Combinatorics and invariants of toric arrangements

Luca Moci

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Abstract

Given the toric (or toral) arrangement defined by a root system Φ, we classify and count its components of each dimension. We show how to reduce to the case of 0-dimensional components, and in this case we give an explicit formula involving the maximal subdiagrams of the affine Dynkin diagram of Φ. Then we compute the Euler characteristic and the Poincaré polynomial of the complement of the arrangement, that is the set of regular points of the torus.

1 Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ and $\Phi^\vee \subset \mathfrak{h}$ respectively its root and coroot systems. The equations $\{\alpha(h) = 0\}_{\alpha \in \Phi}$ define a family $\mathcal{H}$ of intersecting hyperplanes in $\mathfrak{h}$. Let $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots: the quotient $T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$ is a complex torus of rank n. Each root α takes integer values on $\langle \Phi^\vee \rangle$, so it induces a map $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote by e^α. The equations $\{\alpha(h) \in \mathbb{Z}\}_{\alpha \in \Phi}$ define in $\mathfrak{h}$ a periodic family of hyperplanes, or equivalently the equations $\{e^\alpha(t) = 1\}_{\alpha \in \Phi}$ define in T a finite family $\mathcal{T}$ of codimension 1 subtori. $\mathcal{H}$ and $\mathcal{T}$ are called respectively the *hyperplane arrangement* and the *toric arrangement* defined by Φ (see for example [7], [9], [22]). We call *subspaces* of $\mathcal{H}$ the intersections of elements of $\mathcal{H}$, and *components* of $\mathcal{T}$ the connected components of the intersections of elements of $\mathcal{T}$. We denote by $\mathcal{S}(\Phi)$ the set of the subspaces of $\mathcal{H}$, by $\mathcal{C}(\Phi)$ the set of the components of $\mathcal{T}$, and by $\mathcal{S}_d(\Phi)$ and $\mathcal{C}_d(\Phi)$ the sets of $d-$dimensional subspaces and components. Clearly if $\Phi = \Phi_1 \times \Phi_2$ then $\mathcal{S}(\Phi) = \mathcal{S}(\Phi_1) \times \mathcal{S}(\Phi_2)$ and $\mathcal{C}(\Phi) = \mathcal{C}(\Phi_1) \times \mathcal{C}(\Phi_2)$, so from now on we will suppose Φ to be irreducible.

$\mathcal{H}$ is a classical object, whereas De Concini and Procesi [7] recently showed that $\mathcal{T}$ provides a geometric way to compute the values of the Kostant partition function. This function counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots, and plays an important role in representation theory, since (by Kostant's and Steinberg's formulae [18], [25]) it yields efficient computation of weight multiplicities and Littlewood-Richardson coefficients, as shown in [5] using results from [1], [3], [6], [26]. The values of Kostant partition function can be computed as a sum of contributions given by the elements of $\mathcal{C}_0(\Phi)$ (see [5, Teor 3.2]).

Furthermore, let R be the complement in T of the union of all elements of $\mathcal{T}$. R is called the set of the *regular points* of the torus T and has been intensively studied (see in particular [7], [19], [20]). The cohomology of R is direct sum of contributions given by the elements of $\mathcal{C}(\Phi)$ (see for example [7]). Then by describing the action of W on $\mathcal{C}(\Phi)$ we implicitly get a $W-$equivariant decomposition of the cohomology of R, and by counting and classifying the elements of $\mathcal{C}(\Phi)$ we can compute the Poincaré polynomial of R.

