SHELLABLE COMPLEXES AND TOPOLOGY OF DIAGONAL ARRANGEMENTS

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ABSTRACT. We prove that if a simplicial complex Δ is shellable, then the intersection lattice L_Δ for the corresponding diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres. Furthermore, we describe precisely the spheres in the wedge, based on the data of shelling. Also, we give some examples of diagonal arrangements $\mathcal{A}$ where the complement $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$, coming from rank 3 matroids.

CONTENTS

1. INTRODUCTION

Consider $\mathbb{R}^n$ with coordinates $u_1, \ldots, u_n$. A *diagonal subspace* $U_{i_1 \cdots i_r}$ is a linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$. A *diagonal arrangement* (or a *hypergraph arrangement*) $\mathcal{A}$ is a finite set of diagonal subspaces of $\mathbb{R}^n$. Throughout this paper, we assume that for any $H_1, H_2 \in \mathcal{A}$, H_1 is not included in H_2.

For a simplicial complex Δ on a vertex set $[n] := \{1, 2, \ldots, n\}$ such that $\dim \Delta \leq n - 3$, one can associate a diagonal arrangement $\mathcal{A}_\Delta$ in $\mathbb{R}^n$ as follows. For a facet F of Δ, let $U_{\overline{F}}$ be the diagonal subspace $u_{i_1} = \cdots = u_{i_r}$ where

$\overline{F} = [n] - F = \{i_1, \ldots, i_r\}$. Define

$$\mathcal{A}_\Delta = \{U_{\overline{F}} \mid F \text{ is a facet of } \Delta\}.$$

If every subspace in a diagonal arrangement $\mathcal{A}$ in $\mathbb{R}^n$ has the form $U_{i_1 \cdots i_r}$ with $r \geq 2$, one can find a simplicial complex Δ on $[n]$ satisfying $\mathcal{A} = \mathcal{A}_\Delta$.

Two important spaces associated with an arrangement $\mathcal{A}$ of linear subspaces in $\mathbb{R}^n$ are

$$\mathcal{M}_\mathcal{A} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{V}_\mathcal{A}^\circ = \mathbb{S}^{n-1} \cap \bigcup_{H \in \mathcal{A}} H,$$

called the *complement* and the *singularity link*.

We are interested in the topology of $\mathcal{M}_\mathcal{A}$ and $\mathcal{V}_\mathcal{A}^\circ$ for a diagonal arrangement $\mathcal{A}$. Diagonal arrangements arise in connection with important questions in many different fields. In computer science, Björner, Lovász and Yao [5] find lower bounds on the complexity of k-equal problems using the topology of diagonal arrangements (see also [4]). In group cohomology, it is well-known that $\mathcal{M}_{\mathcal{B}_n}$ for the braid arrangement $\mathcal{B}_n$ in $\mathbb{C}^n$ is a $K(\pi, 1)$ space with the fundamental group isomorphic to the pure braid group [10]. Khovanov [13] shows that $\mathcal{M}_{\mathcal{A}_{n,3}}$ for the 3-equal arrangement $\mathcal{A}_{n,3}$ in $\mathbb{R}^n$ is also a $K(\pi, 1)$ space.

Note that $\mathcal{M}_\mathcal{A}$ and $\mathcal{V}_\mathcal{A}^\circ$ are related by Alexander duality as follows:

(1.1) $\qquad H^i(\mathcal{M}_\mathcal{A}; \mathbb{F}) = H_{n-2-i}(\mathcal{V}_\mathcal{A}^\circ; \mathbb{F}) \qquad (\mathbb{F} \text{ is any field})$

In the mid 1980's Goresky and MacPherson [11] found a formula for the Betti numbers of $\mathcal{M}_\mathcal{A}$, while the homotopy type of $\mathcal{V}_\mathcal{A}^\circ$ was computed by Ziegler and Živaljević [19] (see Section 5). The answers are phrased in terms of the lower intervals in the *intersection lattice* $L_\mathcal{A}$ of the subspace arrangement $\mathcal{A}$, that is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion. For general subspace arrangements, these lower intervals in $L_\mathcal{A}$ can have arbitrary homotopy type (see [19, Corollary 3.1]).

Our goal is to find a general sufficient condition for the intersection lattice $L_\mathcal{A}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved. Björner and Welker [8] show that $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres, where $\mathcal{A}_{n,k}$ is the k-equal arrangement consisting of all $U_{i_1 \cdots i_k}$ for all $1 \leq i_1 < \cdots < i_k \leq n$ (see Section 3), and Björner and Wachs [6] show that $L_{\mathcal{A}_{n,k}}$ is shellable. More generally, Kozlov [15] shows that $L_\mathcal{A}$ is shellable if $\mathcal{A}$ satisfies certain technical conditions (see Section 3). Suggested by a homological calculation (Theorem 5.4 below), we prove the following main result, capturing the homotopy type assertion from [15] (see Section 4).

Theorem 1.1. *Let Δ be a shellable simplicial complex. Then the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.*

Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. Let Δ be a simplicial complex on $[n]$ with a shelling order $F_1, \ldots, F_q$ on its facets. Let σ be the intersection of all facets, and $\bar{\sigma}$ its complement. Let $G_1 = F_1$ and for each $i \geq 2$, let G_i be the face of F_i obtained by intersecting the walls of F_i that lie in the subcomplex generated by $F_1, \ldots, F_{i-1}$, where a *wall of F_i* is a codimension 1 face of F_i. An *(unordered) shelling-trapped decomposition (of $\bar{\sigma}$ over Δ)* is defined to be a family $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ such that $\{\bar{\sigma}_1, \ldots, \bar{\sigma}_p\}$ is a decomposition of $\bar{\sigma}$ as a disjoint union

$$\bar{\sigma} = \bigsqcup_{j=1}^{p} \bar{\sigma}_j$$

and $F_{i_1}, \ldots, F_{i_p}$ are facets of Δ such that $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all $j = 1, \ldots, p$. Then the wedge of spheres in Theorem 1.1 consists of $(p-1)!$ copies of spheres of dimension

$$p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition D of $\bar{\sigma}$ and a permutation w of $[p-1]$, there exists a saturated chain $C_{D,w}$ (see Section 4.1) such that removing the simplices corresponding to these chains leaves a contractible simplicial complex.

The following example shows that the intersection lattice in Theorem 1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres.

Example 1.2. Let Δ be a simplicial complex on $[8]$ with a shelling 123456 (short for $\{1, 2, 3, 4, 5, 6\}$), $127, 237, 137, 458, 568, 468$. Then the order complex of the upper interval $(U_{78}, \hat{1})$ is a disjoint union of two circles, hence is not shellable. Therefore, the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is also not shellable. The intersection lattice L_Δ is shown in Figure 1 (thick lines represent the open interval $(U_{78}, \hat{1})$). In Figures, the subspace $U_{i_1 \ldots i_r}$ is labeled by $i_1 \ldots i_r$. Also note that a facet F of Δ corresponds to the subspace $U_{[n]-F}$. For example, the facet 127 corresponds to U_{34568}.

The next example shows that there are nonshellable simplicial complexes whose intersection lattices are shellable.

Example 1.3. Let Δ be a simplicial complex on $[4]$ whose facets are 12 and 34. Then Δ is not shellable. But the order complex of $\overline{L}_\Delta$ consists of two vertices, hence is shellable.
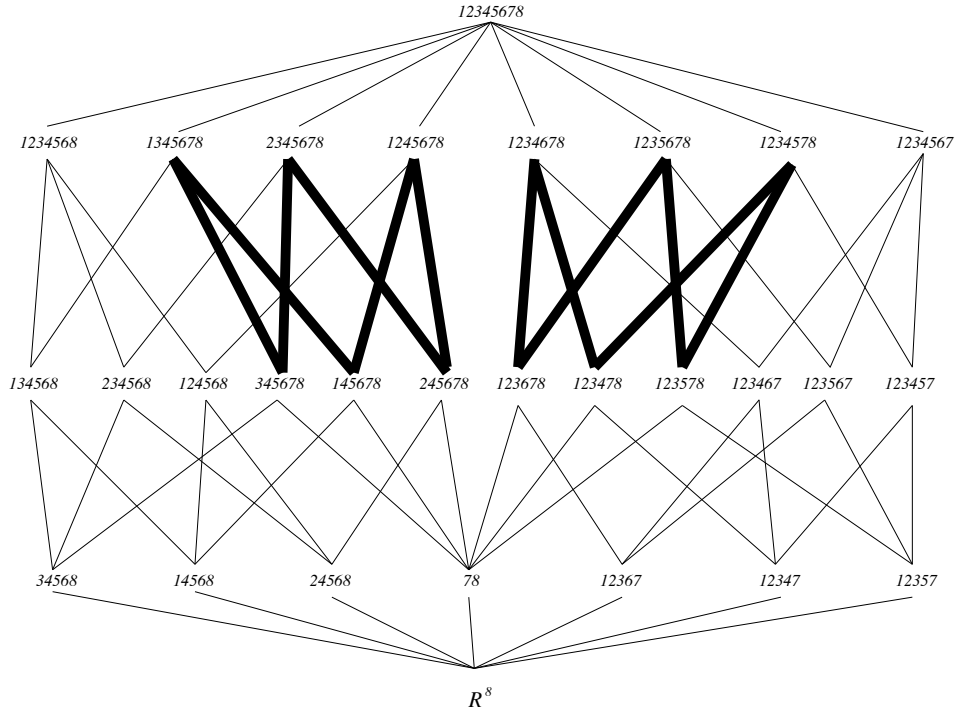
FIGURE 1. The intersection lattice L_Δ for Example 1.2 ($U_{i_1 \ldots i_r}$ is labeled by $i_1 \ldots i_r$.)

