Topology of real arrangements corresponding to shellable complexes

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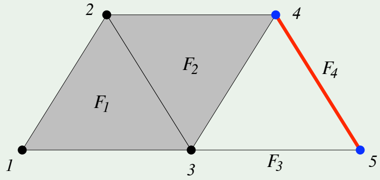
Outline

Shellable simplicial complexes

Definition

A simplicial complex is shellable if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} 2^{F_i}) \cap 2^{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. Such an ordering of facets is called a shelling order or shelling.

Example

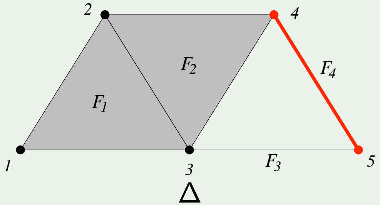


Simplicial complexes and diagonal arrangements

Correspondence

A simplicial complex Δ on $[n]$ $\iff$ A diagonal arrangement $\mathcal{A}_\Delta$: collection of diagonal subspaces $\{x_{i_1} = \cdots = x_{i_k}\}$ of $\mathbb{R}^n$ for all $\{i_1, \ldots, i_k\}$ complementary to facets of Δ

Example



$\iff$

$\{x_4 = x_5\}$	F_1
$\{x_1 = x_5\}$	F_2
$\{x_1 = x_2 = x_4\}$	F_3
$\{x_1 = x_2 = x_3\}$	F_4

$\mathcal{A}_\Delta$

Simplicial complexes and diagonal arrangements

Example

The Braid arrangement $\mathcal{B}_n = \bigcup_{i<j} \{x_i = x_j\}$

$$\Updownarrow$$

$$\Delta_{n,n-2} = \{\sigma \subset [n] : |\sigma| \leq n-2\}$$

Example

The k-equal arrangement $\mathcal{A}_{n,k} = \bigcup_{i_1<\cdots<i_k} \{x_{i_1} = \cdots = x_{i_k}\}$

$$\Updownarrow$$

$$\Delta_{n,n-k} = \{\sigma \subset [n] : |\sigma| \leq n-k\}$$

Two important spaces associated with $\mathcal{A}$

Definition

- The complement of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is

$$\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$$

- The singularity link of a central arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is

$$\mathcal{V}^{\circ}_{\mathcal{A}} = \mathbb{S}^{n-1} \cap \bigcup_{H \in \mathcal{A}} H$$

Fact

By Alexander duality,

$$H^i(\mathcal{M}_{\mathcal{A}}; \mathbb{F}) = H_{n-2-i}(\mathcal{V}^{\circ}_{\mathcal{A}}; \mathbb{F})$$

Application in group cohomology

Definition

An Eilenberg-MacLane space (or a $K(\pi, n)$ space) is a connected cell complex with all homotopy groups except the n-th homotopy group being trivial and the n-th homotopy group isomorphic to π.

Fact

If a CW complex X is a $K(\pi, 1)$ space, then

$$\mathrm{Tor}_n^{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z}) = H_n(X; \mathbb{Z}) \text{ and } \mathrm{Ext}^n_{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z}) = H^n(X; \mathbb{Z}).$$

Theorem (Fadell and Neuwirth, 1962)

Let $\mathcal{B}_n$ be the braid arrangement in $\mathbb{C}^n$. Then $\mathcal{M}_{\mathcal{B}_n}$ is a $K(\pi, 1)$ space.

Theorem (Khovanov, 1996)

Let $\mathcal{A}_{n,3}$ be the 3-equal arrangement in $\mathbb{R}^n$. Then $\mathcal{M}_{\mathcal{A}_{n,3}}$ is a $K(\pi, 1)$ space.

What is the topology of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{V}_{\mathcal{A}}^{\circ}$?

Definition

The intersection lattice $L_{\mathcal{A}}$ of a subspace arrangement $\mathcal{A}$ is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion.