We say that a subset Θ of Φ is a *subsystem* if it satisfies the following conditions:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

For each $t \in T$ let us define the subsystem of Φ

$$\Phi(t) \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1\}.$$

The aim of Section 2 is to describe $\mathcal{C}_0(\Phi)$, that is the set of points $t \in T$ such that $\Phi(t)$ has rank n. Let $\alpha_1, \ldots, \alpha_n$ be simple roots of Φ, α_0 the lowest root, and Φ_p the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$. Let Γ be the affine Dynkin diagram of Φ and $V(\Gamma)$ the set of its vertices (a list of such diagrams can be found for example in [12] or in [17]). $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, so we can identify each vertex p with an integer from 0 to n. The diagram Γ_p that we get by removing from Γ the vertex p (and all adjacent edges) is the (genuine) Dynkin diagram of Φ_p. Let W be the Weyl group of Φ and W_p the Weyl group of Φ_p, i.e. the subgroup of W generated by all the reflections $s_{\alpha_0}, \ldots, s_{\alpha_n}$ except s_{α_p}. Notice that Γ_0 is the Dynkin diagram of Φ and $W_0 = W$. Since W permutes the roots, its natural action on T restricts to an action on $\mathcal{C}_0(\Phi)$. We denote by $W(t)$ the stabilizer of a point $t \in \mathcal{C}_0(\Phi)$. Then we prove the

Theorem 1. *There is a bijection between the $W-$orbits of $\mathcal{C}_0(\Phi)$ and the vertices of Γ, having the property that for every point t in the orbit $\mathcal{O}_p$ corresponding to the vertex p, $\Phi(t)$ is $W-$conjugated to Φ_p and $W(t)$ is $W-$conjugated to W_p.*

As a corollary we get the formula

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}. \tag{1}$$

In Section 3 we deal with components of arbitrary dimension. For each component U of $\mathcal{T}$ we consider the subsystem of Φ

$$\Theta_U \doteq \{\alpha \in \Phi | e^\alpha(t) = 1 \forall t \in U\}$$

and its *completion* $\overline{\Theta_U} \doteq \langle \Theta_U \rangle_\mathbb{R} \cap \Phi$.

Let $\mathcal{K}_d$ be the set of subsystems Θ of Φ of rank $n-d$ that are *complete* (i.e. such that $\Theta = \overline{\Theta}$), and let $\mathcal{C}^\Phi_\Theta$ be the set of components U such that $\overline{\Theta_U} = \Theta$. This gives a partition of the components:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^\Phi_\Theta.$$

Notice that the subsystem of roots vanishing on a subspace of $\mathcal{H}$ is always complete; then $\mathcal{K}_d$ is in bijection with $\mathcal{S}_d$. The elements of $\mathcal{S}_d$ are classified and counted in [21], [22]. Thus the description of the sets $\mathcal{C}^\Phi_\Theta$ that we give in Theorem 11 yields a classification of the components of $\mathcal{T}$. In particular we show that $|\mathcal{C}^\Phi_\Theta| = n_\Theta^{-1}|\mathcal{C}_0(\Theta)|$, where n_Θ is an integer depending only on the conjugacy class of Θ, and then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |\mathcal{C}_0(\Theta)|. \tag{2}$$

In Section 4, using results of [7] and [8], we deduce from Theorem 1 that the Euler characteristic of R is equal to $(-1)^n|W|$. This fact was known as a consequence of a topological construction of Salvetti ([10], [24]). Moreover Corollary 12 yields a formula for the Poincaré polynomial of R:

$$P_\Phi(q) = \sum_{d=0}^{n} (-1)^d (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|. \tag{3}$$

This formula allows to compute explicitly $P_\Phi(q)$.

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2 0-dimensional components

2.1 Statements

For all facts about Lie algebras and root systems we refer to [14]. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the Cartan decomposition of $\mathfrak{g}$, and let us choose nonzero elements $X_0, X_1, \ldots, X_n$ in the $1-$dimensional subalgebras $\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_n}$: since $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] = \mathfrak{g}_{\alpha+\alpha'}$ whenever $\alpha, \alpha', \alpha + \alpha' \in \Phi$, we have that $X_0, X_1, \ldots, X_n$ generate $\mathfrak{g}$. Let $a_0 = 1$ and for $p = 1, \ldots, n$ let a_p be the coefficient of α_p in $-\alpha_0$. For each $p = 0, \ldots, n$ we define an automorphism σ_p of $\mathfrak{g}$ by

$$\sigma_p(X_p) \doteq e^{2\pi i a_p^{-1}} X_p \ \ , \ \ \sigma_p(X_i) = X_i \ \forall i \neq p;$$