In Section 3, we give Kozlov's result and show that its homotopy type consequence is a special case of Theorem 1.1. Also, we give a new proof of Björner and Welker's result using Theorem 1.1. Section 4 proves Theorem 1.1. In Section 5, we deduce from it the homotopy type and the homology of the singularity link (and hence the homology of the complement) of a diagonal arrangement $\mathcal{A}_\Delta$ for a shellable simplicial complex Δ. In Section 6, we give some examples in which $\mathcal{M}_\mathcal{A}$ are $K(\pi, 1)$, coming from matroids.

2. BASIC NOTIONS AND DEFINITIONS

In this section, we give a few definitions used throughout this paper.

First, we start with the definition of (non-pure) shellability of simplicial complexes; see [6], [7] for further background on this notion.

Definition 2.1. A simplicial complex is *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_q$ in such a way that $(\bigcup_{i=1}^{k-1} 2^{F_i}) \cap 2^{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, q$, where $2^F = \{G | G \subseteq F\}$. Such an ordering of facets is called a *shelling order* or *shelling*.

There are several equivalent definitions of shellability. The following restatement of shellability is often useful.

Lemma 2.2. [6, Lemma 2.3] *A linear order $F_1, F_2, \ldots, F_q$ of facets of a simplicial complex is a shelling if and only if for every F_i and F_k with $F_i < F_k$, there is a facet $F_j < F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k \lessdot F_k$, where $G \lessdot F$ means that G has codimension 1 in F.*

It is well-known that a pure d-dimensional shellable simplicial complex has the homotopy type of a wedge of d-spheres. Björner and Wachs [6] generalize this result to the nonpure case, i.e., a nonpure shellable simplicial complex has the homotopy type of a wedge of spheres, however these spheres need not be equidimensional.

The *link* of a face σ of a simplicial complex Δ is

$$\mathrm{link}_\Delta \sigma = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}.$$

Björner and Wachs [7] show that shellability is inherited by all links of faces in a simplicial complex.

Proposition 2.3. *If Δ is shellable, then so is $\mathrm{link}_\Delta \sigma$ for all faces $\sigma \in \Delta$, using the induced order on facets of $\mathrm{link}_\Delta \sigma$.*

Now we give the definition of the order complex of a poset which we will use frequently.

Definition 2.4. The *order complex* $\Delta(P)$ of a poset P is the simplicial complex whose vertices are the elements of P and whose faces are the chains of P.

For the order complex $\Delta((x, y))$ of an open interval (x, y), we will use the notation $\Delta(x, y)$. When we say that a finite lattice L with bottom element $\hat{0}$ and top element $\hat{1}$ has some topological properties, such as purity, shellability and homotopy type, it means the order complex of $\overline{L} := L - \{\hat{0}, \hat{1}\}$ has those properties.

3. Special cases that were known

In this section, we give Kozlov's theorem and show how its consequence for homotopy type follows from Theorem 1.1. Also, we give a new proof of Björner and Welker's theorem about the intersection lattice of the k-equal arrangements using Theorem 1.1.

Kozlov [15] shows that $\mathcal{A}_\Delta$ has shellable intersection lattice if Δ satisfies some conditions. This class includes k-equal arrangements and all other diagonal arrangements for which the intersection lattice was proved shellable.

Theorem 3.1. [15, Corollary 3.2] *Consider a partition of $[n] = E_1 \sqcup \cdots \sqcup E_r$ such that $\max E_i < \min E_{i+1}$ for $i = 1, \ldots, r-1$. Let*

$$f : \{1, 2, \ldots, r\} \to \{2, 3, \ldots\}$$

be a nondecreasing map. Let Δ be a simplicial complex on $[n]$ such that F is a facet of Δ if and only if

(1) $|E_i - F| \leq 1$ for $i = 1, \ldots, r$;
(2) if $\min \overline{F} \in E_i$ then $|F| = n - f(i)$.

Then the intersection lattice for $\mathcal{A}_\Delta$ is shellable. In particular, L_Δ has the homotopy type of a wedge of spheres.

Proposition 3.2. Δ *in Theorem 3.1 is shellable.*

Proof. We claim that a shelling order is $F_1, F_2, \ldots, F_q$ such that the words $w_1, w_2, \ldots, w_q$ are in lexicographic order, where w_i is the increasing array of elements in $\overline{F}_i$. Let F_s, F_t be two facets of Δ with $1 \leq s < t \leq q$. Then $w_s \prec_{lex} w_t$. Let m be the first number in $[r]$ such that $E_m - F_s \neq E_m - F_t$. Construct the word w as follows:

(1) $w \cap E_i = w_s \cap E_i$ for $i = 1, \ldots, m$;
(2) for $i = m+1, \ldots, q$,

$$w \cap E_i = \begin{cases} w_t \cap E_i & \text{if } |w \cap (\cup_{j=1}^i E_j)| \leq f(l); \\ \emptyset & \text{otherwise}, \end{cases}$$

where $\min w_s \in E_l$.

Note that the length of w is $f(l)$ and $w \prec_{lex} w_t$. Let F be the set of all elements which do not appear in w. Since F satisfies the two conditions from Theorem 3.1, F is a facet of Δ. Since $F \cap F_t = F_t - (E_m - F_s)$ and $E_m - F_s$ is a subset of F_t of size 1, $F \cap F_t$ has codimension 1 in F_t. Also $F_s \cap F_t \subseteq F \cap F_t$. Hence $F_1, F_2, \ldots, F_q$ is a shelling by Lemma 2.2. □

Example 3.3. Consider the partition of

$$[7] = \{1\} \sqcup \{2, 3\} \sqcup \{4\} \sqcup \{5, 6, 7\}$$

and the function f given by $f(1) = 2, f(2) = 3, f(3) = 4$, and $f(4) = 5$. Then the facets of the simplicial complex Δ that satisfy the conditions from Corollary 3.1 and the corresponding words can be found in Table 1. Thus the ordering 34567, 24567, 23567, 23467, 23457, 23456, 1367, 1357, 1356, 1267, 1257 and 1256 is a shelling for Δ.

One can also use Theorem 1.1 to recover the following theorem of Björner and Welker [8].

min $\overline{F}$	F	w	min $\overline{F}$	F	w	min $\overline{F}$	F	w
1	23456	17	2	1356	247	3	1256	347
	23457	16		1357	246		1257	346
	23467	15		1367	245		1267	345
	23567	14						
	24567	13						
	34567	12						

TABLE 1. Facets of Δ from Example 3.3 and corresponding words

Theorem 3.4. *The order complex of the intersection lattice $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres consisting of*

$$(p-1)! \sum_{0=i_0 \leq i_1 \leq \cdots \leq i_p = n-pk} \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j}$$

copies of $(n-3-p(k-2))$-dimensional spheres for $1 \leq p \leq \lfloor \frac{n}{k} \rfloor$.

Proof. It is clear that $\mathcal{A}_{n,k} = \mathcal{A}_{\Delta_{n,n-k}}$, where $\Delta_{n,n-k}$ is a simplicial complex on $[n]$ whose facets are all $n-k$ subsets of $[n]$. Here, $\sigma = \emptyset$, and so $\bar{\sigma} = [n]$. By ordering the elements of each facet in increasing order, the lexicographic order of facets of $\Delta_{n,n-k}$ gives a shelling. Also, one can see that the facets of the form $F_i = \{1, 2, \ldots, m, a_{m+1}, \ldots, a_{n-k}\}$, where $m+1 < a_{m+1} < \cdots < a_{n-k}$, have $G_i = \{1, \ldots, m\}$. Thus, $G_i \subseteq \sigma_i \subseteq F_i$ implies $\overline{F}_i \subseteq \bar{\sigma}_i \subseteq \overline{G}_i = \{m+1, \ldots, n\}$. Note that $\min \bar{\sigma} = \min \overline{F} = m+1$ and $\overline{F}$ has k elements. Thus, in any shelling-trapped decomposition $[n] = \sqcup_{j=1}^{p} \bar{\sigma}_j$, one has $p \leq \lfloor \frac{n}{k} \rfloor$.