Theorem (Goresky and Macpherson, 1988)

Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$. Then

$$\widetilde{H}^i(\mathcal{M}_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}} - \{\hat{0}\}} \widetilde{H}_{codim(x)-2-i}(\hat{0}, x).$$

Theorem (Ziegler and Živaljević, 1993)

For every central subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$,

$$\mathcal{V}_{\mathcal{A}}^{\circ} \simeq \bigvee_{x \in L_{\mathcal{A}} - \{\hat{0}\}} (\Delta(\hat{0}, x) * \mathbb{S}^{\dim(x)-1}).$$

What is a general sufficient condition for the intersection lattice $L_{\mathcal{A}}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved?

Theorem (Björner and Welker, 1995)

The order complex of the intersection lattice $L_{\mathcal{A}_{n,k}}$ for the k-equal arrangement $\mathcal{A}_{n,k}$ has the homotopy type of a wedge of spheres.

$\mathcal{A}_{n,k} = \mathcal{A}_{\Delta_{n,n-k}}$ and $\Delta_{n,n-k}$ is shellable.

Theorem (Kozlov, 1999)

Let Δ be a simplicial complex on $[n]$ that satisfies some conditions. Then the intersection lattice for $\mathcal{A}_{\Delta}$ has the homotopy type of a wedge of spheres.

Δ in Kozlov's theorem is shellable.

Homotopy type of L_Δ for shellable Δ

Theorem (K.)

Let Δ be a shellable simplicial complex with $\dim \Delta \leq n - 3$. Then the order complex of the intersection lattice L_Δ of $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.

Homotopy type of L_Δ for shellable Δ(precise version)

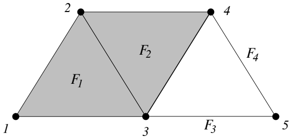
Theorem (K.)

Let Δ be a shellable simplicial complex on $[n]$ with $\dim \Delta \leq n-3$. Let σ be the intersection of all facets and $\bar{\sigma}$ its complement. Then the intersection lattice L_Δ is homotopy equivalent to a wedge of spheres, consisting of $(p-1)!$ copies of spheres of dimension

$$\delta(D) = p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each (unordered) shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$.
Moreover, if one removes the $\delta(D)$-simplex corresponding to a saturated chain $\overline{C}_{D,\omega}$ for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$ and a permutation ω of $[p-1]$, then the remaining simplicial complex $\widehat{\Delta}(\hat{0}, U_{\bar{\sigma}})$ is contractible.

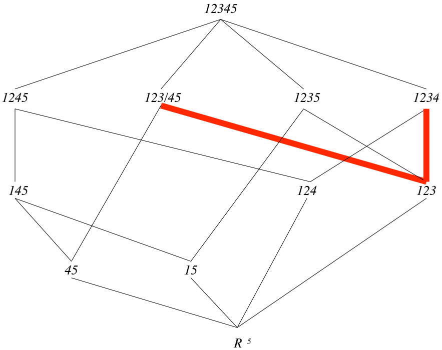
Example



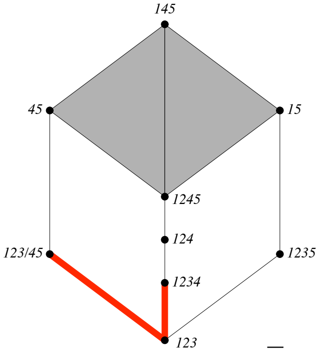
A shellable complex Δ

$\{(12345, F_4)\}$
$\{(45, F_1), (123, F_4)\}$

Shelling-trapped decompositions of [5]



The intersection lattice L_Δ of $\mathcal{A}_\Delta$



The order complex of $\overline{L}_\Delta$

Proof sketch of Main theorem

Lemma (K.)

