf{V}$ is canonically isomorphic to the cohomology of the sheaf $\mathbf{P}^{\bullet}$, so using Theorem 2.4 we obtain:

COROLLARY 5.1. *Let $\mathbf{V}$ be a local system of coefficients on the complement M. Then the cohomology of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$, where $\mathbf{P}^{\bullet} = Ri_{*}\mathbf{V}$, is canonically isomorphic to $H^{*}(M; \mathbf{V})$, the cohomology of M with coefficients in the local system $\mathbf{V}$.*

In this instance, it is possible to give explicit combinatorial descriptions of the terms of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$. First we establish some notation.

On the lattice L of the arrangement $\mathscr{A}$, define the Möbius function $\mu: L \to \mathbb{Z}$ recursively by $\mu(\mathbb{C}^{d}) = 1$, and for $w > \mathbb{C}^{d}$,

$$\mu(w) = -\sum_{v < w} \mu(v).$$

PROPOSITION 5.2. *Let $\mathbf{V}$ be a local system of coefficients on the complement M, and let $\mathbf{P}^{\bullet} = Ri_{*}\mathbf{V}$ be the direct image of $\mathbf{V}$ on X. Then the Morse group $A_{w}(\mathbf{P}^{\bullet})$ associated to the flat $w \in L$ is isomorphic to $V^{|\mu(w)|}$, where V denotes the stalk of the local system $\mathbf{V}$ at the basepoint $x_0 \in M$.*

Proof. Let $f: X \to \mathbb{R}$ be a weakly self-indexing Morse function, and construct the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$ using f as in Section 2. We use induction on the codimension, q, of the flat w. The case $q = 0$ is trivial since, as we noted in the proof of Theorem 2.4, the Morse group corresponding to the unique codimension 0 stratum, $\mathbb{C}^{d}$, is the cohomology of a (nonsingular) disk, and $\mu(\mathbb{C}^{d}) = 1$.

Let w be a flat of positive codimension q, and denote by p the unique minimum of the restriction of f to w. Let $\eta = f(p)$ be the corresponding critical value. For $\varepsilon > 0$ sufficiently small, the Morse group $A_{w}(\mathbf{P}^{\bullet})$ is given by $H^{q}(X_{\leqslant \eta+\varepsilon}, X_{\leqslant \eta-\varepsilon}; \mathbf{P}^{\bullet})$, and as in Section 4, we have

$$A_{w}(\mathbf{P}^{\bullet}) = H^{q}(X_{\leqslant \eta+\varepsilon}, X_{\leqslant \eta-\varepsilon}; \mathbf{P}^{\bullet}) = H^{q}(C, C_{<0}; \mathbf{P}^{\bullet}),$$

where C and $C_{<0}$ are the cylindrical neighborhood and cut off space of the stratum w. Since $\mathbf{P}^{\bullet} = Ri_{*}\mathbf{V}$ is the direct image of $\mathbf{V}$, we have

$$H^{q}(C, C_{<0}; \mathbf{P}^{\bullet}) = H^{q}(M \cap C, M \cap C_{<0}; \mathbf{V}).$$

Using this fact, the case $q = 1$ is straightforward, since one can check that the pair $(M \cap C, M \cap C_{<0})$ associated to a codimension one flat (i.e., a hyperplane) has the homotopy type of the pair $(I, \partial I)$, where I denotes an interval (along which the local system is necessarily trivial), and for any hyperplane $H \in \mathscr{A}$, we have $|\mu(H)| = 1$.

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Inductively assume that the proposition holds for all strata of codimension less than d, and let $w \in L_d$. By working within the cylindrical neighborhood of the point w, we may assume that w is the only codimension d flat, i.e., the unique vertex of the *central* arrangement $\mathscr{A}$.

By Corollary 5.1, the cohomology of M with coefficients in $\mathbf{V}$ is isomorphic to the cohomology of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$. The terms of this complex are direct sums of the Morse groups, so by induction we have

$$\mathbf{K}^q(\mathbf{P}^\bullet) = \bigoplus_{v \in L_q} A_v(\mathbf{P}^\bullet) \approx \bigoplus_{v \in L_q} V^{|\mu(v)|}, \qquad \text{for} \quad q < d.$$

Consequently, we may express the Euler characteristic of $H^*(M; \mathbf{V})$ as

$$\begin{aligned}\chi(M; \mathbf{V}) &= \sum_{q=0}^{d} (-1)^q \dim \mathbf{K}^q(\mathbf{P}^\bullet) \\ &= (-1)^d \dim A_w(\mathbf{P}^\bullet) + \sum_{q=0}^{d-1} \sum_{v \in L_q} (-1)^q \, |\mu(v)| \cdot \dim V.\end{aligned}$$

It is not difficult to verify [Or2, 5.3] that the complement M of a central arrangement in $\mathbb{C}^d$ can be realized as a product $M = \mathbb{C}^* \times M^*$, where M^* is the complement of an (affine) arrangement in $\mathbb{C}^{d-1}$. Thus, $\chi(M; \mathbf{V}) = 0$, and

$$\dim A_w(\mathbf{P}^\bullet) = \left| \sum_{q=0}^{d-1} \sum_{v \in L_q} (-1)^q \, |\mu(v)| \cdot \dim V \right|.$$

The result now follows directly from the definition of the Möbius function. ∎

Remark 5.3. Let w be a stratum of positive codimension q. Since the sheaf $\mathbf{P}^\bullet$ on X is the direct image of the local system $\mathbf{V}$ on M, we have

$$A_w(\mathbf{P}^\bullet) = H^q(C, C_{<0}; \mathbf{P}^\bullet) = H^q(M \cap C, M \cap C_{<0}; \mathbf{V})$$

and

$$H^q(C - \{p\}, C_{<0}; \mathbf{P}^\bullet) = H^q(M \cap (C - \{p\}), M \cap C_{<0}; \mathbf{V}),$$

where C and $C_{<0}$ are the cylindrical neighborhood and cut off space of w at p. The spaces $M \cap C$ and $M \cap (C - \{p\})$ are clearly idential, so it follows from the above canonical identitications that

$$H^q(C, C_{<0}; \mathbf{P}^\bullet) = H^q(C - \{p\}, C_{<0}; \mathbf{P}^\bullet).$$

Thus when $\mathbf{P}^\bullet = Ri_*\mathbf{V}$ is the direct image of the local system $\mathbf{V}$, the variation map var_w and the boundary homomorphism δ_w of the pair $(C, C_{<0})$ are identical (see Remarks 4.2.1 and 4.6).

Remark 5.4. If the local coefficient system $\mathbf{V}$ on M is trivial, then a modification of the techniques of [GM3, III.3] shows that, for every critical point p with corresponding critical value $\eta = f(p)$, the long exact sequence of the pair $(X_{\leqslant \eta + \varepsilon}, X_{\leqslant \eta - \varepsilon})$ splits into short exact sequences. It follows that for each flat w, the variation map, var_w, in ordinary cohomology is trivial. Hence in this case, the differentials in the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$ all vanish, and the cohomology groups of M with coefficients in $\mathbf{V}$ are precisely the terms of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$. Using Proposition 5.2, we recover the formula for the Betti numbers of the complement due to Orlik and Solomon [OS].