Let $1 \leq p \leq \lfloor \frac{n}{k} \rfloor$ and $0 = i_0 \leq i_1 \leq \cdots \leq i_p = n-pk$. We will construct a shelling-trapped family $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ as in Theorem 1.1. Since $F_{i_1} < \cdots < F_{i_p}$, we have $\min \bar{\sigma}_1 > \cdots > \min \bar{\sigma}_p$. In particular, $1 \in \overline{F}_{i_p} \subseteq \bar{\sigma}_p$. Thus there are $\binom{n-1}{k-1}$ ways to pick $\overline{F}_{i_p}$ (equivalently, F_{i_p}). Now suppose that we have chosen $F_{i_p}, \ldots, F_{i_{p-j+1}}$. We pick $F_{i_{p-j}}$ so that $\min \overline{F}_{i_{p-j}} = \min \bar{\sigma}_{i_{p-j}}$ is the $i_j + 1$st element in $[n] - (F_{i_p} \cup \cdots \cup F_{i_{p-j+1}})$. Then we have $\binom{n-jk-i_j-1}{k-1}$ ways to choose $F_{i_{p-j}}$. For each $j = 1, \ldots, p$, there are $i_j - i_{j-1}$ elements in $[n] - (F_{i_p} \cup \cdots \cup F_{i_{p-j+1}})$ which are strictly between $\min \overline{F}_{i_{p-j+1}}$ and $\min \overline{F}_{i_{p-j}}$ and they must be contained in one of $\bar{\sigma}_p, \ldots, \bar{\sigma}_{p-j+1}$ (i.e., there are $j^{i_j-i_{j-1}}$ choices). Therefore there are

$$\prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} \prod_{j=1}^{p} j^{i_j-i_{j-1}} = \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j}$$

shelling-trapped families. By Theorem 1.1, each of those families contributes $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p}(n-k) + n - 3 = n - 3 - p(k-2).$$

$\square$

4. Proof of main theorem

Theorem 1.1 will be deduced from a more general statement about homotopy types of lower intervals in L_Δ, Theorem 4.1 below. Throughout this section, we assume that Δ is a simplicial complex on $[n]$ with $\dim \Delta \leq n-3$.

Theorem 4.1. *Let $F_1, \ldots, F_q$ be a shelling of Δ and $U_{\bar{\sigma}}$ a subspace in L_Δ for some subset $\bar{\sigma}$ of $[n]$. Then $\Delta(\hat{0}, U_{\bar{\sigma}})$ is homotopy equivalent to a wedge of spheres, consisting of $(p-1)!$ copies of spheres of dimension*

$$\delta(D) := p(2-n) + \sum_{j=1}^{p}|F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$.

Moreover, for each such shelling-trapped decomposition D and each permutation w of $[p-1]$, one can construct a saturated chain $C_{D,w}$ (see Section 4.1 below), such that if one removes the corresponding $\delta(D)$-dimensional simplices for all pairs (D, w), the remaining simplicial complex $\widehat{\Delta}(\hat{0}, U_{\bar{\sigma}})$ is contractible.

To prove this result, we begin with some preparatory lemmas.

First of all, one can characterize exactly which subspaces lie in L_Δ when Δ is shellable. Recall that for $\bar{\sigma} = \{i_1, \ldots, i_r\} \subseteq [n]$, we denote by $U_{\bar{\sigma}}$ the diagonal subspace of the form $u_{i_1} = \cdots = u_{i_r}$. We also use the notation $U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$ to denote $U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}$ for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$.

A simplicial complex is called *gallery-connected* if any pair F, F' of facets are connected by a path

$$F = F_0, F_1, \ldots, F_{r-1}, F_r = F'$$

of facets such that $F_i \cap F_{i-1}$ has codimension 1 in F_i for $i = 1, \ldots, r$. Since it is known that Cohen-Macaulay simplicial complexes are gallery-connected, shellable simplicial complexes are gallery-connected.

Lemma 4.2.   *(1) Given any simplicial complex Δ on $[n]$, every subspace H in L_Δ has the form*

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$ such that σ_i can be expressed as an intersection of facets of Δ for $i = 1, 2, \ldots, p$.

(2) *Conversely, when Δ is gallery-connected, every subspace H of $\mathbb{R}^n$ that has the above form lies in L_Δ.*

Proof. To see (1), note that every subspace H in L_Δ has the form

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$. Since $H = \bigcap_{F \in \mathcal{F}} U_{\overline{F}}$ for some family $\mathcal{F}$ of facets of Δ,

$$U_{\bar{\sigma}_j} = \bigcap_{F \in \mathcal{F}_j} U_{\overline{F}}$$

for some subfamily $\mathcal{F}_j$ of $\mathcal{F}$ for all $j = 1, \ldots, p$. Therefore

$$\sigma_j = \bigcap_{F \in \mathcal{F}_j} F$$

for $j = 1, \ldots, p$.

For (2), suppose that H has the form

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$ such that σ_i can be expressed as an intersection of facets of Δ for $i = 1, 2, \ldots, p$. It is enough to show the case when $H = U_{\bar{\sigma}}$. Since gallery-connectedness is inherited by all links of faces in a simplicial complex, we may assume $\sigma = \bigcap_{F \in \mathcal{F}} F$, where $\mathcal{F}$ is the set of all facets of Δ, without loss of generality. Then $\bar{\sigma} = \bigcup_{F \in \mathcal{F}} \overline{F}$.

We claim that the simplicial complex Γ whose facets are $\{\overline{F} \mid F \in \mathcal{F}\}$ is connected. Since $\dim \Delta \leq n - 3$, every facet of Γ has at least two elements. Let $\overline{F}, \overline{F'}$ be two facets of Γ with $F < F'$. Since Δ is gallery-connected, there is $F = F_1, F_2, \ldots, F_k = F'$ such that $F_i \cap F_{i-1}$ has codimension 1 in F_i for all $i = 2, \ldots, k$. Thus $\overline{F}_i$ and $\overline{F}_{i-1}$ share at least one vertex for all $i = 2, \ldots, k$. This implies that $\overline{F}$ and $\overline{F'}$ are connected. Hence Γ is connected.

Therefore $U_{\bar{\sigma}} = \bigcap_{F \in \mathcal{F}} U_{\overline{F}}$. $\qquad\square$

The next example shows that the conclusion of Lemma 4.2(2) can fail when Δ is not assumed to be gallery-connected.

Example 4.3. Let Δ be a simplicial complex with two facets 123 and 345. Then Δ is not gallery-connected. Since L_Δ has only four subspaces $\mathbb{R}^5, U_{12}, U_{45}$ and $U_{12/45}$, it does not have the subspace U_{1245}, even though $\overline{1245} = 3$ is an intersection of facets 123 and 345 of Δ. Thus the conclusion of Lemma 4.2(2) fails for Δ.

The following easy lemma, whose obvious proof is omitted, shows that every lower interval $[\hat{0}, H]$ can be written as a product of lower intervals of the form $[\hat{0}, U_{\bar{\sigma}}]$.

Lemma 4.4. *Let $H \in L_\Delta$ be a subspace of the form*

$$H = U_{\bar{\sigma}_1 / \cdots / \bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$. Then

$$[\hat{0}, H] = [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

In particular,

$$\Delta(\hat{0}, H) = \Delta(\hat{0}, U_{\bar{\sigma}_1}) * \cdots * \Delta(\hat{0}, U_{\bar{\sigma}_p}) * \mathbb{S}^{p-2},$$

where " $*$ " denotes join of spaces.

Note that the join of a space X with an empty set equals X.

The next lemma, whose proof is completely straightforward and omitted, shows that the lower interval $[\hat{0}, U_{\bar{\sigma}}]$ is isomorphic to the intersection lattice for the diagonal arrangement corresponding to $\mathrm{link}_\Delta \sigma$.

Lemma 4.5. *Let $U_{\bar{\sigma}}$ be a subspace in L_Δ for some face σ of Δ. Then the lower interval $[\hat{0}, U_{\bar{\sigma}}]$ is isomorphic to the intersection lattice of the diagonal arrangement $\mathcal{A}_{\mathrm{link}_\Delta(\sigma)}$ corresponding to $\mathrm{link}_\Delta(\sigma)$ on the vertex set $\bar{\sigma}$.*

The following lemma shows that upper intervals in L_Δ are at least still *homotopy equivalent* to the intersection lattice of a diagonal arrangement.

Lemma 4.6. *Let $U_{\bar{\sigma}}$ be a subspace in L_Δ for some face σ of Δ. Then the upper interval $[U_{\bar{\sigma}}, \hat{1}]$ is homotopy equivalent to the intersection lattice of the diagonal arrangement $\mathcal{A}_{\Delta_\sigma}$ corresponding to the simplicial complex Δ_σ on the vertex set $\sigma \cup \{v\}$ whose facets are obtained in the following ways:*

    (A) If $F \cap \sigma$ is maximal among

$$\{F \cap \sigma \mid F \text{ is a facet of } \Delta \text{ with } \sigma \nsubseteq F \text{ and } F \cup \sigma \neq [n]\},$$

    then $\widetilde{F} = F \cap \sigma$ is a facet of Δ_σ.
    (B) If a facet F of Δ satisfies $F \cup \sigma = [n]$, then $\widetilde{F} = (F \cap \sigma) \cup \{v\}$ is a facet of Δ_σ.

