PICTURES
and LINE ARRANGEMENTS

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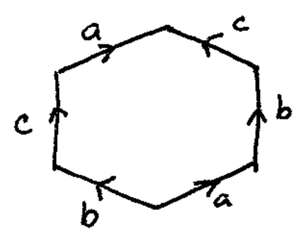
What is a picture?

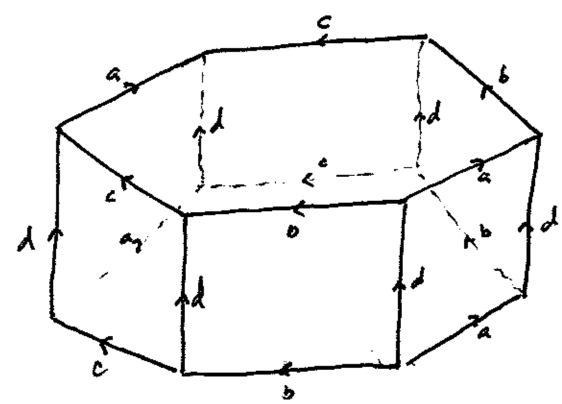
- Begin with any finitely presented group, for example:

$$G = \langle a, b, c, d \| [a, b, c], [a, d], [b, d], [c, d] \rangle$$

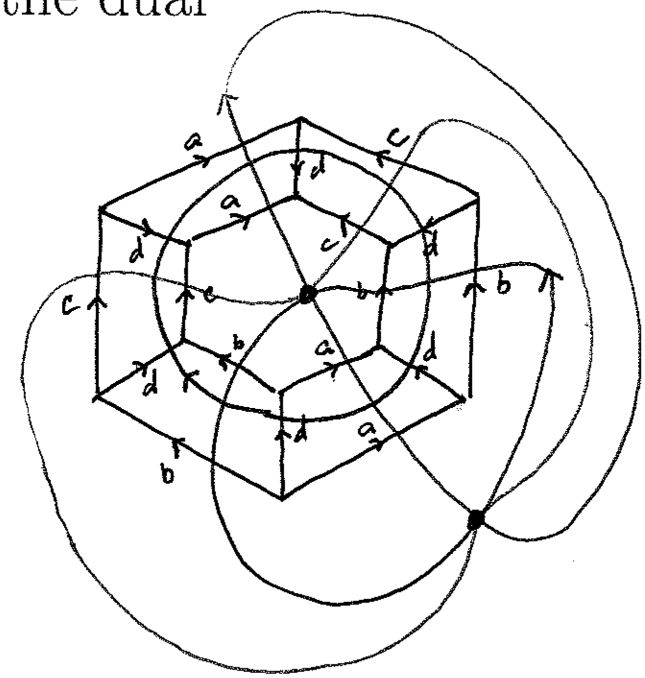
- View relations as 2-cells equipped with attaching maps.
  $r = abca^{-1}c^{-1}b^{-1}$



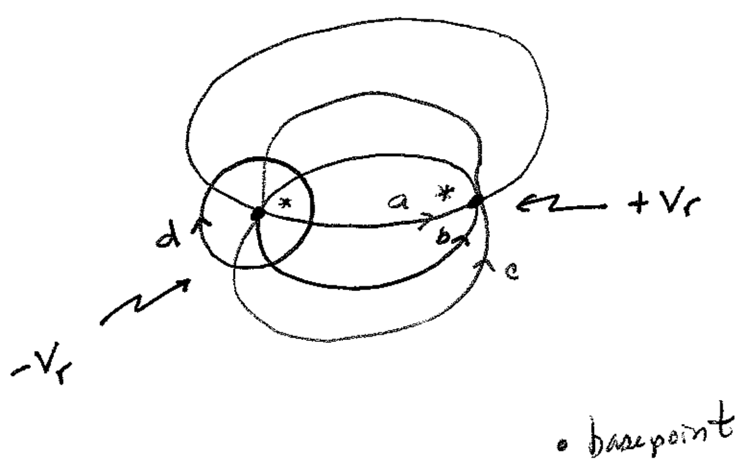
- Glue 2-cells together to cover a sphere.