Remark 5.5. Using a technical Morse-theoretic argument, one can show that, for any flat $w \in L$, the homology groups $H_i(M \cap C, M \cap C_{<0}; \mathbf{V})$ vanish in all degrees but one (the codimension q of the stratum w), and that

$$A_w(\mathbf{V}) := H_q(M \cap C, M \cap C_{<0}; \mathbf{V}) \approx V^{|\mu(w)|},$$

where V is the stalk of the local system. By modifying the construction of Section 2 and the proof of Theorem 2.4, one may construct a (chain) complex $\mathbf{K}_{\bullet}(\mathbf{V})$, whose terms are given by direct sums of Morse groups $A_w(\mathbf{V})$ in homology with local coefficients,

$$\mathbf{K}_q(\mathbf{V}) = \bigoplus_{w \in L_q} A_w(\mathbf{V}) \approx \bigoplus_{w \in L_q} V^{|\mu(w)|},$$

the homology of which is isomorphic to $H_*(M; \mathbf{V})$, the homology of M with coefficients in $\mathbf{V}$. As in Remarks 4.6 and 5.3, the differential $\Phi_q \colon \mathbf{K}_q(\mathbf{V}) \to \mathbf{K}_{q-1}(\mathbf{V})$ of this complex may be realized as the sum over all codimension q flats w of the variation maps

$$\mathrm{var}_w \colon A_w(\mathbf{V}) = H_q(M \cap C, M \cap C_{<0}; \mathbf{V}) \to H_{q-1}(M \cap C_{<0}; \mathbf{V}) \hookrightarrow \mathbf{K}_{q-1}(\mathbf{V}),$$

in homology with local coefficients (compare [GM3, II.6.3]).

If the local coefficient system $\mathbf{V}$ on M is trivial, the argument of [GM3, III.3] shows that each of these variation maps var_w in homology with local coefficients is trivial. It follows that all of the differentials Φ_q vanish, and that the terms of the complex $\mathbf{K}_{\bullet}(\mathbf{V})$ are the homology groups of M with coefficients in the (trivial) local system $\mathbf{V}$.

6. Consequences in Intersection Cohomology

Intersection cohomology with coefficients in a local system requires only that a local system be defined on the nonsingular stratum of the singular space in question [GM1, 2.2]. So, given a local system $\mathbf{V}$ on the complement M of $\mathscr{A}$, the smooth stratum of $X = \mathbb{C}^d$ stratified by $\mathscr{A}$, we now study the intersection cohomology of X with coefficients in $\mathbf{V}$.

Let $\bar{p}$ be a perversity which lies between the "sub-logarithmic" ($p(2k) = k - 2$) and the "logarithmic" ($p(2k) = k$) perversities. Such perversities are those for which the intersection cohomology sheaves, $\mathbf{I}^{\bar{p}}\mathbf{C}^{\bullet}(\mathbf{V})$, are perverse [BBD]. In this setting, Theorem 2.4 yields:

COROLLARY 6.1. *Let $\mathbf{V}$ be a local system of coefficients on the complement M of $\mathscr{A}$ in $\mathbb{C}^d$. Let $\mathbf{P}^{\bullet} = \mathbf{I}^{\bar{p}}\mathbf{C}^{\bullet}(\mathbf{V})$ denote the intersection cohomology sheaf on X. Then the cohomology of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$ is canonically isomorphic to $I^{\bar{p}}H^*(X; \mathbf{V})$, the intersection cohomology of X with coefficients in $\mathbf{V}$.*

In particular, Corollary 6.1 holds when $\mathbf{P}^{\bullet} = \mathbf{I}^{\bar{m}}\mathbf{C}^{\bullet}(\mathbf{V})$ is the "middle" intersection cohomology sheaf ($m(2k) = k - 1$). Unless otherwise noted, we subsequently refer to middle intersection cohomology as merely intersection cohomology, and write $\mathbf{IC}^{\bullet}$ and IH^* in place of $\mathbf{I}^{\bar{m}}\mathbf{C}^{\bullet}$ and $I^{\bar{m}}H^*$. In this case, the terms and differentials of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$ may be described in terms of the local geometry of X.

Remark 6.2. Let $w \in L$ be a codimension $q > 0$ stratum of X, and let p be the unique critical point of the restriction of a weakly self-indexing Morse function f to w. The Morse group $A_w(\mathbf{P}^{\bullet})$ in intersection cohomology associated to the flat w is canonically isomorphic to the image of the variation map in intersection cohomology,

$$A_w(\mathbf{P}^{\bullet}) = \mathrm{Image}(\mathrm{var}_w : IH^{q-1}(C_{<0}; \mathbf{V}) \to IH^{q-1}(C - \{p\}, C_{<0}; \mathbf{V}))$$

(compare Section 4, and [GM2, II.6.3, II.6.4]), where C and $C_{<0}$ are the cylindrical neighborhood and cut off space of w at p. Thus the terms of the complex $\mathbf{K}^{\bullet}(\mathbf{P}^{\bullet})$ are given by

$$\mathbf{K}^q(\mathbf{P}^{\bullet}) = \bigoplus_{w \in L_q} \mathrm{Image}(\mathrm{var}_w).$$

Furthermore, as in Remark 4.2.1, we have the following commutative diagram

$$\begin{array}{ccc} & IH^q(C, C_{<0}; \mathbf{V}) & \\ {}^{\delta_w}\nearrow & & \searrow^{=} \\ IH^{q-1}(C_{<0}; \mathbf{V}) & \xrightarrow{\mathrm{var}_w} & \mathrm{Image}(\mathrm{var}_w). \end{array}$$

Thus the boundary homomorphism δ_w of the pair $(C, C_{<0})$ is equal to the variation map in intersection cohomology, and is surjective. It follows that the component Φ_w^q of the differential of $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ associated to the flat w is also surjective in this case (see Remark 4.6).

The Morse group in intersection cohomology at a nonsingular point is given by the stalk V of the local system $\mathbf{V}$ at that point (compare [GM3, II.6.4]). Hence $\mathbf{K}^0(\mathbf{P}^\bullet) = V$.

Recall that $i: M \to X$ denotes the natural inclusion. We subsequently denote the direct image of the local system $\mathbf{V}$ by $\mathbf{Q}^\bullet := Ri_*\mathbf{V}$. It is clear from Deligne's construction of the intersection cohomology sheaf [Bo, V.2; GM1, 3.1] that there is a canonical morphism

$$\mathbf{P}^\bullet = \mathbf{IC}^\bullet(\mathbf{V}) \to Ri_*\mathbf{V} = \mathbf{Q}^\bullet.$$

This morphism induces a map from the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ to the complex $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$, which (by naturality) commutes with the differentials. We denote this induced homomorphism by $\tau: \mathbf{K}^\bullet(\mathbf{P}^\bullet) \to \mathbf{K}^\bullet(\mathbf{Q}^\bullet)$.

THEOREM 6.3. *The homomorphism $\tau: \mathbf{K}^\bullet(\mathbf{P}^\bullet) \to \mathbf{K}^\bullet(\mathbf{Q}^\bullet)$ is injective. In other words, the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ in (middle) intersection cohomology, is a subcomplex of the complex $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$ in ordinary cohomology.*

Proof. Let $f: X \to \mathbb{R}$ be a weakly self-indexing Morse function, and construct the complexes $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ and $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$ using f. By induction on the codimension of the stratum w, we show that the canonical morphism $\mathbf{P}^\bullet \to \mathbf{Q}^\bullet$ induces an injection from the Morse group $A_w(\mathbf{P}^\bullet)$ in intersection cohomology to the Morse group $A_w(\mathbf{Q}^\bullet)$ in ordinary cohomology. This is immediate if $\operatorname{codim}(w) = 0$, so assume that w has positive codimension $q+1$.