Proof. We apply a standard crosscut/closure lemma ([3, Theorem 10.8]) saying that a finite lattice L is homotopy equivalent to the sublattice consisting of the joins of subsets of its atoms. By the closure relation ψ_1 on $[U_{\bar{\sigma}}, \hat{1}]$ which sends a subspace to the intersection of all subspaces that lie weakly below it and cover $U_{\bar{\sigma}}$ in $[U_{\bar{\sigma}}, \hat{1}]$, one can see that $[U_{\bar{\sigma}}, \hat{1}]$ is homotopy equivalent to the sublattice L_σ generated by the subspaces of $[U_{\bar{\sigma}}, \hat{1}]$ that cover $U_{\bar{\sigma}}$. Using the map ψ_2 defined by

$$\psi_2(U_{\bar{\tau}}) = \begin{cases} U_{(\bar{\tau} - \bar{\sigma}) \cup \{v\}} & \text{if } \bar{\sigma} \cap \bar{\tau} \neq \emptyset, \\ U_{\bar{\tau}} & \text{otherwise,} \end{cases}$$
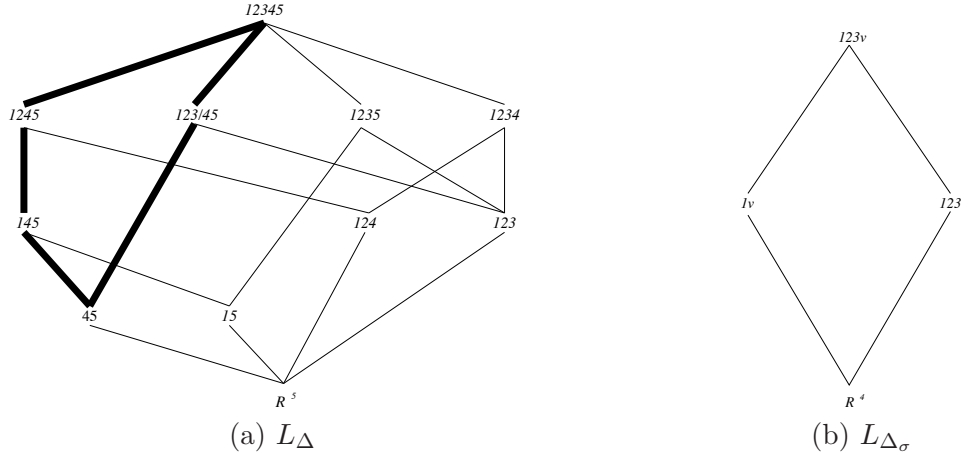
FIGURE 2. The intersection lattices for Δ and Δ_σ

one can see that L_σ is isomorphic to the intersection lattice L_{Δ_σ} for a simplicial complex Δ_σ on the vertex set $\sigma \cup \{v\}$. The facets of Δ_σ correspond to the subspaces that cover $U_{\bar\sigma}$, giving the claimed characterization of facets of Δ_σ. Therefore the map $\psi := \psi_2 \circ \psi_1$ gives the homotopy equivalence between $[U_{\bar\sigma}, \hat{1}]$ and L_{Δ_σ}. $\qquad\qquad\square$

Facets of Δ_σ that do not contain v are called *facets of type (A)*, and facets of Δ_σ containing v are called *facets of type (B)*.

Example 4.7. Let Δ be a simplicial complex on [5] with facets 123, 234, 35, 45 and let $\sigma = \{1, 2, 3\}$. Then Δ_σ is a simplicial complex on $\{1, 2, 3, v\}$ and its facets are 23 and v. The intersection lattices L_Δ and L_{Δ_σ} are shown in Figure 2 and it is easy to see that the order complex for $\overline{L}_{\Delta_\sigma}$ is homotopy equivalent to the order complex for the open interval $(U_{45}, \hat{1})$ in L_Δ. Note that the thick lines in Figure 2 (a) represent the closed interval $[U_{45}, \hat{1}]$ in L_Δ.

In general, the simplicial complex Δ_σ from Lemma 4.6 is not shellable, even though Δ is shellable (see Example 1.2). However, the next lemma shows that Δ_σ is shellable if σ is the last facet in the shelling order.

Lemma 4.8. *Let Δ be a shellable simplicial complex. If F is the last facet in a shelling order of Δ, then Δ_F is also shellable. Moreover, if $\widetilde{F}_i$ is a facet of Δ_F of type (B), then $\widetilde{G}_i = G_i \cap F$.*

Proof. One can check that the following gives a shelling order on the facets of Δ_F. First list the facets of type (A) according to the order of their earliest corresponding facet of Δ, followed by the facets of type (B) according to the order of their corresponding facet of Δ.

To see the second assertion, let $\widetilde{F}_i$ be a facet of Δ_F of type (B), i.e.,

$$\widetilde{F}_i = (F_i \cap F) \cup \{v\}$$

for some facet F_i of Δ such that $F \cup F_i = [n]$. Then $\widetilde{G}_i = G_i \cap F$ follows from the observations that $F_i \cap F$ is an old wall of $\widetilde{F}_i$, and all other old walls of $\widetilde{F}_i$ are $\widetilde{F}_k \cap \widetilde{F}_i$ for some facets $\widetilde{F}_k$ of Δ_F of type (B) such that $F_k \cap F_i$ is an old wall of F_i. $\square$

Example 4.9. Let Δ be a shellable simplicial complex on [7] with a shelling $12367, 12346, 13467, 34567, 13457, 14567, 12345$ and let $F = 12345$. Then Δ_F is a simplicial complex on $\{1, 2, 3, 4, 5, v\}$ and its facets are $123v, 1234, 134v, 345v, 1345, 145v$. Since $1234, 1345$ are facets of Δ_F of type (A) and $123v, 134v, 345v, 145v$ are facets of Δ_F of type (B), the ordering $1234, 1345, 123v, 134v, 345v, 145v$ is a shelling of Δ_F.

We next construct the saturated chains appearing in the statement of Theorem 4.1.

4.1. Constructing the chains $C_{D,w}$.

Let Δ be a simplicial complex on $[n]$ with a shelling $F_1, F_2, \ldots, F_q$ and let $U_{\bar{\sigma}}$ be a subspace in L_Δ. Let

$$D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$$

be a shelling-trapped decomposition of $\bar{\sigma}$ with $i_1 < i_2 < \cdots < i_p$, and let w be a permutation on $[p-1]$.

It is well known that the lattice Π_p of partitions of the set $[n]$ ordered by refinement is homotopy equivalent to a wedge of $(p-1)!$ spheres of dimension $p-3$ and there is a saturated chain C_w in Π_p for each permutation w of $[p-1]$ such that removing $\{\overline{C}_w = C_w - \{\hat{0}, \hat{1}\} | w \in \mathfrak{S}_{p-1}\}$ from the order complex of $\overline{\Pi}_p$ gives a contractible subcomplex (see [1, Example 2.9]).

We construct a chain $C_{D,w}$ in $[\hat{0}, U_{\bar{\sigma}}]$ as follows:

(1) Since $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all $j = 1, 2, \ldots, p$, Lemma 4.2 shows that the interval $[U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}, U_{\bar{\sigma}}]$ contains the subspaces $U_{B_1/\cdots/B_r}$, where $B_m = \cup_{k \in K_m} \bar{\tau}_k$ such that
   - $(K_1/K_2/\cdots/K_r)$ is a partition of $[p]$, and
   - $\overline{F}_{i_k} \subseteq \bar{\tau}_k \subseteq \bar{\sigma}_k$ for all $k = 1, 2, \ldots, p$.

   Choose a saturated chain $\widehat{C}_w$ in $[U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}, U_{\bar{\sigma}}]$ whose covering relations are one of the following types (see [15]):
   (a) $U_{B_1/B_2/\cdots/B_r} \lessdot U_{B_1 \cup B_2/B_3/\cdots/B_r}$,
   (b) $U_{B_1/\cdots/B_r} \lessdot U_{B_1 \cup \{a\}/B_2/\cdots/B_r}$, where $a \in \bar{\sigma}_k - \overline{F}_{i_k}$ such that $\overline{F}_{i_k} \subseteq B_1$, where the covering relations of type (a) correspond to the saturated chain C_w in Π_p, i.e., the covering relation $U_{B_1/B_2/\cdots/B_r} \lessdot U_{B_1 \cup B_2/B_3/\cdots/B_r}$

appears exactly where $(K_1/K_2/\cdots/K_r) \lessdot (K_1 \cup K_2/K_3/\cdots/K_r)$ appears in Π_p (where $\lessdot$ means the covering relation).

(2) Define a saturated chain $C_{D,w}$ by

$$\hat{0} \lessdot U_{\overline{F}_{i_p}} \lessdot U_{\overline{F}_{i_{p-1}}/\overline{F}_{i_p}} \lessdot \cdots \lessdot U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}$$

followed by the chain $\widehat{C}_w$.