- Project to the plane and construct the dual

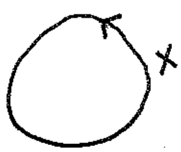


- Here is the picture



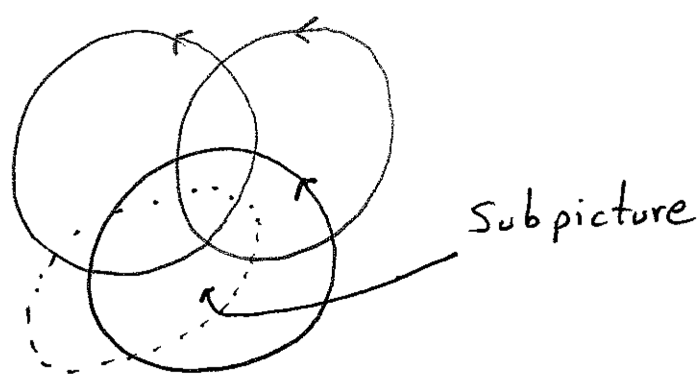
- Or we could assemble a picture directly!

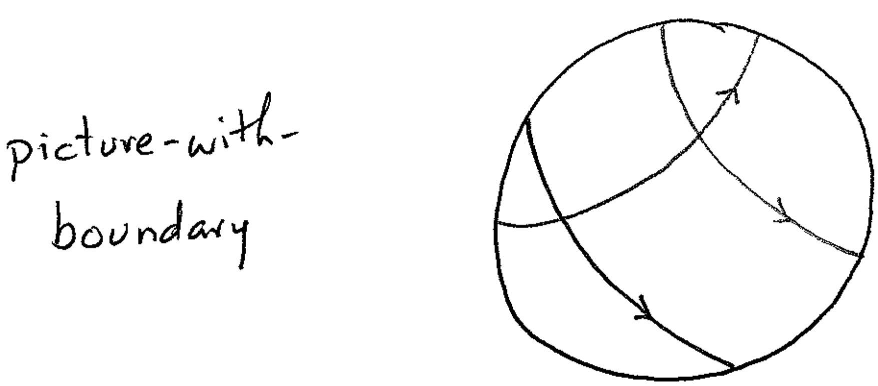
<u>Definition</u>: Given a finitely presented group G, a picture is a finite, oriented plane graph whose vertices represent relations (or their inverses) explicitly as they are written in the list of relations.



- A closed arc containing no vertices, labeled by a generator is a picture.
- Otherwise edges terminate at vertices.
- A picture-with-boundary is a picture embedded in a disk with one or more edges terminating in the boundary of the disk.

- A subpicture is the interior of an unlabeled, closed arc that contains no vertices and that meets edges transversely.
- Note that boundary words of subpictures or of pictures-with-boundary are trivial in the group G.





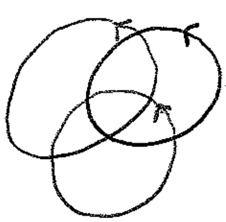
Defining the Picture Group

We first define the extended picture group E_G.

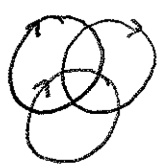
- Binary operation is disjoint union (obviously commutative).

- Identity is the empty picture.