As noted above, the Morse group $A_w(\mathbf{P}^\bullet)$ in intersection cohomology is given by the image of the variation map var_w in intersection cohomology, hence is a subgroup of $IH^{q+1}(C - \{p\}, C_{<0}; \mathbf{V})$, where p is the unique critical point of $f \mid w$, and C and $C_{<0}$ are the cylindrical neighborhood and cut off space of w at p. By [GM2, 3.5], the group $IH^{q+1}(C - \{p\}, C_{<0}; \mathbf{V})$ is canonically identified with $IH^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{V})$, where $\mathscr{L}$ is the complex link of w at p. Thus $A_w(\mathbf{P}^\bullet)$ is a subgroup of $IH^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{V}) = H^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{P}^\bullet)$. By Remark 5.3 and [GM2, 3.5], the Morse group in ordinary cohomology is given by

$$\begin{aligned} A_w(\mathbf{Q}^\bullet) &= H^{q+1}(C, C_{<0}; \mathbf{Q}^\bullet) = H^{q+1}(C - \{p\}, C_{<0}; \mathbf{Q}^\bullet) \\ &= H^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{Q}^\bullet). \end{aligned}$$

Thus it suffices to show that $H^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{P}^\bullet)$ is a subgroup of $H^q(\mathscr{L}, \partial\mathscr{L}; \mathbf{Q}^\bullet)$.

Without loss of generality, assume that w is the only codimension $q+1 = d$ stratum of X (i.e., that w is the unique vertex of the central arrangement $\mathscr{A}$). It is also no loss of generality to assume that the boundary of the complex link of w is given by $\partial\mathscr{L} = f^{-1}(\lambda) \cap \mathscr{L}$, for some $\lambda \in \mathbb{R}$.

The Morse function $g := \lambda - f : \mathscr{L} \to \mathbb{R}$ has a (degenerate) minimum on $\partial\mathscr{L}$, a single maximum on each stratum of $\mathscr{L}$, and no other critical points. Since $f \mid \mathscr{L}$ is weakly self-indexing, there are positive constants $\zeta_0 < \zeta_1 < \cdots < \zeta_q < \zeta_{q+1} = \lambda$ so that for each i,

$$\max_{v \in L_{q-i+1}} \{\text{critical value of } g \mid v \cap \mathscr{L}\} < \zeta_i < \min_{u \in L_{q-i}} \{\text{critical value of } g \mid u \cap \mathscr{L}\}.$$

The level sets $\mathscr{L}_i = \{x \in \mathscr{L} \mid g(x) \leqslant \zeta_i\}$ form a filtration of the pair $(\mathscr{L}, \partial\mathscr{L})$. Note that the set $\mathscr{L}_0$ is a collared neighborhood of the boundary $\partial\mathscr{L}$, and that $\mathscr{L}_{q+1} = \mathscr{L}$.

For each s, $0 \leqslant s \leqslant q$, the level set $\mathscr{L}_{s+1}$ is obtained from $\mathscr{L}_s$ by attatching the disjoint union of the Morse data (for the function g) corresponding to all strata of $\mathscr{L}$ of codimension $q-s$. The Morse data associated to any one codimension $q-s$ flat $v \cap \mathscr{L}$ is the product of the tangential and normal Morse data [GM3, I.3.5.4]. Since the Morse function g has a maximum on each stratum $v \cap \mathscr{L}$, the Morse index of $g \mid v \cap \mathscr{L}$ is $2s$, and the tangential Morse data of $v \cap \mathscr{L}$ is (topologically) given by the pair $(D^{2s}, \partial D^{2s})$ on which the local system $\mathbf{V}$ is necessarily trivial. For either sheaf $\mathbf{P}^\bullet$ or $\mathbf{Q}^\bullet$, the cohomology of the normal Morse data associated to $v \cap \mathscr{L}$ is, by definition, the Morse group $\tilde{A}_v(\mathbf{P}^\bullet)$ or $\tilde{A}_v(\mathbf{Q}^\bullet)$ of $v \cap \mathscr{L}$ (constructed using the function g). Choosing orientations for the tangential Morse data, by excision and the Künneth formula, we have

$$H^i(\mathscr{L}_{s+1}, \mathscr{L}_s; \mathbf{P}^\bullet) = \begin{cases} \bigoplus_{v \in L_{q-s}} \tilde{A}_v(\mathbf{P}^\bullet) & \text{if } i = q+s \\ 0 & \text{if } i \neq q+s \end{cases}$$

and

$$H^i(\mathscr{L}_{s+1}, \mathscr{L}_s; \mathbf{Q}^\bullet) = \begin{cases} \bigoplus_{v \in L_{q-s}} \tilde{A}_v(\mathbf{Q}^\bullet) & \text{if } i = q+s \\ 0 & \text{if } i \neq q+s. \end{cases}$$

Since the tangential Morse data associated to any codimension 0 stratum of $\mathscr{L}$ is topologically trivial, the identifications

$$H^q(\mathscr{L}_1, \mathscr{L}_0; \mathbf{P}^\bullet) = \bigoplus_{v \in L_q} \tilde{A}_v(\mathbf{P}^\bullet) \qquad \text{and} \qquad H^q(\mathscr{L}_1, \mathscr{L}_0; \mathbf{Q}^\bullet) = \bigoplus_{v \in L_q} \tilde{A}_v(\mathbf{Q}^\bullet)$$

are canonical.

Using the inductive hypothesis, we observe that, for each flat $v \in L_q$, the group $A_v(\mathbf{P}^\bullet)$ is a subgroup of $A_v(\mathbf{Q}^\bullet)$, where $A_v(\mathbf{P}^\bullet)$ and $A_v(\mathbf{Q}^\bullet)$ are the

Morse groups of v constructed using the original function f. The choice of a path between the functions g and $f \mid \mathscr{L}$ in the space of all Morse functions on $\mathscr{L}$ determines isomorphisms $A_v(\mathbf{P}^\bullet) \approx \tilde{A}_v(\mathbf{P}^\bullet)$ and $A_v(\mathbf{Q}^\bullet) \approx \tilde{A}_v(\mathbf{Q}^\bullet)$ for each stratum $v \in L_q$. It follows that $H^q(\mathscr{L}_1, \mathscr{L}_0; \mathbf{P}^\bullet)$ is a subgroup of $H^q(\mathscr{L}_1, \mathscr{L}_0; \mathbf{Q}^\bullet)$.