Let G be the semisimple and simply connected algebraic group having root system Φ; $\mathfrak{g}$ and T are respectively the Lie algebra and a maximal torus of G (see for example [13]). G acts on itself by conjugacy, i.e. for each $g \in G$ the map $k \mapsto gkg^{-1}$ is an automorphism of G. Its differential $Ad(g)$ is an automorphism of $\mathfrak{g}$.

Remark 2. Let $t \in \mathcal{C}_0(\Phi)$ and $\mathfrak{g}^{Ad(t)}$ be the subalgebra of elements fixed by $Ad(t)$. For each $\alpha \in \Phi$ and for each $X_\alpha \in \mathfrak{g}_\alpha$ we have that $Ad(t)(X_\alpha) = e^\alpha(t) X_\alpha$, thus

$$\mathfrak{g}^{Ad(t)} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(t)} \mathfrak{g}_\alpha.$$

Moreover $\mathfrak{g}^{\sigma_p}$ is generated by the subalgebras $\{\mathfrak{g}_{\alpha_i}\}_{0 \leq i \leq n, i \neq p}$. Then $\mathfrak{g}^{Ad(t)}$ and $\mathfrak{g}^{\sigma_p}$ are semisimple algebras having root system respectively $\Phi(t)$ and Φ_p. Our strategy will be to prove that for each $t \in \mathcal{C}_0(\Phi)$, $Ad(t)$ is conjugated to some σ_p. This implies that $\mathfrak{g}^{Ad(t)}$ is conjugated to $\mathfrak{g}^{\sigma_p}$ and then $\Phi(t)$ to Φ_p, as claimed in Theorem 1.

Then we want give a bijection between vertices of Γ and $W-$orbits of $\mathcal{C}_0(\Phi)$ showing that, for every t in the orbit $\mathcal{O}_p$, $Ad(t)$ is conjugated to σ_p. However, since some of the σ_p (as well as the corresponding Φ_p) are themselves conjugate, this bijection is not going to be canonical. To make it canonical we should merge the orbits corresponding to conjugate automorphisms: for this we consider the action of a larger group.

Let $\Lambda(\Phi) \subset \mathfrak{h}$ be the lattice of the coweights of Φ, i.e.

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}.$$

The lattice spanned by the coroots $\langle \Phi^\vee \rangle$ is a sublattice of $\Lambda(\Phi)$; set

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle}.$$

This finite subgroup of T coincides with $Z(G)$, the *center* of G. It is well known ([13, 13.4]) that

$$Ad(g) = id_{\mathfrak{g}} \Leftrightarrow g \in Z(\Phi). \tag{4}$$

Notice that

$$Z(\Phi) = \{t \in T | \Phi(t) = \Phi\}$$

thus $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$. Moreover, for each $z \in Z(\Phi), t \in T, \alpha \in \Phi$,

$$e^\alpha(zt) = e^\alpha(z)e^\alpha(t) = e^\alpha(t)$$

and therefore $\Phi(zt) = \Phi(t)$. In particular $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$. Clearly this action commutes with that of W and we get an action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$.

Let Q be the set of the $Aut(\Gamma)$-orbits of $V(\Gamma)$. If $p, p' \in V(\Gamma)$ are two representatives of $q \in Q$, then $\Gamma_p \simeq \Gamma_{p'}$, thus $W_p \simeq W_{p'}$. Moreover we will see (Corollary 7.2) that σ_p is conjugated to $\sigma_{p'}$. Then we can restate Theorem 1 as follows.