- Form the inverse by constructing the mirror image and then reversing all arrows.



p



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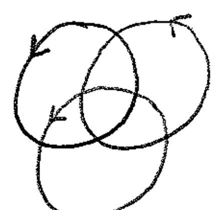
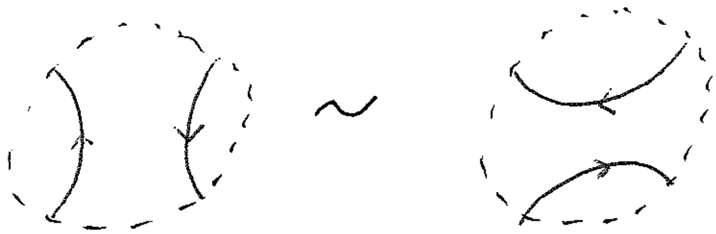
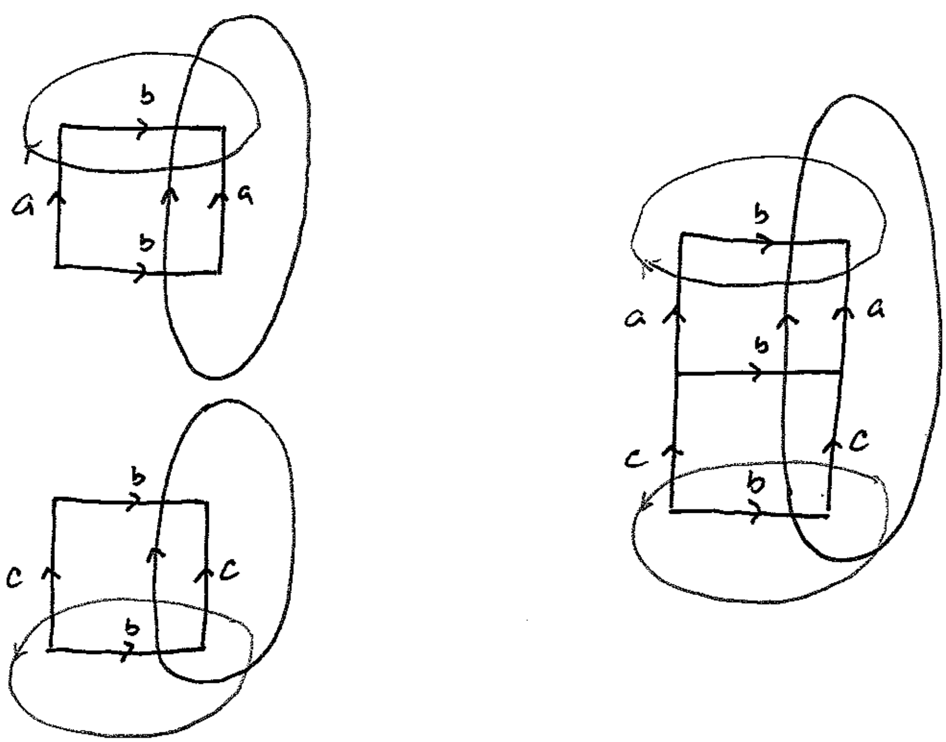