For each s, $1 \leqslant s \leqslant q$, the canonical morphism $\mathbf{P}^\bullet \to \mathbf{Q}^\bullet$ induces homomorphisms from the groups in the exact sequence of the triple $(\mathscr{L}_{s+1}, \mathscr{L}_s, \mathscr{L}_0)$ in intersection cohomology to the corresponding groups in the exact sequence in ordinary cohomology. By naturality, these homomorphisms commute with the differentials of these sequences. Using this fact and the above computations, we observe that $H^q(\mathscr{L}_{s+1}, \mathscr{L}_0; \mathbf{P}^\bullet)$ is a subgroup of $H^q(\mathscr{L}_s, \mathscr{L}_0; \mathbf{P}^\bullet)$, which in turn is a subgroup of $H^q(\mathscr{L}_s, \mathscr{L}_0; \mathbf{Q}^\bullet)$ for each s, $1 \leqslant s \leqslant q$, and the result follows immediately. ∎

Remark 6.4. It follows from Remark 6.2 and the above theorem that the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ for the intersection cohomology of X with coefficients in $\mathbf{V}$ is completely determined by the complex $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$ for the ordinary cohomology of the complement M with coefficients in $\mathbf{V}$.

Remark 6.5. If the local coefficient system $\mathbf{V}$ on the complement M is trivial, then it follows from Remark 5.4 and Theorem 6.3 that the Morse groups in intersection cohomology associated to all strata of positive codimension vanish. Thus the terms, $\mathbf{K}^q(\mathbf{P}^\bullet)$, of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ are trivial for $q > 0$, and the intersection cohomology of X with coefficients in $\mathbf{V}$ is merely the (ordinary) cohomology of $\mathbb{C}^d$ with coefficients in the (trivial) local system $\mathbf{V}$.

Remark 6.6. For perversities which do not lie between the sublogarithmic and logarithmic perversities, there is no analogue of Corollary 6.1. Since the intersection cohomology sheaves, $\mathbf{I}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{V})$, are not perverse in this case, the Morse groups, $A^i_w(\mathbf{I}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{V})) = I^{\bar{p}}H^i(C, C_{<0}; \mathbf{V})$, may be nontrivial for many different degrees. Careful examination of a braid diagram analogous to that of [GM2, II.6.7] shows that

$$A^i_w(\mathbf{I}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{V})) = \begin{cases} I^{\bar{p}}H^{i-1}(\mathscr{L}, \partial\mathscr{L}; \mathbf{V}) & \text{if } i \leqslant 2q - p(2q) - 2 \\ \text{Image}(\text{var}^j_w) & \text{if } i = 2q - p(2q) - 1 \\ I^{\bar{p}}H^{i-1}(\mathscr{L}; \mathbf{V}) & \text{if } i \geqslant 2q - p(2q), \end{cases}$$

where $\mathscr{L}$ is the complex link of the codimension q flat w, $j = 2q - p(2q) - 1$, and

$$\text{var}^j_w : I^{\bar{p}}H^{j-1}(C_{<0}; \mathbf{V}) \to I^{\bar{p}}H^j(C, C_{<0}; \mathbf{V})$$

is the corresponding variation map. For an arbitrary local coefficient system $\mathbf{V}$, there is no a priori reason for all of the Morse groups

$A^i_w(\mathbf{I}^p\mathbf{C}^\bullet(\mathbf{V}))$ with $i \neq q$ to vanish. For instance, if $\mathscr{A}$ consists of the coordinate hyperplanes in $\mathbb{C}^3$, then the fundamental group of the complement M is free abelian of rank 3. The rank 3 representation $\rho: \pi_1(M, *) = \mathbb{Z}^3 \to \mathrm{Aut}(\mathbb{C}^3)$ given by

$$\rho(1,0,0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho(0,1,0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\rho(0,0,1) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

determines a local system $\mathbf{V}$ for which the Morse group $A^2_w(\mathbf{I}^t\mathbf{C}^\bullet(\mathbf{V}))$ in "top" perversity intersection cohomology at the origin (the codimension 3 stratum w) is nontrivial.

7. General Position Arrangements

Definition 7.1. The arrangement $\mathscr{A}$ in $\mathbb{C}^d$ is said to be a *general position* arrangement if for every subset $\{H_1, ..., H_p\}$ of hyperplanes in $\mathscr{A}$ with $p \leqslant d$

$$\operatorname{codim} H_1 \cap \cdots \cap H_p = p,$$

and when $p > d$,

$$H_1 \cap \cdots \cap H_p = \varnothing.$$

Fix a general position arrangement $\mathscr{A} = \{H_1, ..., H_n\}$ in $\mathbb{C}^d$. Recall that M denotes the complement of $\mathscr{A}$, that X denotes $\mathbb{C}^d$ stratified by $\mathscr{A}$, and that $i: M \to X$ denotes the natural inclusion. Fix a basepoint x_0 in M.

The fundamental group of the complement is generated by loops $\gamma_1, ..., \gamma_n$, based at x_0, about the missing hyperplanes. If $d = 1$, there are no relations among these generators, and $\pi_1(M, x_0)$ is a free group on n letters. If $d \geqslant 2$, the generators commute pairwise, and the fundamental group of M is free abelian of rank n [Ha, 3],

$$\pi_1(M, x_0) = \mathbb{Z}^n.$$

A complex representation $\rho: \pi_1(M, x_0) \to \mathrm{Aut}(V)$ of the fundamental group of M is determined by the choice of (conjugacy classes of) n monodromy transformations $\rho(\gamma_j) = T_j \in \mathrm{Aut}(V)$, where V is a complex vector space.

If $d \geqslant 2$, these monodromies necessarily commute. Let n^* denote the cardinality of the set

$$\{T_j \mid \det(Id - T_j) \neq 0\}.$$

Let $\mathbf{V}$ denote the local system on M determined by the representation ρ, let $\mathbf{P}^\bullet = \mathbf{IC}^\bullet(\mathbf{V})$ be the middle intersection cohomology sheaf on X, and let $\mathbf{Q}^\bullet = Ri_*\mathbf{V}$ be the direct image of $\mathbf{V}$ on X.

Let $\mathbf{n} = \{1, 2, ..., n\}$. The lattice of $\mathscr{A}$ contains one flat of codimension q for each subset $J \subset \mathbf{n}$ of cardinality $|J| = q \leqslant d$, where $J = \{j_1, j_2, ..., j_q\}$ and $1 \leqslant j_1 < j_2 < \cdots < j_q \leqslant n$. Let w_J denote the flat associated to J,

$$w_J = H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_q},$$

and denote the Morse groups in intersection cohomology and cohomology associated to w_J by $A_J(\mathbf{P}^\bullet)$ and $A_J(\mathbf{Q}^\bullet)$, respectively. It follows from [Or2, 2.20] that $\mu(w_J) = (-1)^{|J|}$ for each flat $w_J \in L$. Using this fact and Proposition 5.2 we observe that the Morse group in ordinary cohomology $A_J(\mathbf{Q}^\bullet)$ is isomorphic to V. Denote this Morse group by $V_J := A_J(\mathbf{Q}^\bullet)$.

If $w_I < w_J$ is a codimension one inclusion of flats in the lattice of $\mathscr{A}$, then we have

$$I = \{i_1, i_2, ..., i_{q-1}\} \subset \{j_1, j_2, ..., j_q\} = J,$$

i.e., $I = \{j_1, ..., \hat{j}_p, ..., j_q\} = J - \{j_p\}$ for some p. Denote the homomorphism

$$\Phi_{w_I, w_J} : V_I \to V_J$$

in ordinary cohomology associated to this codimension one inclusion (Definition 2.2) by simply $\Phi_{I,J}$.