Theorem 3. *There is a canonical bijection between Q and the set of $W \times Z(\Phi)-$orbits in $\mathcal{C}_0(\Phi)$, having the property that if $p \in V(\Gamma)$ is a representative of $q \in Q$, then:*

1. *every point t in the corresponding orbit $\mathcal{O}_q$ induces an automorphism conjugated to σ_p;*
2. *the stabilizer of $t \in \mathcal{O}_q$ is isomorphic to $W_p \times Stab_{Aut(\Gamma)}p$.*

This theorem implies immediately the formula:

$$|\mathcal{C}_0(\Phi)| = \sum_{q \in Q} n_q \frac{|W|}{|W_p|} \tag{5}$$

where n_q is the cardinality of the $Aut(\Gamma)-$orbit q. This is clearly equivalent to formula (1).

Remark 4. If we see the elements of $\Lambda(\Phi)$ as translations, we can define a group of isometries of $\mathfrak{h}$

$$\widetilde{W} \doteq W \ltimes \Lambda(\Phi).$$

$\widetilde{W}$ is called the *extended affine Weyl group* of Φ and contains the affine Weyl group $\widehat{W} \doteq W \ltimes \langle \Phi^\vee \rangle$ (see [15], [23]).

The action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$ can be lifted to an action of $\widetilde{W}$. Indeed $\widetilde{W}$ preserves the lattice $\langle \Phi^\vee \rangle$ of $\mathfrak{h}$, and thus acts on $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ and on $\mathcal{C}_0(\Phi) \subset T$. Since the semidirect factor $\langle \Phi^\vee \rangle$ acts trivially, $\widetilde{W}$ acts as its quotient

$$\frac{\widetilde{W}}{\langle \Phi^\vee \rangle} \simeq W \times Z(\Phi).$$

2.2 Examples

In the following examples we denote by $\mathfrak{S}_n$, $\mathfrak{D}_n$, $\mathfrak{C}_n$ respectively the symmetric, dihedral and cyclic group on n letters.

1. **Case** C_n The roots $2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$ $(i = 1, \ldots, n)$ take integer values on the points $[\alpha_1^\vee/2], \ldots, [\alpha_n^\vee/2] \in \mathfrak{h}/\langle\Phi^\vee\rangle$, and thus on their sums, for a total of 2^n points of $\mathcal{C}_0(\Phi)$. Indeed let us introduce the following notation. If we fix a basis $h_1^*, \ldots, h_n^*$ of $\mathfrak{h}^*$, we can write the simple roots of C_n as

$$\alpha_i = h_i^* - h_{i+1}^* \text{ for } i = 1, \ldots, n-1\ , \text{ and } \alpha_n = 2h_n^*. \tag{6}$$

Then $\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\}$ $(i, j = 1, \ldots, n\, ,\, i \neq j)$, and if we write t_i for $e^{h_i^*}$, we have that

$$e^\Phi \doteq \{e^\alpha, \alpha \in \Phi\} = \{t_i t_j^{-1}\} \cup \{\, t_i t_j\} \cup \{\, t_i^{\pm 2}\}.$$

The system of n independent equations $t_1^2 = 1, \ldots, t_n^2 = 1$ has 2^n solutions: $(\pm 1, \ldots, \pm 1)$, and it is easy to see that all other systems does not have other solutions. $W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on $T = (\mathbb{C}^*)^n$ by permuting and inverting its coordinates; the second operation is trivial on $\mathcal{C}_0(\Phi)$. Then two elements of $\mathcal{C}_0(\Phi)$ are in the same $W-$orbit if and only if they have the same number of negative coordinates. So we can define the $p-$th $W-$orbit $\mathcal{O}_p$ as the set of points with p negative coordinates. (This choice is not canonical: we may choose the set of points with p positive coordinates as well). Clearly if $t \in \mathcal{O}_p$ then

$$W(t) \simeq (\mathfrak{S}_p \times \mathfrak{S}_{n-p}) \ltimes (\mathfrak{C}_2)^n.$$