p^{-1}

- Equivalence relations:
  - Isotopy in the plane
  - Bridge moves

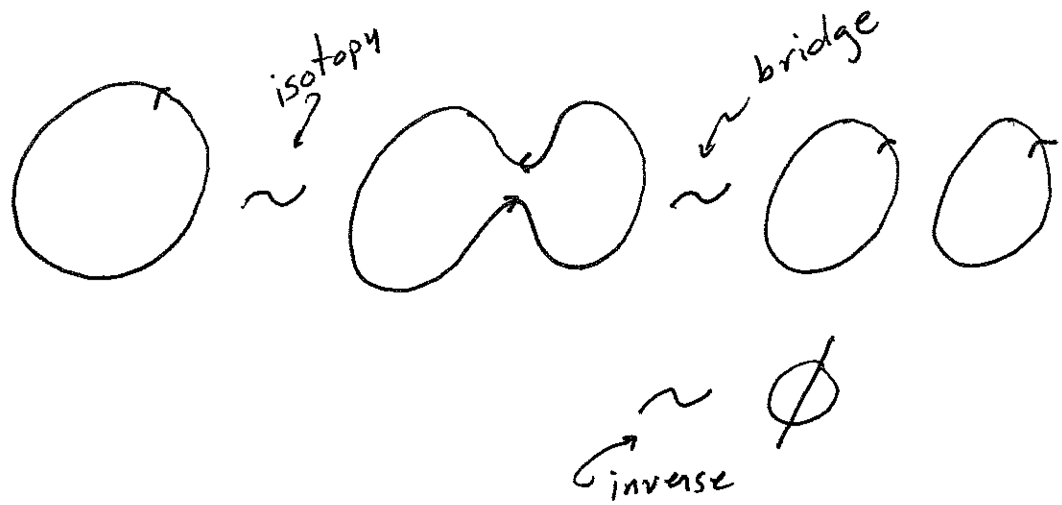


Why do we want bridge moves?

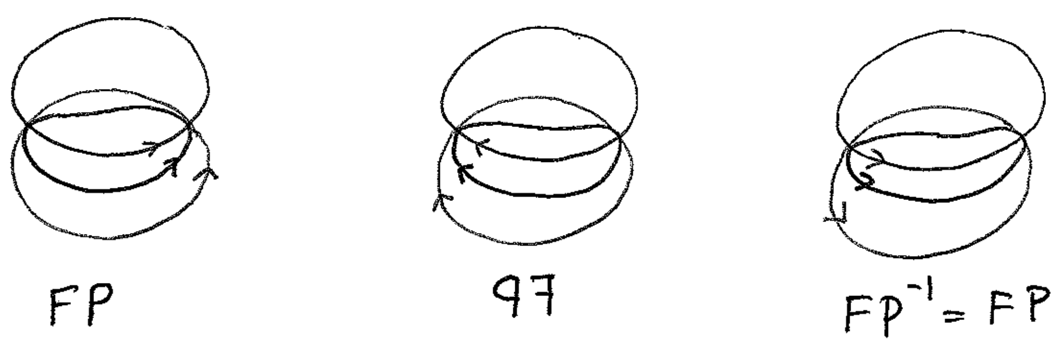


Two immediate consequences:

- A closed arc with empty interior is trivial in E_G.



- Floating pairs (two-vertex pictures) are torsion elements.

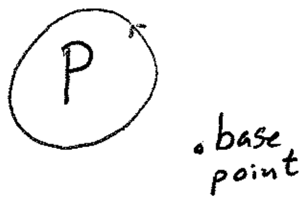


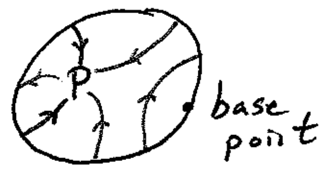
Denote the subgroup generated by all floating pairs as T_G. If the presentation for G is minimal, and all relations are commutators, then T_G contains all the order two elements of E_G.

Now define $P_G = E_G/T_G$

Proposition P_G is isomorphic to the picture group as defined by Fenn [1983] or Bogley and Pride [1993].

G acts on E_G, T_G and P_G.





Here's the main point: For a CW complex with no cells in dimension greater than 2,

$$P_G \cong \pi_2(X)$$

where $G = \pi_1(X)$. See Loday [2000]

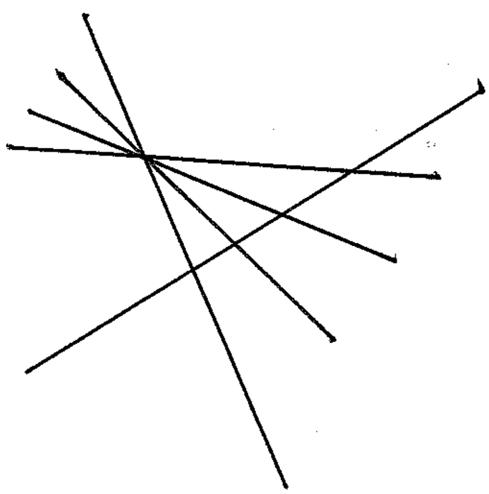
Affine Transformation of a Line Arrangement

Let $\mathcal{A} = \{l_1, \ldots, l_n\}$ be an arrangement of n lines in $\mathbb{C}^2$. For each i, define affine transformation $\phi_i : l_i \rightarrow \mathbb{C}^2$. The by $\Phi(\mathcal{A})$ we mean the arrangement $\{\phi_1(l_1), \ldots, \phi_n(l_n)\}$. Note that $|\Phi(\mathcal{A})| \leq n$.