Proposition 7.2. *With notation as above, for each codimension one inclusion $w_I < w_J$ in the lattice of $\mathscr{A}$, the homomorphism $\Phi_{I,J}: A_I(\mathbf{Q}^\bullet) \to A_J(\mathbf{Q}^\bullet)$ is given by*

$$\Phi_{I,J} = (-1)^{p-1}(Id - T_{j_p}) : V_I \to V_J,$$

where $I = J - \{j_p\}$.

The proof of this proposition is given in Section 8.

The next result follows from Theorem 6.3, Proposition 7.2, and an easy inductive arguement.

Corollary 7.3. *With notation as above, the Morse group in intersection cohomology associated to the stratum w_J is given by*

$$A_J(\mathbf{P}^\bullet) = (Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_q}) V,$$

where $J = \{j_1, j_2, ..., j_q\}$, and $(Id - T)V$ denotes the image of $Id - T: V \to V$.

Remark 7.4. For the general position arrangement $\mathscr{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^d$, we give explicit combinatorial descriptions of the complexes $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ and $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$, where $\mathbf{P}^\bullet = \mathbf{IC}^\bullet(\mathbf{V})$ is the (middle) intersection cohomology sheaf with coefficients in the local system $\mathbf{V}$, and $\mathbf{Q}^\bullet = Ri_* \mathbf{V}$ is the direct image of the local system. The complexes have terms

$$\mathbf{K}^q(\mathbf{P}^\bullet) = \bigoplus_{\substack{J \subset \mathbf{n} \\ |J| = q}} (Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_q}) V,$$

$$\mathbf{K}^q(\mathbf{Q}^\bullet) = \bigoplus_{\substack{J \subset \mathbf{n} \\ |J| = q}} V_J,$$

and differentials

$$\Phi^q = \bigoplus_{\substack{J \subset \mathbf{n} \\ |J| = q}} \sum_{\substack{I \subset J \\ |I| = q-1}} \Phi_{I,J},$$

as calculated in Proposition 7.3. By Theorem 6.3, the differential of the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ is identical to that of $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$. Hence we present algorithms for computing $H^*(M; \mathbf{V})$ and $IH^*(X; \mathbf{V})$, the cohomology of the complement of $\mathscr{A}$ and the intersection cohomology of $\mathbb{C}^d$ stratified by $\mathscr{A}$ with coefficients in the local system $\mathbf{V}$.

Using the above remark, we obtain the following result of Hattori [Ha, 4], as well as its analogue in intersection cohomology, which is a generalization of an observation of Lusztig [Lu, 1.6].

PROPOSITION 7.5. *If $\rho: \pi_1(M, x_0) \to \mathrm{Aut}(V)$ is a nontrivial, rank one, complex representation of the fundamental group of the complement of the general position arrangement $\mathscr{A} = \{H_1, \ldots, H_n\}$, and $\mathbf{V}$ is the local coefficient system on M determined by the representation ρ, then*

(i) *The cohomology groups $H^p(M; \mathbf{V})$ vanish for $p \neq d$, and*

$$\dim H^d(M; \mathbf{V}) = \sum_{k=1}^{n-d} (-1)^{k+1} \binom{n}{d+k}.$$

(ii) *The intersection cohomology groups $IH^p(X; \mathbf{V})$ vanish for $p \neq d$, and*

$$\dim IH^d(X; \mathbf{V}) = \begin{cases} \binom{n^*-1}{d} & \text{if } n^* > d \\ 0 & \text{if } n^* \leqslant d. \end{cases}$$

Notice that, for a rank one representation ρ, the number n^* is merely the number of nontrivial monodromy transformations.

Proof. Since the representation ρ is nontrivial, at least one of the monodromies, $T_j = \rho(\gamma_j)$ (where γ_j is the fixed loop about $H_j \in \mathscr{A}$), is nontrivial. So without loss of generality, assume that $T_n \neq 1$.

For each codimension one inclusion $w_I < w_J$ in the lattice of $\mathscr{A}$ define $\partial_{I,J}: V_J \to V_I$ by

$$\partial_{I,J} = \begin{cases} (-1)^{|J|}(Id - T_n)^{-1} & \text{if } \ I = J - \{n\} \\ 0 & \text{otherwise.} \end{cases}$$

Then define

$$\partial^p = \bigoplus_{\substack{I \subset \mathbf{n} \\ |I| = p-1}} \sum_{\substack{J \supset I \\ |J| = p}} \partial_{I,J}: \mathbf{K}^p(\mathbf{Q}^\bullet) \to \mathbf{K}^{p-1}(\mathbf{Q}^\bullet).$$

Checking that $\partial^{p-1} \circ \Phi^{p-1} + \Phi^p \circ \partial^p = 1$ for each $p \leq d$, we have

$$H^p(M; \mathbf{V}) = H^p(\mathbf{K}^\bullet(\mathbf{Q}^\bullet)) = 0 \qquad \text{for} \quad p \neq d.$$

Thus the dimension of $H^d(M; \mathbf{V})$ is given by the Euler characteristic of the complex $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$,

$$(-1)^d \dim H^d(M; \mathbf{V}) = \chi(\mathbf{K}^\bullet(\mathbf{Q}^\bullet)) = \sum_{k=0}^{d} (-1)^k \binom{n}{k},$$

and it follows immediately that

$$\dim H^d(M; \mathbf{V}) = \sum_{k=1}^{n-d} (-1)^{k+1} \binom{n}{d+k}.$$

Since $\partial(\mathbf{K}^\bullet(\mathbf{P}^\bullet)) \subset \mathbf{K}^\bullet(\mathbf{P}^\bullet)$, the above argument also shows that

$$IH^p(X; \mathbf{V}) = H^p(\mathbf{K}^\bullet(\mathbf{P}^\bullet)) = 0 \qquad \text{for} \quad p \neq d,$$

and one can easily check that

$$\begin{aligned} \dim IH^d(X; \mathbf{V}) &= (-1)^d \chi(\mathbf{K}^\bullet(\mathbf{P}^\bullet)) \\ &= \sum_{\substack{|J| = d \\ n \notin J}} \dim(Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_d}) V. \end{aligned}$$

Since precisely $\binom{n^* - 1}{d}$ of the groups $(Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_d}) V$ with $1 \leq j_1 \leq \cdots \leq j_d \leq n-1$ are nontrivial, this completes the proof. ∎

Remarks. 7.6.1. If the arrangement $\mathscr{A}$ is the complexification of a real arrangement $\mathscr{A}_{\mathbb{R}}$, then the complement of $\mathscr{A}_{\mathbb{R}}$ in $\mathbb{R}^d$ has $\sum_{k=1}^{n-d} (-1)^{k+1} \binom{n}{d+k}$ bounded components.

7.6.2. If the cardinality of $\mathcal{A}$ is equal to the dimension of the ambient space (i.e., $n=d$), then the above argument shows that $H^p(M;\mathbf{V})=0$ and $IH^p(X;\mathbf{V})=0$ for all p.