Thus $|\mathcal{O}_p| = \binom{n}{p}$ and we get:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

Notice that if $t \in \mathcal{O}_p$ then $-t \in \mathcal{O}_{n-p}$ and $Ad(t) = Ad(-t)$ since $Z(\Phi) = \{\pm(1, \ldots, 1)\}$. In fact Γ has a symmetry exchanging the vertices p and $n-p$. Finally notice that $\mathcal{C}_0(\Phi)$ is a subgroup of T isomorphic to $(\mathfrak{C}_2)^n$ and generated by the elements

$$\delta_i \doteq (1, \ldots, 1, -1, 1, \ldots, 1) \text{ (with the } -1 \text{ at the } i-th \text{ place)}.$$

Then we can come back to the original coordinates observing that δ_i is the nontrivial solution of the system ${t_i}^2 = 1$, $t_j = 1 \forall j \neq i$, and using (6) to get:

$$\delta_i \leftrightarrow \left[\sum_{k=i}^{n} \alpha_k^\vee/2\right].$$

2. **Case** D_n We can write $\alpha_n = h^*_{n-1} + h^*_n$ and the others α_i as before, so $e^\Phi = \{t_i t_j^{-1}\} \cup \{ t_i t_j\}$. Then each system of n independent equations is $W-$conjugated to

   $$t_1 = t_2, \dots, t_{p-1} = t_p, t_{p-1} = t_p^{-1},\ t_{p+1}^{\pm 1} = t_{p+2}, \dots, t_{n-1} = t_n, t_{n-1} = t_n^{-1}$$

   for some $p \neq 1, n-1$. Then we get the subset of $(\mathfrak{C}_2)^n$ composed by the following $n-$ples: $\{(\pm 1, \dots, \pm 1)\} \backslash \{\pm \delta_i\}_{i=1,\dots,n}$, $2^n - 2n$ in number. However reasoning as before we see that each one represents two points in $\mathfrak{h}/\langle \Phi^\vee \rangle$. Namely, the correspondence is given by:

   $$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_{n-1}^\vee - \alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i.$$

   From a geometric point of view, the t_is are coordinates of a maximal torus of the orthogonal group, while $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ is a maximal torus of its two-sheets universal covering. Each one of the $W-$orbits corresponding to the four extremal vertices of Γ is composed by one of the four points over $\pm(1, \dots, 1)$, all inducing the identity automorphism: indeed $Aut(\Gamma)$ acts transitively on these points. The other orbits are defined as in the case C_n.

3. **Case** B_n This case is very similar to the previous one, but now $\alpha_n = h^*_n$, $e^\Phi = \{t_i t_j^{-1}\} \cup \{ t_i t_j\} \cup \{ t_i^{\pm 1}\}$, and then we get the points $\{(\pm 1, \dots, \pm 1)\} \backslash \{\delta_i\}_{i=1,\dots,n}$. Here the projection is

   $$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i$$

   so we have $2^n - n$ pairs of points in $\mathcal{C}_0(\Phi)$.

4. **Case** A_n If we see $\mathfrak{h}^*$ as the subspace of $\langle h_1^*, \dots, h^*_{n+1}\rangle$ of equation $\sum h_i^* = 0$, and T as the subgroup of $(\mathbb{C}^*)^{n+1}$ of equation $\prod t_i = 1$, we can write all the simple roots as $\alpha_i = h_i^* - h_{i+1}^*$; then $e^\Phi = \{t_i t_j^{-1}\}$. In this case Φ has no proper subsystem of its same rank, then all the coordinates must be identical. Therefore

   $$\mathcal{C}_0(\Phi) = Z(\Phi) = \big\{(\zeta, \dots, \zeta) | \zeta^{n+1} = 1\big\} \simeq \mathfrak{C}_{n+1}.$$