Note that the length of the chain $\overline{C}_{D,w} = C_{D,w} - \{\hat{0}, U_{\bar{\sigma}}\}$ is

$$l(\overline{C}_{D,w}) = p + (p-1) + \sum_{j=1}^{p} \left(|\bar{\sigma}_j| - |\overline{F}_{i_j}| \right) - 2$$

$$= p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3.$$

Example 4.10. Let Δ be the shellable simplicial complex from Example 4.9. Then one can see that

$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$

is a shelling-trapped decomposition of $[7]$. Let w be the permutation in $\mathfrak{S}_2$ with $w(1) = 2$ and $w(2) = 1$. Then the maximal chain C_w in Π_3 corresponding to w is $(1/2/3) \lessdot (1/23) \lessdot (123)$. One can choose

$$\widehat{C}_w = U_{45/23/67} \lessdot U_{45/123/67} \lessdot U_{45/12367} \lessdot U_{1234567},$$

where the covering $U_{45/123/67} \lessdot U_{45/12367}$ corresponds to $(1/2/3) \lessdot (1/23)$, and $U_{45/12367} \lessdot U_{1234567}$ corresponds to $(1/23) \lessdot (123)$. Thus $C_{D,w}$ is the chain

$$\hat{0} \lessdot U_{67} \lessdot U_{23/67} \lessdot U_{45/23/67} \lessdot U_{45/123/67} \lessdot U_{45/12367} \lessdot U_{1234567}.$$

The upper interval $(U_{67}, \hat{1})$ is shown in Figure 3 and the chain $\overline{C}_{D,w}$ is represented by thick lines.

The following lemma gives the relationship between the shelling-trapped decompositions of $[n]$ containing F and the shelling-trapped decompositions of $F \cup \{v\}$.

Lemma 4.11. *Let Δ be a shellable simplicial complex such that the intersection of all facets is empty. If F is the last facet in the shelling order of Δ, then there is a one-to-one correspondence between*

- *pairs (D, w) of shelling-trapped decompositions D of $[n]$ over Δ containing F and $w \in \mathfrak{S}_{|D|-1}$, and*
- *pairs $(\widetilde{D}, \widetilde{w})$ of shelling-trapped decompositions $\widetilde{D}$ of $F \cup \{v\}$ over Δ_F and $\widetilde{w} \in \mathfrak{S}_{|\widetilde{D}|-1}$.*
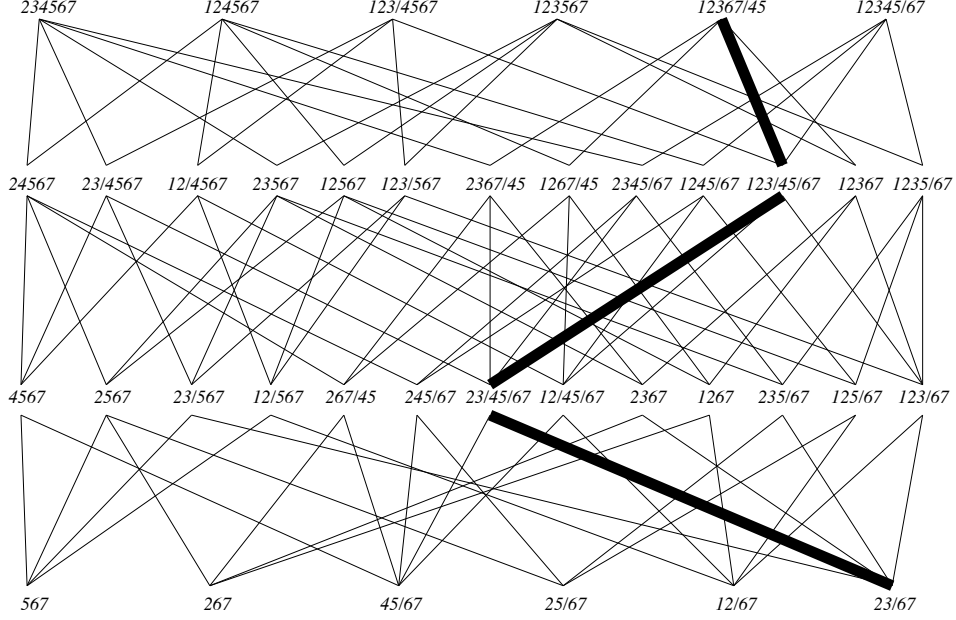
FIGURE 3. The upper interval $(U_{67}, \hat{1})$ in L_Δ

Moreover, one can choose $\overline{C}_{D,w}$ and $\overline{C}_{\widetilde{D},\widetilde{w}}$ in such a way that the homotopy equivalence ψ appearing in the proof of Lemma 4.6 maps the chain $\overline{C}_{D,w} - U_{\overline{F}}$ to the chain $\overline{C}_{\widetilde{D},\widetilde{w}}$.

Proof. Let $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ be a shelling-trapped decomposition of $[n]$ over Δ with $F_{i_p} = F$ and let w be a permutation in $\mathfrak{S}_{p-1}$. Then $\widetilde{F}_{i_j} = (F_{i_j} \cap F) \cup \{v\}$ are facets of Δ_F of type (B) for $j = 1, \ldots, p-1$ and $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_{p-1}}$. By Lemma 4.8, $\widetilde{G}_{i_j} = G_{i_j} \cap F$ for all $j = 1, \ldots, p-1$.

There are two cases to consider:

Case 1. $\sigma_p \neq F$.

In this case, we will show

$$\widetilde{D} = \{([\bar{\sigma}_p \cap F] \cup \{v\}, \widetilde{F}), (\bar{\sigma}_1, \widetilde{F}_{i_1}), \ldots, (\bar{\sigma}_{p-1}, \widetilde{F}_{i_{p-1}})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$ over Δ_F ($\widetilde{F}$ will be defined later).

Define

$$\tilde{\sigma}_j = \begin{cases} (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, p-1, \\ \sigma_j & \text{for } j = p. \end{cases}$$

For $j = 1, \ldots, p-1$, $\widetilde{G}_{i_j} \subseteq \tilde{\sigma}_j \subseteq \widetilde{F}_{i_j}$ since $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$.

Since $G_{i_p} \subseteq \sigma_p$, it must be that σ_p is an intersection of some old walls of F. Thus one can find a family $\mathcal{G}$ of facets of Δ such that $\sigma_p = \cap_{F' \in \mathcal{G}}(F' \cap F)$ and $F' \cap F \prec F$ for all $F' \in \mathcal{G}$. Since $|F' \cup F| = |F'| + 1 < n$, one knows that $F' \cap F$ is a facet of Δ_F of type (A) for all $F' \in \mathcal{G}$. Let $\widetilde{F} = F_k \cap F$ be the last facet in the family $\{F' \cap F | F' \in \mathcal{G}\}$ (pick k as small as possible). Since all facets of Δ_F occurring earlier than $\widetilde{F}$ have the form $F \cap F_i$ such that $F_i < F_k$ and $F_i \cap F \prec F$, one can see $\widetilde{G} \subseteq \tilde{\sigma}_p \subseteq \widetilde{F}$. Thus $\widetilde{D}$ is a shelling-trapped decomposition of $F \cup \{v\}$ over Δ_F. Also one can define $\widetilde{w} \in \mathfrak{S}_{p-1}$ by

$$\widetilde{w}(j) = \begin{cases} w(j-1) & \text{if } 1 < j \leq p-1, \\ w(p-1) & \text{if } j = 1. \end{cases}$$

Case 2. $\sigma_p = F$.

In this case, we claim that

$$\widetilde{D} = \{(\bar{\sigma}_1, \widetilde{F}_{i_1}), \ldots, (\bar{\sigma}_k \cup \{v\}, \widetilde{F}_{i_k}), \ldots, (\bar{\sigma}_{p-1}, \widetilde{F}_{i_{p-1}})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$.

Let $k = w(1)$. Define

$$\tilde{\sigma}_j = \begin{cases} \sigma_j \cap F & \text{for } j = k, \\ (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, \hat{k}, \ldots, p-1. \end{cases}$$

Then

$$\widetilde{G}_{i_k} \subseteq \tilde{\sigma}_k = \sigma_k \cap F \subseteq \widetilde{F}_{i_k}$$

and

$$\widetilde{G}_{i_j} \subseteq \tilde{\sigma}_j = (\sigma_j \cap F) \cup \{v\} \subseteq \widetilde{F}_{i_j}$$

for $j = 1, \ldots, \hat{k}, \ldots, p-1$. Thus $\widetilde{D}$ is a shelling-trapped decomposition of $F \cup \{v\}$. Also one can define $\widetilde{w} \in \mathfrak{S}_{p-2}$ as follows:

$$\widetilde{w}(j) = \begin{cases} w(j+1) & \text{if } w(j+1) < k, \\ w(j+1) - 1 & \text{if } w(j+1) > k. \end{cases}$$

Conversely, let

$$\widetilde{D} = \{([F \cup \{v\}] - \tilde{\sigma}_1, \widetilde{F}_{i_1}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, \widetilde{F}_{i_p})\}$$

be a shelling-trapped decomposition of $F \cup \{v\}$, where $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_p}$ are facets of Δ_F, and let $\widetilde{w}$ be a permutation in $\mathfrak{S}_{p-1}$. There is at most one facet of Δ_F of type (A) because $\widetilde{F}_{i_j} \cup \widetilde{F}_{i_k} = F \cup \{v\}$ for all $j \neq k$. Since $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_p}$ and the facets of type (A) appear earlier than the ones of type (B), there are two possible cases.