PROPOSITION 7.7. *If $\rho:\pi_1(M,x_0)\to \mathrm{Aut}(V)$ is a complex representation (of arbitrary rank) of the fundamental group of the complement of the general position arrangement $\mathcal{A}=\{H_1,...,H_n\}$ with the property that $\det(Id-T_n)\neq 0$, where $T_n=\rho(\gamma_n)$ is the monodromy about $H_n\in\mathcal{A}$, and $\mathbf{V}$ is the local coefficient system on M determined by the representation ρ, then*

(i) *The cohomology groups $H^p(M;\mathbf{V})$ vanish for $p\neq d$, and*

$$\dim H^d(M;\mathbf{V})=\dim V\cdot\sum_{k=1}^{n-d}(-1)^{k+1}\binom{n}{d+k}.$$

(ii) *The intersection cohomology groups $IH^p(X;\mathbf{V})$ vanish for $p\neq d$, and*

$$\dim IH^d(X;\mathbf{V})=\sum_{\substack{|J|=d\\ n\notin J}}\dim(Id-T_{j_1})(Id-T_{j_2})\cdots(Id-T_{j_d})V.$$

The proof of this result is similar to that of Proposition 7.5.

EXAMPLE 7.8. Let $\mathcal{A}$ be a general position arrangement consisting of d hyperplanes in $\mathbb{C}^d$, and let $\mathbf{V}$ be an arbitrary local system on the complement M of $\mathcal{A}$. We may assume that $\mathcal{A}$ consists of the coordinate hyperplanes in $\mathbb{C}^d$, and that $\mathbf{n}=\mathbf{d}=\{1,2,...,d\}$. The complement of $\mathcal{A}$ is the complex d-torus, $M=(\mathbb{C}^*)^d$. Thus the complex $\mathbf{K}^\bullet(\mathbf{Q}^\bullet)$ of Remark 7.4 computes the cohomology of the d-torus with coefficients in $\mathbf{V}$. Similarly, the complex $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ of Remark 7.4 computes the intersection cohomology of $\mathbb{C}^d$ stratified by the coordinate hyperplanes with coefficients in $\mathbf{V}$.

In this instance, there is a stratum-preserving retraction from $X=\mathbb{C}^d$ stratified by $\mathcal{A}$ to a d-fold product of disks $U=\Delta\times\Delta\times\cdots\times\Delta$, where the singular stratum of each of the disks Δ in $\mathbb{C}$ consists only of the origin, and U has the product stratification. Following [CKS], we assume that the monodromy transformations T_j, $1\leqslant j\leqslant d$, are unipotent, so that the endomorphisms $Id-T_j$ are nilpotent. Let N_j denote the logarithm of T_j.

In [CKS], Cattani, Kaplan, and Schmid construct a complex $\mathbf{B}^\bullet$ which computes the intersection cohomology of U with coefficients in $\mathbf{V}$. This complex has terms

$$\mathbf{B}^q=\bigoplus_{\substack{J\subset\mathbf{d}\\ |J|=q}}N_{j_1}N_{j_2}\cdots N_{j_q}V,$$

and differentials

$$\Psi^q = \bigoplus_{\substack{J \subset \mathbf{d} \\ |J| = q}} \sum_{\substack{I \subset J \\ |I| = q-1}} \Psi_{I,J},$$

where $\Psi_{I,J} = (-1)^{p-1} N_{j_p}$ if $I = J - \{j_p\}$.

PROPOSITION 7.9. *The complexes* $\mathbf{K}^\bullet(\mathbf{P}^\bullet)$ *and* $\mathbf{B}^\bullet$ *are isomorphic.*

Proof. Using the assumption that the monodromy transformations T_j are unipotent, one can show that there are automorphisms S_j so that $N_j = (Id - T_j) S_j$, for each j, $1 \leqslant j \leqslant n$. For each $J \subset \mathbf{d}$, define

$$\alpha_J : (Id - T_{j_1})(Id - T_{j_2}) \cdots (Id - T_{j_q}) V \to N_{j_1} N_{j_2} \cdots N_{j_q} V$$

by $\alpha_J = S_{j_1} S_{j_2} \cdots S_{j_q}$. When $J = \varnothing$, set $\alpha_J = Id: V \to V$. Since each of the transformations S_j is an automorphism, the map α_J is an isomorphism for each $J \subset \mathbf{d}$.

For each q, $0 \leqslant q \leqslant d$, define transformations $\alpha^q: \mathbf{K}^q(\mathbf{P}^\bullet) \to \mathbf{B}^q$ by

$$\alpha^q = \bigoplus_{\substack{J \subset \mathbf{d} \\ |J| = q}} \alpha_J.$$

By construction, each of the maps α^q is an isomorphism. It is standard to check that the transformations α^q commute with the differentials Φ and Ψ, i.e., that $\alpha^q \circ \Phi^q = \Psi^q \circ \alpha^{q-1}$. ∎

8. PROOF OF PROPOSITION 7.2

Let S^1 denote the unit circle in $\mathbb{C}$, and let $\vartheta = \{z \in S^1 \mid \mathrm{Re}(z) > 0\}$. Note that ϑ is contractible. Let $T^n = S^1 \times \cdots \times S^1$ denote the n-dimensional torus.

DEFINITION 8.1. For each $J \subset \mathbf{n}$, define the *subtorus* T_J of T^n by

$$T_J = \{(z_1, ..., z_n) \in T^n \mid z_j = 1 \text{ for } j \notin J\},$$

and define the *thickened subtorus* $\tilde{T}_J$ of T^n by

$$\tilde{T}_J = \{(z_1, ..., z_n) \in T^n \mid z_j \in \vartheta \text{ for } j \notin J\}.$$

Remarks. 8.1.1. The subtorus T_J is a deformation retract of the thickened subtorus $\tilde{T}_J$.

8.1.2. For each k, $1 \leqslant k \leqslant n$, the union of subtori $\bigcup_{|J| = k} T_J$ is the k-skeleton of the n-torus in the standard (product) CW-decomposition.

8.1.3. For each k, $1 \leqslant k \leqslant n$, the k-skeleton of the n-torus is a deformation retract of the union of thickened subtori $\bigcup_{|J|=k} \tilde{T}_J$.

8.1.4. In [Ha], Hattori showed that the complement of a general position arrangement $\mathscr{A}$ consisting of n hyperplanes in $\mathbb{C}^d$ has the homotopy type of $\bigcup_{|J|=d} T_J$, the d-skeleton of the n-torus.

Without loss of generality, assume that w_J is the only codimension d stratum of X. Then $\mathscr{A}$ is a central general position arrangement, $|\mathscr{A}| = n = d$, and $J = \{1, 2, ..., n\}$. We may take the hyperplanes of $\mathscr{A}$ to be the coordinate hyperplanes, $H_j = \ker z_j$, in $\mathbb{C}^n$. It follows that the complement of $\mathscr{A}$ is the complex n-torus, $M = (\mathbb{C}^*)^n$. It is also no loss of generality to assume that $I \subset J$ is given by $I = \{1, 2, ..., n-1\}$.

Write $z_j = x_j + \sqrt{-1}\, y_j$ for each j, and define $f_{s,t} : X \to \mathbb{R}$ by

$$f_{s,t}(x, y) = \left(1 + \frac{s}{n}\right) \sum_{j=1}^{n-1} (x_j - 1)^2 + (x_n - 1)^2 + t \sum_{i=1}^{n} y_j^2,$$

where s and t are real numbers. It is a straightforward exercise to check that $f_{1,1} : X \to \mathbb{R}$ is a Morse function, which is weakly self-indexing with respect to the strata w_I (the z_n-axis) and w_J (the origin). Fix ε, $0 < \varepsilon \ll 1/2n$, and let $a = n - 1/n - \varepsilon$, and $b = n - \varepsilon$. Then the level set $X_a = \{(x, y) \in X \mid f_{1,1}(x, y) \leqslant a\}$ meets all strata of X except w_I and w_J, and the level set $X_b = \{(x, y) \in X \mid f_{1,1}(x, y) \leqslant b\}$ meets all strata of X except w_J. Hence we have the following canonical identifications,

$$A_I(\mathbf{Q}^\bullet) = H^{n-1}(X_b, X_a; \mathbf{Q}^\bullet), \qquad A_J(\mathbf{Q}^\bullet) = H^n(X, X_b; \mathbf{Q}^\bullet),$$

and the homomorphism $\Phi_{I,J}$ is the boundary map of the triple (X, X_b, X_a).