For the upper interval, there is a simplicial complex whose intersection lattice is isomorphic to $[U_{\bar{\sigma}}, \hat{1}]$*. If* F *is the last facet in the shelling order, the simplicial complex which corresponds to* $[U_{\overline{F}}, \hat{1}]$ *is shellable.*

Proof sketch of Main theorem

If F is the last facet in the shelling of Δ, one can consider the following decomposition of $\widehat{\Delta}(\overline{L})$:

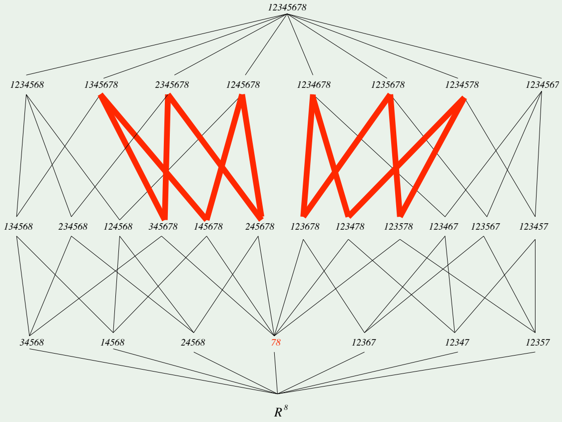
$$\widehat{\Delta}(\overline{L}) = \widehat{\Delta}(\overline{L} - \{H\}) \cup \widehat{\Delta}(\overline{L}_{\geq H}),$$

where $\widehat{\Delta}(\overline{L} - \{H\})$ is obtained by removing all chains $\overline{C}_{D,\omega}$ not containing H from $\Delta(\overline{L} - \{H\})$ and $\widehat{\Delta}(\overline{L}_{\geq H})$ is obtained by removing $\overline{C}_{D,\omega}$ and $\overline{C}_{D,\omega} - H$ from $\Delta(\overline{L}_{\geq H})$ for all $\overline{C}_{D,\omega}$ containing H. Then one can show that all three spaces $\widehat{\Delta}(\overline{L} - \{H\})$, $\widehat{\Delta}(\overline{L}_{\geq H})$ and their intersection are contractible, and hence $\widehat{\Delta}(\overline{L})$ is also contractible.

L_Δ is not shellable in general

Example

Let Δ be a shellable complex with a shelling
$123456, 127, 137, 237, 458, 468, 568$.



Corollary (K.)

Let Δ be a shellable simplicial complex with $\dim \Delta \leq n-3$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension $n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$ for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of some subset of $[n]$.

Theorem (K.)

Let Δ be a shellable simplicial complex with $\dim \Delta \leq n-3$. Then $\dim_{\mathbb{F}} H_i(\mathcal{V}^{\circ}_{\mathcal{A}_\Delta}; \mathbb{F})$ is the number of ordered shelling-trapped decompositions $((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))$ of some subset of $[n]$ with $i = n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$.

Diagonal arrangement $\mathcal{A}$ such that $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$

Theorem (Davis, Januszkiewicz and Scott, 1998)

Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{A}$ be any arrangement of codimension-2 intersection subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$.

Proposition

Let $\mathcal{A}$ be a subarrangement of 3-equal arrangement of $\mathbb{R}^n$ so that

$$\mathcal{A} = \left\{ \{x_i = x_j = x_k\} \mid \{i, j, k\} \in T_{\mathcal{A}} \right\},$$

for some collection $T_{\mathcal{A}}$ of 3-element subsets of $[n]$. Then $\mathcal{A}$ satisfies the hypothesis of DJS's theorem (and hence $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$) if and only if every permutation ω in $\mathfrak{S}_n$ has at least one triple in $T_{\mathcal{A}}$ consecutive.

DJS matroids

The matroid complexes $\Delta = \mathcal{I}(M)$ are a natural class of shellable complexes.

Definition

Say a rank 3 matroid M on $[n]$ is DJS if every permutation ω in $\mathfrak{S}_n$ has at least one triple in $\mathcal{B}(M)$ consecutive.

Proposition (K.)

Rank 3 Matroids without parallel elements are DJS.
In particular, rank 3 simple matroids are DJS.

Proposition (K.)

Let M be a rank 3 matroid on the ground set $[n]$ with no circuits of size 3. Let $P_1, \ldots, P_k$ be distinct parallel classes which have more than one element and let N be the set of all elements which are not parallel with anything else. Then, M is DJS if and only if
$\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k < |N| - 2.$

DJS matroids

Definition

- A simplicial complex Δ on $[n]$ is shifted if, for any face of Δ, replacing any vertex i by a vertex $j(< i)$ gives another face in Δ.
- The Gale ordering on all k element subsets of $[n]$ is given by $\{x_1 < \cdots < x_k\}$ is less than $\{y_1 < \cdots < y_k\}$ if $x_i \leq y_i$ for all i and $\{x_1, \ldots, x_k\} \neq \{y_1, \ldots, y_k\}$.