   Then $W \simeq \mathfrak{S}_{n+1}$ acts on $\mathcal{C}_0(\Phi)$ trivially and $Z(\Phi)$ transitively, as expected since $Aut(\Gamma) \simeq \mathfrak{D}_{n+1}$ acts transitively on the vertices of Γ. We can write more explicitly $\mathcal{C}_0(\Phi) \subseteq \mathfrak{h}/\langle \Phi^\vee\rangle$ as

   $$\mathcal{C}_0(\Phi) = \left\{ \left[\frac{k}{n+1} \sum_{i=1}^n i \alpha_i^\vee \right], k = 0, \dots, n \right\}.$$

2.3 Proof of Theorem 3

Motivated by Remark 2, we start to describe the automorphisms of $\mathfrak{g}$ that are induced by the points of $\mathcal{C}_0(\Phi)$.

Lemma 5. *If $t \in \mathcal{C}_0(\Phi)$, then $Ad(t)$ has finite order.*

Proof. Let $\beta_1, \dots, \beta_n$ linearly independent roots such that $e^{\beta_i}(t) = 1$: then for each root $\alpha \in \Phi$ we have that $m\alpha = \sum c_i \beta_i$ for some m and $c_i \in \mathbb{Z}$, thus

$$e^{\alpha}(t^m) = e^{m\alpha}(t) = \prod_{i=1}^{n} (e^{\beta_i})^{c_i}(t) = 1.$$

Then $Ad(t^m)$ is the identity on $\mathfrak{g}$, so by (4) $t^m \in Z(\Phi)$. $Z(\Phi)$ is a finite group, so t^m and t have finite order. $\square$

The previous lemma allows us to apply the following

Theorem 6 (Kač)**.**

1. *Each inner automorphism of $\mathfrak{g}$ of finite order m is conjugated to an automorphism σ of the form*

$$\sigma(X_i) = \zeta^{s_i} X_i$$

with ζ fixed primitive $m-th$ root of unity and $(s_0, \dots, s_n)$ nonnegative integers without common factors such that $m = \sum s_i a_i$.

2. *Two such automorphisms are conjugated if and only if there is an automorphism of Γ sending the parameters $(s_0, \dots, s_n)$ of the first in the parameters $(s'_0, \dots, s'_n)$ of the second.*

3. *Let $(i_1, \dots, i_r)$ be all the indices for which $s_{i_1} = \dots = s_{i_r} = 0$. Then $\mathfrak{g}^\sigma$ is the direct sum of an $(n-r)-$dimensional center and of a semisimple Lie algebra whose Dynkin diagram is the subdiagram of Γ of vertices $i_1, \dots, i_r$.*

This is a special case of a theorem proved in [16] and more extensively in [12, X.5.15 and 16]. We only need the following

Corollary 7.

1. *Let σ be an inner automorphism of $\mathfrak{g}$ of finite order m such that $\mathfrak{g}^\sigma$ is semisimple. Then there is $p \in V(\Gamma)$ such that σ is conjugated to σ_p. In particular $m = a_p$ and the Dynkin diagram of $\mathfrak{g}^\sigma$ is Γ_p.*

2. *Two automorphisms σ_p, $\sigma_{p'}$ are conjugated if and only if p, p' are in the same $Aut(\Gamma)-$orbit.*

Proof. If $\mathfrak{g}^\sigma$ is semisimple, then in Theorem 6.3 $n = r$, hence all but one parameters of σ are equal to 0, and the nonzero parameter s_p must be equal to 1, otherwise there would be a common factor, contradicting Theorem 6.1. So we get the first statement. Then the second statement follows from Theorem 6.2. $\square$

Let be $t \in \mathcal{C}_0(\Phi)$: by Remark 2 $\mathfrak{g}^{Ad(t)}$ is semisimple, so by Corollary 7.1 $Ad(t)$ is conjugated to some σ_p. Then there is a canonical map

$$\psi : \mathcal{C}_0(\Phi) \longrightarrow Q \tag{7}$$

sending t in $\psi(t) = \{p \in V(\Gamma) \text{ such that } \sigma_p \text{ is conjugated to } Ad(t)\}$. Notice that $\psi(t)$ is a well-defined element of Q by Corollary 7.2.