Since $\mathbf{Q}^\bullet = Ri_* \mathbf{V}$ is the direct image of the local system $\mathbf{V}$, writing $M_a = M \cap X_a$ and $M_b = M \cap X_b$, we have

$$A_I(\mathbf{Q}^\bullet) = H^{n-1}(M_b, M_a; \mathbf{V}), \qquad A_J(\mathbf{Q}^\bullet) = H^n(M, M_b; \mathbf{V}),$$

and we identify the homomorphism $\Phi_{I,J}$ with the boundary map of the triple (M, M_b, M_a). We prove the proposition by identifying the level sets M_a and M_b with the appropriate subsets of the CW-decomposition of the n-torus described above.

Let $r : (\mathbb{C}^*)^n \to T^n$ denote the usual projection,

$$r(z_1, ..., z_n) = \left(\frac{z_1}{|z_1|}, ..., \frac{z_n}{|z_n|}\right),$$

let

$$K = \bigcup_{|J|=n-1} T_J, \qquad \text{and} \qquad \tilde{K} = \bigcup_{|J|=n-1} \tilde{T}_J$$

denote the $(n-1)$-skeleton and the thickened $(n-1)$-skeleton of T^n, and let

$$K_n = \bigcup_{\substack{|J|=n-1\\ n\in J}} T_J, \quad \text{and} \quad \tilde{K}_n = \bigcup_{\substack{|J|=n-1\\ n\in J}} \tilde{T}_J.$$

Check that

$$M_a \subset r^{-1}(\tilde{K}_n) = \bigcup_{j=1}^{n-1} \{(x, y) \in M \mid x_j > 0\}$$

and

$$M_b \subset r^{-1}(\tilde{K}) = \bigcup_{j=1}^{n} \{(x, y) \in M \mid x_j > 0\}.$$

LEMMA 8.2. *The inclusions* $M_a \subset r^{-1}(\tilde{K}_n)$ *and* $M_b \subset r^{-1}(\tilde{K})$ *are homotopy equivalences.*

Proof. Let $i: (M_b, M_a) \to (r^{-1}(\tilde{K}), r^{-1}(\tilde{K}_n))$ denote the inclusion. We first consider the level set M_b. Let

$$\begin{aligned}\tilde{X}_b &= \{(x, y) \in X \mid f_{1,0}(x, y) \leqslant b\}\\ &= \left\{(x, y) \in X \,\middle|\, \left(1+\frac{1}{n}\right) \sum_{j=1}^{n-1} (x_j - 1)^2 + (x_n - 1)^2 \leqslant b\right\}.\end{aligned}$$

Then $X_b \subset \tilde{X}_b$, and the function $g: \tilde{X}_b \to X_b$ defined by

$$g(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in X_b \\ (x, \lambda(x, y) \cdot y) & \text{otherwise,} \end{cases}$$

where $\lambda(x, y) = \sqrt{(b - f_{1,0}(x, y))/(f_{1,1}(x, y) - f_{1,0}(x, y))}$, is a stratum-preserving retraction. Thus M_b is a deformation retract of $\tilde{M}_b = M \cap \tilde{X}_b$.

Let

$$B = \{(x, y) \in X \mid f_{0,0}(x, y) \leqslant b\} = \left\{(x, y) \in X \,\middle|\, \sum_{j=1}^{n} (x_j - 1)^2 \leqslant b\right\},$$

and let $B_M = M \cap B$. Notice that the spaces $\tilde{X}_b$ and B are products,

$$\tilde{X}_b = \mathrm{Re}(\tilde{X}_b) \times \mathrm{Im}(\tilde{X}_b), \qquad B = \mathrm{Re}(B) \times \mathrm{Im}(B),$$

and that $\mathrm{Im}(\tilde{X}_b) = \mathrm{Im}(B)$. The set $\mathrm{Re}(B)$ is compact. Using the technique of *Moving the Wall* [GM3, I. 4], we construct a stratum-preserving homeomorphism between $\mathrm{Re}(\tilde{X}_b)$ and $\mathrm{Re}(B)$ as follows.

Define $F_1: \mathrm{Re}(B) \to \mathbb{R}$ and $F_2: \mathrm{Re}(B) \to \mathbb{R}$ by

$$F_1(x) = \sum_{j=1}^{n-1} (x_j - 1)^2, \qquad \text{and} \qquad F_2(x) = (x_n - 1)^2,$$

and define $F: \mathrm{Re}(B) \to \mathbb{R}^2$ by $F = (F_1, F_2)$. With this notation, the set $\mathrm{Re}(\tilde{X}_b)$ is given by the pre-image under the map F of the region $\{(u, v) \in \mathbb{R}^2 \mid (1 + 1/n)u + v \leq b\}$, and $\mathrm{Re}(B)$ is the pre-image of $\{(u, v) \in \mathbb{R}^2 \mid u + v \leq b\}$. Checking that the map F has no characteristic covectors [GM3, I.1.9] on the line $\{(u, v) \in \mathbb{R}^2 \mid (1 + t/n)u + v = b\}$ for each $t \in [0, 1]$, we show that $\mathrm{Re}(\tilde{X}_b)$ is homeomorphic to $\mathrm{Re}(B)$ by moving the wall in the uv-plane as indicated in Fig. 1. It follows that $\tilde{M}_b$ is a deformation retract of B_M.

Now let

$$H^+ = \left\{(x, y) \in M \,\middle|\, \sum_{j=1}^{n} x_j > 0\right\} \qquad \text{and} \qquad H_\varepsilon = \left\{(x, y) \in M \,\middle|\, \sum_{j=1}^{n} x_j = \varepsilon\right\}.$$

The map $\mu: H^+ \to H_\varepsilon$ defined by

$$\mu(x, y) = (\sigma(x) \cdot x, y), \qquad \text{where} \quad \sigma(x) = \frac{\varepsilon}{\sum_{j=1}^{n} x_j},$$

is a retraction, as is its restriction $\mu: B_M \to B_M \cap H_\varepsilon$. It follows from [Or1, 2.9] that $B_M \cap H_\varepsilon$ is a deformation retract of H_ε. Combining these results, we observe that the inclusion $B_M \subset H^+$ is a homotopy equivalence.