Case 1. $v \notin \widetilde{F}_{i_1}$ and $v \in \widetilde{F}_{i_j}$ for $j = 2, \ldots, p$.

In this case, $v \notin \tilde{\sigma}_p$, i.e., $v \in [F \cup \{v\}] - \tilde{\sigma}_p$. One can show that a family

$$D = \{([F \cup \{v\}] - \tilde{\sigma}_2, F_{i_2}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, F_{i_p}), ([n] - \tilde{\sigma}_1, F)\},$$

where $F_{i_j} = (\widetilde{F}_{i_j} \cup F) - \{v\}$ for $j = 2, \ldots, p$, is a shelling-trapped decomposition of $[n]$ and $w \in \mathfrak{S}_{p-1}$ is defined by

$$w(j) = \begin{cases} \widetilde{w}(j+1) & \text{if } 1 \le j < p - 1, \\ \widetilde{w}(1) & \text{if } j = p - 1. \end{cases}$$

Case 2. $v \in \widetilde{F}_{i_j}$ for $j = 1, \ldots, p$.

In this case, there is a k such that $v \in [F \cup \{v\}] - \tilde{\sigma}_k$. One can show that the family

$$D = \{(F - \tilde{\sigma}_1, F_{i_1}), \ldots, (F - \tilde{\sigma}_p, F_{i_p}), ([n] - F, F)\},$$

where $F_{i_j} = (\widetilde{F}_{i_j} \cup F) - \{v\}$ for $j = 2, \ldots, p$, is a shelling-trapped decomposition of $[n]$ and $w \in \mathfrak{S}_p$ can be defined by

$$w(j) = \begin{cases} \widetilde{w}(j-1) & \text{if } 1 < j \text{ and } \widetilde{w}(j-1) < k, \\ \widetilde{w}(j-1) + 1 & \text{if } 1 < j \text{ and } \widetilde{w}(j-1) \ge k, \\ k & \text{if } j = 1. \end{cases}$$

For the second assertion, one can show that each subspace in $\overline{C}_{D,w}$ is not changed under the map ψ_1 in the proof of Lemma 4.6 since it is the intersection of all subspaces that lie weakly below it and cover $U_{\overline{F}}$. We will show that $\psi(\overline{C}_{D,w} - U_{\overline{F}}) = \psi_2(\overline{C}_{D,w} - U_{\overline{F}})$ is a chain satisfying all conditions for $\overline{C}_{\widetilde{D},\widetilde{w}}$. If $\bar{\tau}_j$ is a set satisfying $\overline{F}_{i_j} \subseteq \bar{\tau}_j \subseteq \bar{\sigma}_j$, then

$$\psi(U_{\bar{\tau}_j}) = \begin{cases} U_{\bar{\tau}_j} & \text{if } j = 1, 2, \ldots, p - 1, \\ U_{[\bar{\tau}_p \cap F] \cup \{v\}} & \text{if } j = p \text{ and } \bar{\tau}_p \ne \overline{F}, \\ \mathbb{R}^{|F|+1} & \text{if } j = p \text{ and } \bar{\tau}_p = \overline{F}, \end{cases}$$

where the coordinates of $\mathbb{R}^{|F|+1}$ are indexed by $F \cup \{v\}$.

There are two cases:

Case 1. $\sigma_p \ne F$.

In this case, one can see that $F - \widetilde{F}$ has only one element, say x, and $\bar{\sigma}_p$ contains x. Thus one can choose $U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_{p-1}}/\overline{F} \cup \{x\}}$ as the second subspace in the chain $\widehat{C}_w$. Then $\psi(U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_{p-1}}/\overline{F} \cup \{x\}}) = U_{[F \cup \{v\}] - \widetilde{F}/[F \cup \{v\}] - \widetilde{F}_{i_1}/\cdots/[F \cup \{v\}] - \widetilde{F}_{i_{p-1}}}$ since $\overline{F}_{i_j} = [F \cup \{v\}] - \widetilde{F}_{i_j}$ for $j = 1, 2, \ldots, p - 1$ and $[F \cup \{v\}] - \widetilde{F} = \{v, x\}$. Moreover, the image of $\widehat{C}_w - U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}$ under ψ is a saturated chain in $[U_{[F \cup \{v\}] - \widetilde{F}/[F \cup \{v\}] - \widetilde{F}_{i_1}/\cdots/[F \cup \{v\}] - \widetilde{F}_{i_{p-1}}}, U_{F \cup \{v\}}]$ whose covering relations of type

(a) (in Step (1) of Section 4.1) correspond to the covering relations in the chain $C_{\widetilde{w}}$ in Π_p. Therefore, the image of $\overline{C}_{D,w}$ under ψ can be chosen as $\overline{C}_{\widetilde{D},\widetilde{w}}$.

Case 2. $\sigma_p = F$.

In this case, it is not hard to see that the image of $\widehat{C}_w$ under ψ is the saturated chain in $[U_{[F\cup\{v\}]-\widetilde{F}_{i_1}/\cdots/[F\cup\{v\}]-\widetilde{F}_{i_{p-1}}}, U_{F\cup\{v\}}]$ whose covering relations of type (a) (in Step (1) of Section 4.1) correspond to the covering relations in the chain $C_{\widetilde{w}}$ in Π_{p-1}. Thus the image of $\overline{C}_{D,w}$ under the map ψ can be chosen as $\overline{C}_{\widetilde{D},\widetilde{w}}$. $\qquad\qquad\square$

Example 4.12. Let Δ be the shellable simplicial complex from Example 4.9. In Example 4.10, $C_{D,w}$ is the chain

$$\hat{0} \lessdot U_{67} \lessdot U_{23/67} \lessdot U_{45/23/67} \lessdot U_{45/123/67} \lessdot U_{45/12367} \lessdot U_{1234567}$$

for the shelling-trapped decomposition

$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$

of $[7]$ and the permutation w in $\mathfrak{S}_2$ with $w(1) = 2$ and $w(2) = 1$.

Since $67 = \overline{F}_7$, the corresponding shelling-trapped decomposition $\widetilde{D}$ of the set $\{1, 2, 3, 4, 5, v\}$ is

$$\widetilde{D} = \{(45, \widetilde{F}_1 = 123v), (123v, \widetilde{F}_6 = 145v)\}$$

and the corresponding permutation $\widetilde{w} \in \mathfrak{S}_1$ is the identity.

The map ψ from the proof of Lemma 4.6 sends the chain

$$U_{23/67} \lessdot U_{45/23/67} \lessdot U_{45/123/67} \lessdot U_{45/12367}$$

to the chain

$$U_{23} \lessdot U_{45/23} \lessdot U_{45/123} \lessdot U_{45/123v}.$$

and this chain satisfies the conditions for $\overline{C}_{\widetilde{D},\widetilde{w}}$.

The intersection lattice for Δ_F is shown in Figure 4 and the chain $\overline{C}_{\widetilde{D},\widetilde{w}}$ is represented by thick lines.

Proof of Theorem 4.1. By Lemma 4.5, it is enough to show the assertion for the case when $\bar{\sigma} = \cup_{i=1}^q \overline{F}_i = [n]$. Since every chain $\overline{C}_{D,w}$ is saturated, it is enough to show that $\widehat{\Delta}(\overline{L}_\Delta)$, the simplicial complex obtained after removing the corresponding simplices for all pairs (D, w), is contractible. We use induction on the number q of facets of Δ.

Base case: $q = 2$. If Δ has only two facets F_1 and F_2 and $\overline{F}_1 \cup \overline{F}_2 = [n]$, then F_2 has only one element and $G_2 = \emptyset$. It is easy to see that the order complex $\Delta(\overline{L}_\Delta)$ is homotopy equivalent to $\mathbb{S}^0$ and $D = \{([n], F_2)\}$ is the only shelling-trapped decomposition of $[n]$, while $\overline{C}_{D,\emptyset} = (U_{\overline{F}_2})$ is the corresponding saturated chain. Therefore, $\widehat{\Delta}(\overline{L}_\Delta)$ is contractible when $q = 2$.
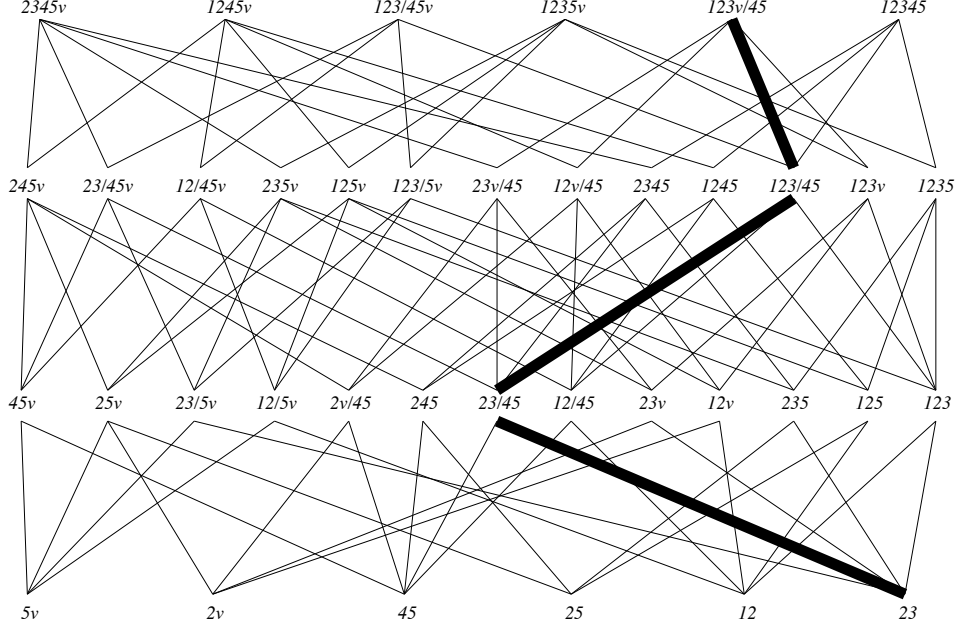
FIGURE 4. The interval $(\hat{0}, \hat{1})$ in L_{Δ_F}