Theorem (Klivans)

Let M be a matroid whose independent set complex is shifted. Then its bases $\mathcal{B}(M)$ is the principal order ideal of Gale ordering.

Proposition (K.)

Let M be the rank 3 matroid on the ground set $[n]$ corresponding to the principal order ideal generated by $\{a, b, n\}$. Then, M is DJS if and only if $\lfloor \frac{n-b}{2} \rfloor < a$.

Simplicial complexes and Coordinate arrangements

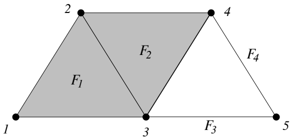
Correspondence

A simplicial complex Δ on $[n]$ $\Longleftrightarrow$ A coordinate arrangement $\mathcal{A}^0_\Delta$: collection of coordinate subspaces $\{x_{i_1} = \cdots = x_{i_k} = 0\}$ of $\mathbb{R}^n$ for all $\{i_1, \ldots, i_k\}$ complementary to facets of Δ

Theorem (K.)

Let Δ be a shellable simplicial complex on $[n]$. Then the intersection lattice L^0_Δ of $\mathcal{A}^0_\Delta$ is homotopy equivalent to $\mathrm{link}_\Delta \sigma$, where σ is the intersection of all facets. Hence the intersection lattice L^0_Δ is homotopy equivalent to a wedge of spheres.

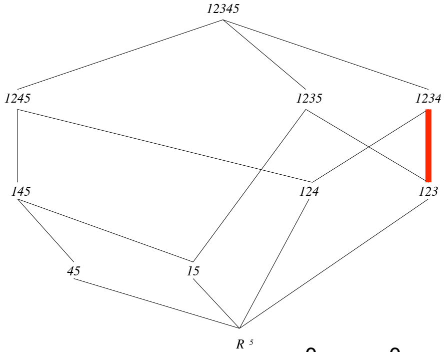
Example



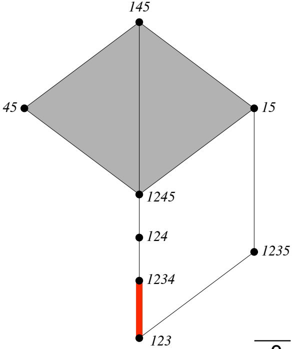
A shellable complex Δ

$$\{x_4 = x_5 = 0\}$$
$$\{x_1 = x_5 = 0\}$$
$$\{x_1 = x_2 = x_4 = 0\}$$
$$\{x_1 = x_2 = x_3 = 0\}$$

$\mathcal{A}^0_\Delta$



The intersection lattice L^0_Δ of $\mathcal{A}^0_\Delta$



The order complex of $\overline{L^0_\Delta}$

Homotopy type of $\mathcal{V}^{\circ}_{\mathcal{A}^0_\Delta}$ and $\mathcal{M}_{\mathcal{A}^0_\Delta}$

Corollary (K.)

If Δ is a shellable simplicial complex, then the singularity link of $\mathcal{A}^0_\Delta$ is homotopy equivalent to a wedge of spheres.

Theorem (Welker)

If Δ is a shifted simplicial complex, then the complement of $\mathcal{A}^0_\Delta$ is homotopy equivalent to a wedge of spheres.

Conjecture (Welker)

If Δ is a shellable simplicial complex, then the complement of $\mathcal{A}^0_\Delta$ is homotopy equivalent to a wedge of spheres.

Question/Problem

Questions

- When is the intersection lattice for the diagonal arrangement shellable?
- When is the complement for the diagonal arrangement homotopy equivalent to a wedge of spheres?

Problems

- Generalize to the case of B_n and D_n.
- Characterize the rank 3 matroids which are DJS.