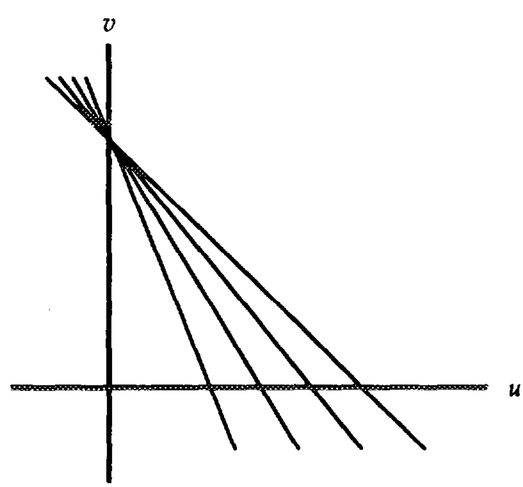

FIG. 1. Moving the wall.

We have constructed the sequence of inclusions,

$$M_b \subset \tilde{M}_b \subset B_M \subset H^+,$$

each of which is a homotopy equivalence. So to show that $M_b \subset r^{-1}(\tilde{K})$ is a homotopy equivalence, it remains to prove that $H^+ \subset r^{-1}(\tilde{K})$ is a homotopy equivalence. Check that $r^{-1}(\tilde{K})$ may be expressed as

$$r^{-1}(\tilde{K}) = \bigcup_{\substack{J \subset \mathbf{n} \\ J \neq \varnothing}} \{(x, y) \in M \mid x_j > 0 \text{ if } j \in J \text{ and } x_j < 0 \text{ if } j \notin J\} = \bigcup_{\substack{J \subset \mathbf{n} \\ J \neq \varnothing}} Q_J.$$

For each nonempty $J \subset \mathbf{n}$, define $h_J\colon Q_J \to Q_J \cap H^+$ by

$$h_J(x, y) = (h_J^1(x), ..., h_J^n(x), y),$$

where

$$h_J^k(x) = \begin{cases} x_k & \text{if } \ k \notin J \\ x_k - (1/|J|) \sum_{j \notin J} x_j & \text{if } \ k \in J, \end{cases}$$

and define $h\colon r^{-1}(\tilde{K}) \to H^+$ by $h \mid Q_J = h_J$. The map h is easily seen to be a retraction.

We now turn our attention to the level set M_a. Let $\tilde{X}_a = \{(x, y) \in X \mid f_{1,0}(x) \leqslant a\}$, and let $\tilde{M}_a = M \cap \tilde{X}_a$. As above, X_a is a stratum-preserving retraction of $\tilde{X}_a$, and $\tilde{X}_a = \mathrm{Re}(\tilde{X}_a) \times \mathrm{Im}(\tilde{X}_a)$ is a product. We now use moving the wall to show that the set $\mathrm{Re}(\tilde{X}_a)$ is homeomorphic to $\mathrm{Re}(C)$, where

$$C = \left\{(x, y) \in X \;\middle|\; \left(1 + \frac{1}{n}\right) \sum_{j=1}^{n-1} (x_j - 1)^2 \leqslant a \text{ and } (x_n - 1)^2 \leqslant a\right\}.$$

With notation as above, the set $\mathrm{Re}(\tilde{X}_a)$ is given by the pre-image under the map $F = (F_1, F_2)$ of the region $\{(u, v) \in \mathbb{R}^2 \mid (1 + 1/n)u + v \leqslant a\}$, and $\mathrm{Re}(C)$ is the pre-image of $\{(u, v) \in \mathbb{R}^2 \mid (1 + 1/n)u \leqslant a$ and $v \leqslant a\}$. Checking that the map F has no characteristic covectors on the line segments

$$\{(u, v) \in \mathbb{R}^2 \mid (a + (1 + \varepsilon - a)t)u + (1 + \varepsilon)tv \\ = a(a + (1 + \varepsilon - a)t) \text{ and } v < a + (1 + \varepsilon - a)t\}$$

and

$$\{(u, v) \in \mathbb{R}^2 \mid (1 + \varepsilon - a)\, tu + ((1 + \varepsilon)t - a)v \\ = a((1 + \varepsilon)t - a) \text{ and } u < a - (1 + \varepsilon)t\},$$

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or at the point $(a-(1+\varepsilon)t,\ a+(1+\varepsilon-a)t)$ for each $t\in[0,1]$, the required wall motion in the uv-plane is as indicated in Fig. 2. Thus $\tilde{M}_a$ is a deformation retract of $C_M = M\cap C$.

Checking that C is a stratum-preserving retraction of

$$\tilde{C}=\left\{(x,y)\in X \;\middle|\; \left(1+\frac{1}{n}\right)\sum_{j=1}^{n-1}(x_j-1)^2\leqslant a\right\},$$

the argument now proceeds as above.

Let

$$V^+=\left\{(x,y)\in M \;\middle|\; \left(1+\frac{1}{n}\right)\sum_{j=1}^{n-1}x_j>0\right\}.$$

The inclusions $M\cap\tilde{C}\subset V^+$ and $V^+\subset r^{-1}(\tilde{K}_n)$ are homotopy equivalences, so we conclude that $i: M_a\to r^{-1}(\tilde{K}_n)$ is also homotopy equivalence. This completes the proof of the lemma. ∎

It follows from Lemma 8.2 that the inclusion $i:(M_b, M_a)\to(r^{-1}(\tilde{K}), r^{-1}(\tilde{K}_n))$ induces isomorphisms in absolute cohomology:

$$i^*: H^*(r^{-1}(\tilde{K}_n);\mathbf{V})\xrightarrow{\simeq} H^*(M_a;\mathbf{V}),$$

and

$$i^*: H^*(r^{-1}(\tilde{K});\mathbf{V})\xrightarrow{\simeq} H^*(M_b;\mathbf{V}).$$

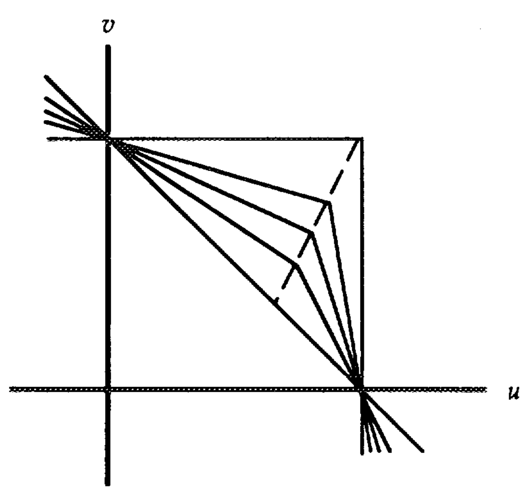

FIG. 2. Moving the wall.

By the five-lemma, the map i also induces isomorphisms in relative cohomology

$$i^*: H^*(M, r^{-1}(\tilde{K}); \mathbf{V}) \xrightarrow{\sim} H^*(M, M_b; \mathbf{V}),$$

and

$$i^*: H^*(r^{-1}(\tilde{K}), r^{-1}(\tilde{K}_n); \mathbf{V}) \xrightarrow{\sim} H^*(M_b, M_a; \mathbf{V}).$$

With these identifications, we realize the homomorphism $\Phi_{I,J}$ as the boundary map of the triple $(M, r^{-1}(\tilde{K}), r^{-1}(\tilde{K}_n))$, or equivalently, the triple (T^n, K, K_n). Using the CW-decomposition of the n-torus, this differential may be found by standard techniques, and we conclude that when $I = \{1, 2, ..., n-1\} \subset \{1, 2, ..., n\} = J$, the homomorphism $\Phi_{I,J}$ is given by

$$\Phi_{I,J} = (-1)^{n-1}(Id - T_n) : V_I \to V_J.$$

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