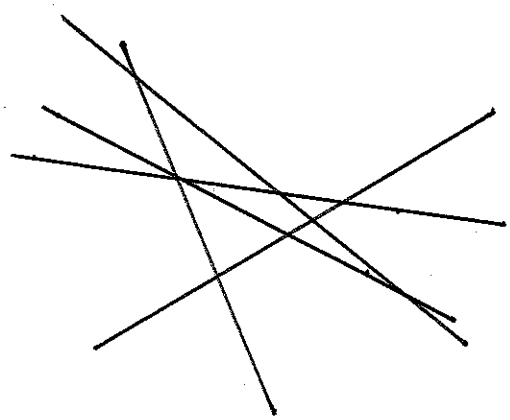
Example:

$G = \langle x_1, \ldots, x_5 | [x_1, x_2, x_3, x_4], [x_i, x_5], i \in \{1, 2, 3, 4\}\rangle$

$G' = \langle x_1, \ldots, x_5 | [x_1, x_2, x_3]; [x_i, x_j] \; i = 1, 2, 3 \; j = 4, 5; [x_4, x_5]\rangle$



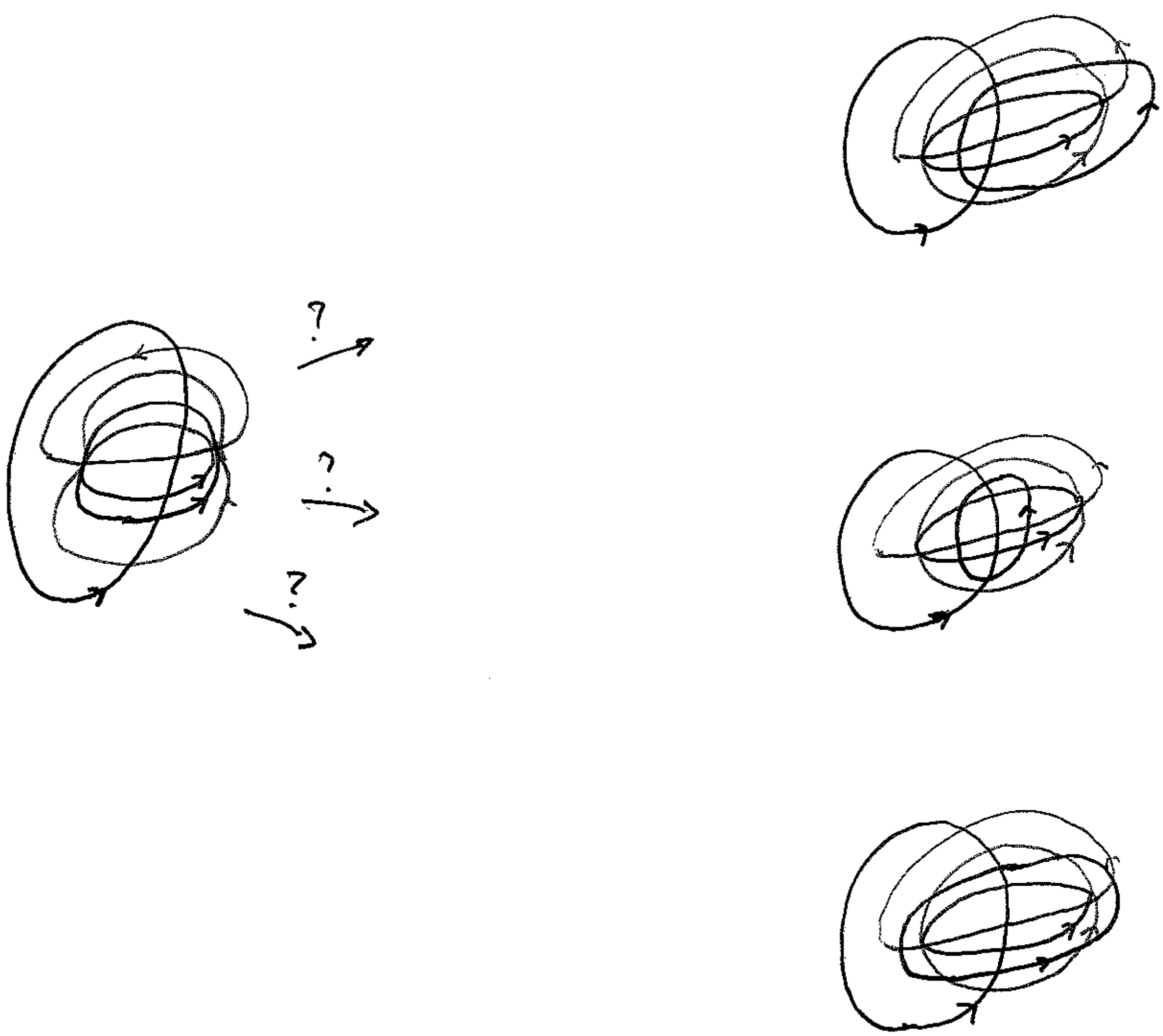
$\mathcal{A}$



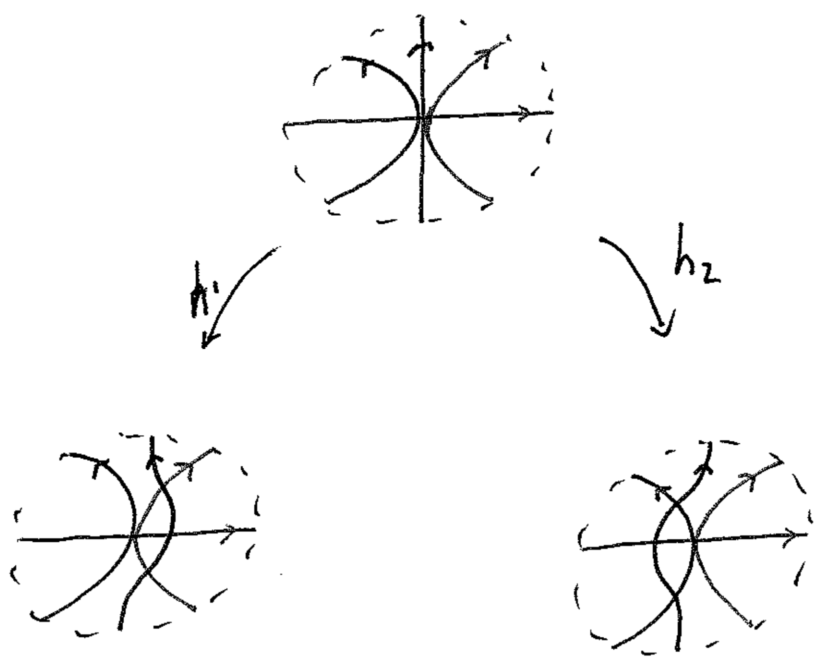
$\Phi(\mathcal{A})$

Homomorphisms Between Picture Groups

Here are pictures that are obviously related to each other, from P_G on the left and from $P_{G'}$ on the right.



The homomorphism is defined by enclosing each vertex in a one vertex subpicture via nonintersecting closed arcs and then stating how each one-vertex subpicture in P_G changes to become a subpicture in $P_{G'}$, maintaining "the same" boundary word. Given $h(v_r)$, then $h(-v_r)$ must be its mirror image, but with arrows reversed.