Inductive step. Now, assume that $\widehat{\Delta}(\overline{L}_\Delta)$ is contractible for all shellable simplicial complexes Δ with less than q facets. For simplicity, denote $L = L_\Delta$. Let $F = F_q$ be the last facet in the shelling order of Δ and $H = U_{\overline{F}}$. Let L' be the intersection lattice for Δ', where Δ' is the simplicial complex generated by the facets $F_1, \ldots, F_{q-1}$.

Let $\overline{L}_{\geq H}$ denote the subposet of elements in $\overline{L}$ which lie weakly above H. Consider the decomposition of $\widehat{\Delta}(\overline{L}) = X \cup Y$, where X is the simplicial complex obtained by removing all simplices corresponding to chains $\overline{C}_{D,w}$ and $\overline{C}_{D,w} - H$ from $\Delta(\overline{L}_{\geq H})$ for all $\overline{C}_{D,w}$ containing H, and Y is the simplicial complex obtained by removing all simplices corresponding to chains $\overline{C}_{D,w}$ not containing H from $\Delta(\overline{L} - \{H\})$. Our goal will be to show that X, Y and $X \cap Y$ are all contractible, and hence so is $X \cup Y (= \widehat{\Delta}(L))$.

Step 1. Contractibility of X

Since X has a cone point H, it is contractible.

Step 2. Contractibility of Y

Define the closure relation π on L which sends a subspace to the join of the elements covering $\hat{0}$ which lie below it except H. Then the closed sets form a sublattice of L, which is the intersection lattice L' for the diagonal arrangement corresponding to Δ'. It is well-known that the inclusion of closed

sets $\overline{L} \cap L' \hookrightarrow \overline{L} - \{H\}$ is a homotopy equivalence (see [6, Lemma 7.6]). We have to consider the following two cases:

Case 1. $S := \cup_{i=1}^{q-1} \overline{F}_i \neq [n]$,

Then $\overline{L} \cap L' = L' - \{\hat{0}\}$ since $U_S \in \overline{L}$. Since $\overline{L} \cap L'$ has a cone point U_S, it is contractible. Since $S \neq [n]$, there is no shelling-trapped decomposition of $[n]$ for Δ'. Thus Y is contractible.

Case 2. $S := \cup_{i=1}^{q-1} \overline{F}_i = [n]$,

In this case, $\overline{L} \cap L' = \overline{L'}$. Moreover, $\widehat{\Delta}(\overline{L'})$ is homotopy equivalent to Y since every element in a chain $\overline{C}_{D,w}$ in $\widehat{\Delta}(\overline{L} - \{H\})$ is fixed under π. Since Δ' has $q - 1$ facets, the induction hypothesis implies that $\widehat{\Delta}(\overline{L'})$ is contractible and hence so is Y.

Step 3. Contractibility of $X \cap Y$

Note that $X \cap Y$ is obtained by removing simplices corresponding to $\overline{C}_{D,w} - H$ for all $\overline{C}_{D,w}$ containing H from $\overline{L}_{>H}$. By Lemma 4.8, $(\overline{L})_{>H}$ is isomorphic to the proper part of the intersection lattice L_F for the diagonal arrangement corresponding to Δ_F on $F \cup \{v\}$. Also, Lemma 4.11 implies that $X \cap Y$ is isomorphic to $\widehat{\Delta}(\overline{L}_F)$, where $\widehat{\Delta}(\overline{L}_F)$ is obtained by removing simplices corresponding to $\overline{C}_{\widetilde{D},\widetilde{w}}$ for all shelling-trapped decomposition $\widetilde{D}$ of $F \cup \{v\}$ and $\widetilde{w} \in \mathfrak{S}_{|\widetilde{D}|-1}$ from $\overline{L}_F$. Since Δ_F has fewer facets than Δ, the induction hypothesis implies $\widehat{\Delta}(\overline{L}_F)$ is contractible and hence $X \cap Y$ is also contractible. $\square$

Example 4.13. Let Δ be a simplicial complex from Example 4.7. Then one can see that

$$F_1 = 123, \quad F_2 = 234, \quad F_3 = 35, \quad F_4 = 45$$

is a shelling and

$$G_1 = 123, \quad G_2 = 23, \quad G_3 = 3, \quad G_4 = \emptyset.$$

Let $\bar{\sigma} = 12345$. Then there are two possible (unordered) shelling-trapped decompositions of $\bar{\sigma}$ (see Table 2).

Thus Theorem 4.1 implies $\Delta(\hat{0}, U_{12345})$ is homotopy equivalent to a wedge of two circles. The intersection lattice L_Δ and the order complex for its proper part are shown in Figure 5. Note that the chains $\overline{C}_{D,w}$ and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines.

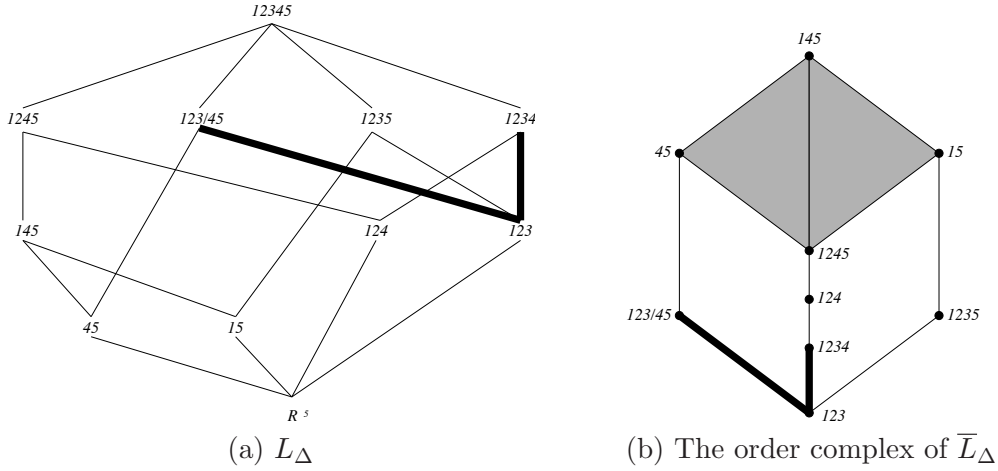(a) L_Δ (b) The order complex of $\overline{L}_\Delta$

FIGURE 5. The intersection lattice L_Δ of $\mathcal{A}_\Delta$ and the order complex of its proper part

5. THE HOMOTOPY TYPE OF THE SINGULARITY LINK OF $\mathcal{A}_\Delta$

In this section, we give the corollary about the homotopy type of the singularity link of $\mathcal{A}_\Delta$ when Δ is shellable. Also we give the homology version of the corollary.

Ziegler and Živaljević [19] show the following theorem about the homotopy type of $\mathcal{V}_\mathcal{A}^\circ$.

Theorem 5.1. *For every subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$,*

$$\mathcal{V}_\mathcal{A}^\circ \simeq \bigvee_{x \in L_\mathcal{A} - \{\hat{0}\}} (\Delta(\hat{0}, x) * \mathbb{S}^{\dim(x)-1}).$$

From this and our results in Section 4, one can deduce the following.

Corollary 5.2. *Let Δ be a shellable simplicial complex on $[n]$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension*

$$n + p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| - 2$$

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of some subset of $[n]$.