Theorem: Let $G = \langle X |\, R \rangle$ and $G' = \langle X' |\, R' \rangle$ be finitely presented groups, and let $f : F_X \to F_{X'}$ be an injective homomorphism between the respective free groups on generators. If $f(r) \in N(R')$ for each $r \in R$, then there exists a homomorphism $h : P_G \to P_{G'}$, as just defined.

The idea of the proof:

- Well defined because bridge moves occur entirely outside the closed arcs that define the one-vertex subpictures.
- The image of T_G must be contained in $T_{G'}$.

Sufficient conditions for defining injective homomorphisms between picture groups of line arrangement complements

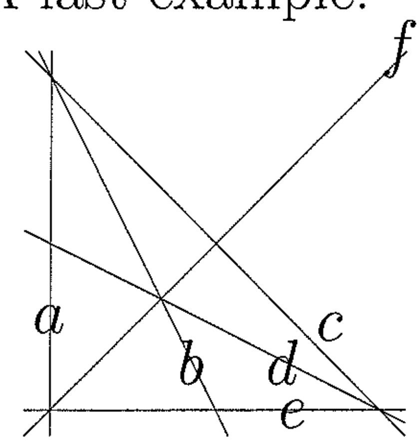
- No new parallel lines:
  $\phi_i l_i \cap \phi_j l_j = \emptyset \Rightarrow l_i \cap l_j = \emptyset$
- Tend toward increased generalization:
  $l_i \cap l_j \cap l_k \neq \emptyset \Rightarrow \phi_i l_i \cap \phi_j l_j \cap \phi_k l_k \neq \emptyset$

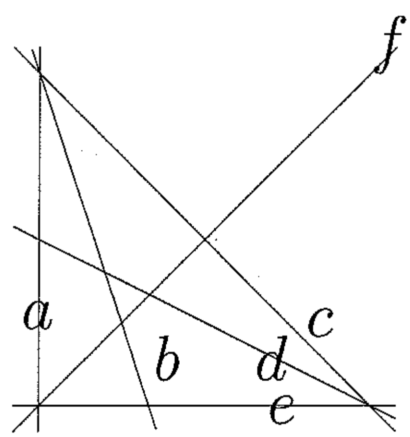
If there exists a transformation Φ such that $\mathcal{B} = \Phi(\mathcal{A})$ meeting the two conditions, we say $\mathcal{A} \geq \mathcal{B}$.

Theorem

If $\mathcal{A} \geq \mathcal{B}$, then there exists a well-defined homomorphism from the picture group of the complement of $\mathcal{A}$ to the picture group of the complement of $\mathcal{B}$. Furthermore, if $|\mathcal{B}| = |\mathcal{A}|$, then the homomorphism is injective.

A last example:





More pictures from respective picture groups:

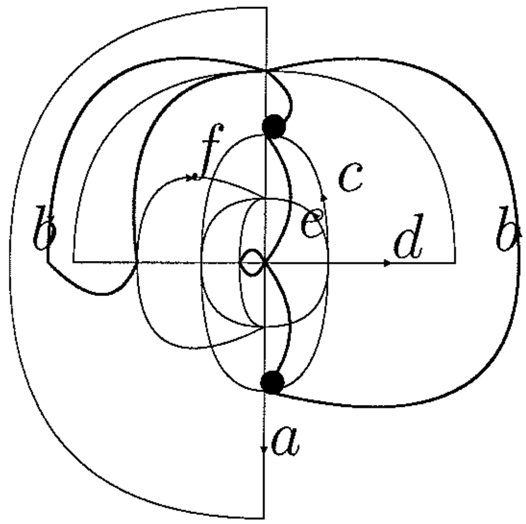


FIGURE 3. A second picture for arrangement $\mathcal{A}$

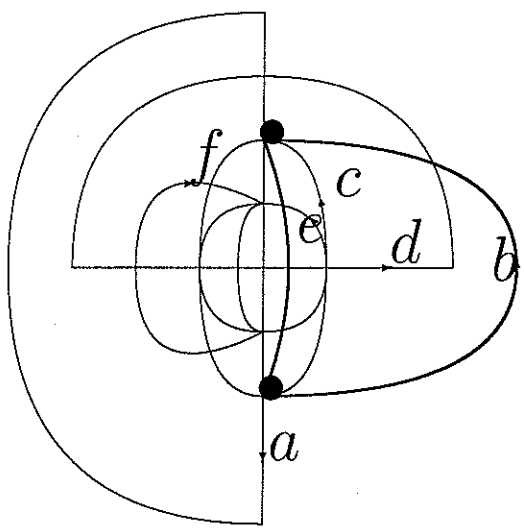


FIGURE 4. A second picture for arrangement $\mathcal{A}'$ obtained from the picture in figure 3 via homomorphism

Pictures from each group with obvious homomorphism:

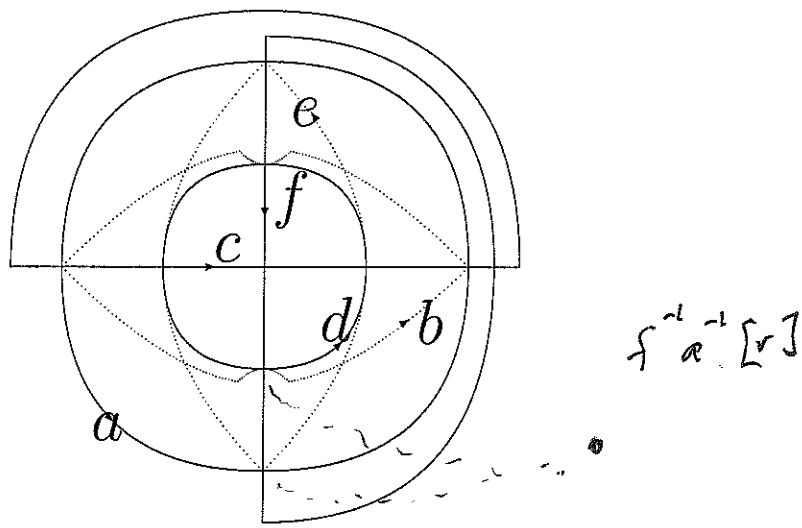


FIGURE 1. A picture in the picture group associated with arrangement $\mathcal{A}$

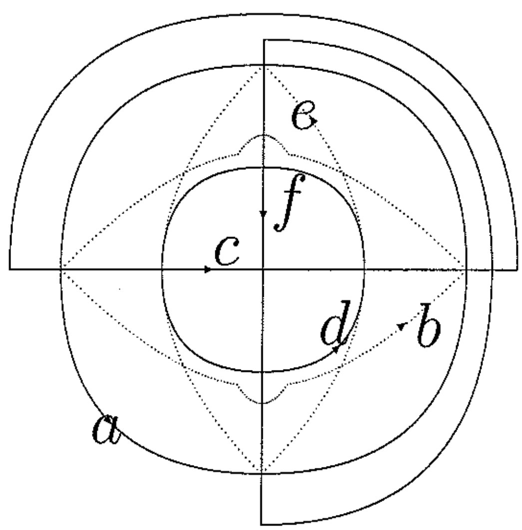


FIGURE 2. A picture in the picture group associated with arrangement $\mathcal{A}'$, obtained from the picture in figure 1 by resolving the intersection of the lines labeled b, d, f

References

W. A. Bogley and J. S. Pride, *Calculating Generators of* π_2, London Mathematical Society Lecture Note Series 197, pp. 157-188, Ed. C. Hog-Angeloni, et. al., Cambridge Univ. Press, Cambridge (1993).

R. Fenn, *Techniques of Geometric Topology*, London Math. Soc. Lecture Notes Series 57, Cambridge Univ. Press, Cambridge (1983).

J. Loday, *Homotopical Syzygies*, Amer. Math. Soc, **265** (2000), pp. 97-127.