Proof sketch. This is a straightforward, but tedious, calculation. By Theorem 5.1, one needs to understand homotopy types of $\Delta(\hat{0}, H)$ for $H \in L_\Delta$. Lemmas 4.2 and 4.4 reduce this to the case of $\Delta(\hat{0}, U_{\bar{\sigma}})$, which is described

shelling-trapped decomp.	dim	shelling-trapped decomp.	dim
$\{(45, F_1)\}$	3	$\{(15, F_2)\}$	3
$\{(145, F_2)\}$	3	$\{(124, F_3)\}$	2
$\{(1245, F_3)\}$	2	$\{(123, F_4)\}$	2
$\{(1234, F_4)\}$	2	$\{(1235, F_4)\}$	2
$\{(12345, F_4)\}$	2	$\{(45, F_1), (123, F_4)\}$	2

TABLE 2. Shelling-trapped decompositions for Δ

fully by Theorem 4.1. The rest is some bookkeeping about shelling-trapped decompositions. $\qquad\square$

Example 5.3. Let Δ be a simplicial complex from Example 4.7. In Example 4.13, we show

$$F_1 = 123, F_2 = 234, F_3 = 35, F_4 = 45$$

is a shelling of Δ and

$$G_1 = 123, G_2 = 23, G_3 = 3, G_4 = \emptyset.$$

Table 2 shows shelling-trapped decompositions $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of subsets of $[5]$ with corresponding dimensions

$$n + p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| - 2.$$

Therefore Corollary 5.2 shows that the singularity link of $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of three 3-dimensional spheres and eight 2-dimensional spheres.

The following theorem is a homology version of Corollary 5.2.

Theorem 5.4. *Let Δ be a shellable simplicial complex and $F_1, \ldots, F_q$ be a shelling of Δ. Then $\dim_\mathbb{F} H_i(\mathcal{V}^\circ_{\mathcal{A}_\Delta}; \mathbb{F})$ is the number of ordered shelling-trapped decompositions $((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))$ with*

$$i = n + p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| - 2.$$

This can be proven without Theorem 4.1 by combining a result of Peeva, Reiner and Welker [16, Theorem 1.3] with results of Herzog, Reiner and Welker [12, Theorem 4, Theorem 9] along with the theory of *Golod rings*. It is what motivated us to prove the stronger Corollary 5.2 and eventually Theorem 1.1.

6. $K(\pi, 1)$ EXAMPLES FROM MATROIDS

In this section, we give some examples of diagonal arrangements $\mathcal{A}$ where the complement $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$, coming from rank 3 matroids.

One should note that an arrangement having any subspace of real codimension 1 (hyperplane) will have $\mathcal{M}_{\mathcal{A}}$ disconnected. So one may assume without loss of generality that all subspaces have real codimension at least 2. Furthermore, if any maximal subspace U in $\mathcal{A}$ has codimension at least 3, then it is not hard to see that $\mathcal{M}_{\mathcal{A}}$ is not $K(\pi, 1)$. Hence we may assume without loss of generality that all maximal subspaces have real codimension 2.

A hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ is *simplicial* if every chamber in $M_{\mathcal{H}}$ is a simplicial cone. Davis, Januszkiewicz and Scott [9] show the following theorem.

Theorem 6.1. *Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{A}$ be any arrangement of codimension-2 subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$.*

Remark 6.2. In order to apply this to diagonal arrangements, we need to consider hyperplane arrangements $\mathcal{H}$ which are subarrangements of the real braid arrangement $\mathcal{B}_n$ in $\mathbb{R}^n$ and also simplicial. It turns out (and we omit the straightforward proof) that all such arrangements $\mathcal{H}$ are direct sums of smaller braid arrangements. So we only consider $\mathcal{H} = \mathcal{B}_n$ itself here.

Corollary 6.3. *Let $\mathcal{A}$ be a subarrangement of the 3-equal arrangement of $\mathbb{R}^n$ so that*
$$\mathcal{A} = \{U_{ijk} \mid \{i, j, k\} \in T_{\mathcal{A}}\},$$
for some collection $T_{\mathcal{A}}$ of 3-element subsets of $[n]$. Then $\mathcal{A}$ satisfies the hypothesis of Theorem 6.1 (and hence $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$) if and only if every permutation $w = w_1 w_2 \ldots w_n$ in $\mathfrak{S}_n$ has at least one triple in $T_{\mathcal{A}}$ consecutive, i.e., there exists j such that $\{w_j, w_{j+1}, w_{j+2}\} \in T_{\mathcal{A}}$.

Proof. It is easy to see that there is a bijection between chambers of the real Braid arrangement $\mathcal{B}_n$ in $\mathbb{R}^n$ and permutations $w = w_1 \cdots w_n$ in $\mathfrak{S}_n$. Moreover, each chamber has the form $x_{w_1} > \cdots > x_{w_n}$ with bounding hyperplanes
$$x_{w_1} = x_{w_2}, x_{w_2} = x_{w_3}, \ldots, x_{w_{n-1}} = x_{w_n}$$
and intersects the 3-equal subspaces of the form $x_{w_i} = x_{w_{i+1}} = x_{w_{i+2}}$ for $i = 1, 2, \ldots, n-2$. $\square$

We seek shellable simplicial complexes Δ for which $\mathcal{A}_{\Delta}$ satisfies this condition.

If Δ is the independent set complex $\mathcal{I}(M)$ for some matroid M (see [17] for the definition of independent sets and further background on matroids), then facets of Δ are bases of M. Simplicial complexes of this kind are called *matroid*

complexes, and they are known to be shellable [2]. For a rank 3 matroid M on $[n]$, consider

$$\mathcal{A}_{\mathcal{I}(M^\perp)} = \{U_{ijk} \mid \{i,j,k\} = [n] - B \text{ for some } B \in \mathcal{B}(M^\perp)\}$$
$$= \{U_{ijk} \mid \{i,j,k\} \in \mathcal{B}(M)\},$$

where $M^\perp$ is the dual matroid of M. Note that adding a loop to M does not change the structure of the intersection lattice for $\mathcal{A}_{\mathcal{I}(M^\perp)}$. Thus, if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 6.3, then the diagonal arrangement corresponding to the matroid in which all loops have been deleted has $K(\pi, 1)$ complement, and hence $\mathcal{A}_{\mathcal{I}(M^\perp)}$ has $K(\pi, 1)$ complement.

Definition 6.4. Let M be a rank 3 matroid on $[n]$. Say M is *DJS* if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 6.3.

The following example shows that matroid complexes are not DJS in general. Thus we look for some subclasses of matroid complexes which are DJS, and hence whose corresponding diagonal arrangements have $K(\pi, 1)$ complements; for these, Theorem 5.4 gives us the group cohomology $H^\bullet(\pi, \mathbb{Z})$.

Example 6.5. Let M be a matroid on $[6]$ which has three distinct parallel classes $\{1, 6\}$, $\{2, 4\}$ and $\{3, 5\}$. Then M is self-dual and $\mathcal{I}(M^\perp)$ is a simplicial complex on $[6]$ whose facets are $123, 134, 145, 125, 236, 256, 346$ and 456. But $w = 124356$ is a permutation that does not satisfy the condition of Corollary 6.3.

Recall that a matroid is *simple* if it has no loops nor parallel elements. The following proposition shows that rank 3 simple matroids are DJS.

Proposition 6.6. *Let M be a matroid of rank 3 on $[n]$. If M does not have parallel elements, then M is DJS. In particular, rank 3 simple matroids are DJS.*

Proof. Without loss of generality, we may assume that M is simple. M is not DJS if and only if there is a permutation $w \in \mathfrak{S}_n$ such that every consecutive triple is not in $\mathcal{B}(M)$. Since M is simple, the latter statement is true if and only if each consecutive triple in w forms a circuit, i.e., all elements lie on a rank 2 flat. But this is impossible since M has rank 3. $\square$

The following two propositions give some subclasses of matroids with parallel elements which are DJS.

Proposition 6.7. *Let M be a rank 3 matroid on $[n]$ with no circuits of size 3. Let $P_1, \ldots, P_k$ be the distinct parallel classes which have more than one*

element, and let N be the set of all non-loop elements which are not parallel with anything else. Then, M is DJS if and only if

$$\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k < |N| - 2.$$

Proof. We may assume that M does not have loops. Since M does not have loops nor circuits of size 3, M is not DJS if and only if one can construct a permutation $w \in \mathfrak{S}_n$ such that for each consecutive triple in w there are at least two elements which are parallel. This means if $w_i \in N$, then $w_{i-2}, w_{i-1}, w_{i+1}$ and w_{i+2} (if they exist) must be in the same parallel class. Such a w can be constructed if and only if $\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k \geq |N| - 2$. $\qquad\square$

A simplicial complex Δ on $[n]$ is *shifted* if, for any face σ of Δ, replacing any vertex $i \in \sigma$ by a vertex $j < i$ with $j \notin \sigma$ gives another face in Δ. A matroid M is *shifted* if its independent set complex is shifted. Klivans [14] shows that a rank 3 shifted matroid on the ground set $[n]$ is indexed by some set $\{a, b, c\}$ with $1 \leq a < b < c \leq n$ as follows:

$$\mathcal{B}(M) = \{(a', b', c') : 1 \leq a' < b' < c' \leq n, a' \leq a, b' \leq b, c' \leq c\}.$$

It is not hard to check the following.

Proposition 6.8. *Let M be the shifted rank 3 matroid on the ground set $[n]$ indexed by $\{a, b, c\}$. Then, M is DJS if and only if $\lfloor \frac{c-b}{2} \rfloor < a$.*

We have not yet been able to characterize all rank 3 matroids which are DJS.

ACKNOWLEDGMENTS

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