We now prove the fundamental

Lemma 8. *Two points in $\mathcal{C}_0(\Phi)$ induce conjugated automorphisms if and only if they are in the same $W \times Z(\Phi)-$orbit.*

Proof. Let N be the normalizer of T in G. We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugation action of N; it is also well known that two points of T are $G-$conjugated if and only if they are $W-$conjugated. Then $W-$conjugated points induce conjugated automorphisms. Moreover by (4)

$$Ad(t) = Ad(s) \Leftrightarrow Ad(ts^{-1}) = id_{\mathfrak{g}} \Leftrightarrow ts^{-1} \in Z(\Phi).$$

Finally suppose that $t, t' \in \mathcal{C}_0(\Phi)$ induce conjugated automorphisms, i.e.

$$\exists g \in G | Ad(t') = Ad(g)Ad(t)Ad(g^{-1}) = Ad(gtg^{-1}).$$

Then $zt' = gtg^{-1}$ for some $z \in Z(\Phi)$. Thus zt' and t are $G-$conjugated elements of T, and so they are $W-$conjugated, proving the claim.

$\square$

We can now prove the first part of Theorem 3. Indeed by the previous lemma there is a canonical injective map defined on the set of the orbits of $\mathcal{C}_0(\Phi)$:

$$\overline{\psi} : \frac{\mathcal{C}_0(\Phi)}{W \times Z(\Phi)} \longrightarrow Q.$$

We must show that this map is surjective. The system

$$\alpha_i(h) = 1 (\forall i \neq 0, p) \ , \ \alpha_p(h) = a_p^{-1}$$

is composed of n linearly independent equations, then it has a solution $h \in \mathfrak{h}$. Notice that $\alpha_0(h) \in \mathbb{Z}$. Let t be the class of h in T; then $e^\alpha(t) = 1 \Leftrightarrow \alpha \in \Phi_p$. Then by Remark 2 $Ad(t)$ is conjugated to σ_p and $\Phi(t)$ to Φ_p.

In order to relate the action of $Z(\Phi)$ with that of $Aut(\Gamma)$, we introduce the following subset of W. For each $p \neq 0$ such that $a_p = 1$, set $z_p \doteq w_0^p w_0$, where w_0 is the longest element of W and w_0^p is the longest element of the parabolic subgroup of W generated by all the simple reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ except s_{α_p}. Then we define

$$W_Z \doteq \{1\} \cup \{ z_p\}_{p=1,\dots,n|a_p=1}$$

W_Z has the following properties (see [15, 1.7 and 1.8]):

Theorem 9 (Iwahori-Matsumoto)**.**

1. *W_Z is a subgroup of W isomorphic to $Z(\Phi)$.*
2. *For each $z_p \in W_Z$, $z_p.\alpha_0 = \alpha_p$. This defines an injective map $W_Z \hookrightarrow Aut(\Gamma)$, and the W_Z-orbits of $V(\Gamma)$ coincide with the $Aut(\Gamma)-$orbits.*

Therefore Q is the set of W_Z-orbits of $V(\Gamma)$, and the bijection $\overline{\psi}$ between Q and the set of $Z(\Phi)-$orbits of $\mathcal{C}_0(\Phi)/W$ can be lifted to a noncanonical bijection between $V(\Gamma)$ and $\mathcal{C}_0(\Phi)/W$. Then we just have to consider the action of W on $\mathcal{C}_0(\Phi)$ and show the

Lemma 10. *If $t \in \mathcal{O}_p$, then $W(t)$ is conjugated to W_p.*

Proof. Notice that the centralizer $C_N(t)$ of t in N is the normalizer of $T = C_T(t)$ in $C_G(t)$. Then $W(t) = C_N(t)/T$ is the Weyl group of $C_G(t)$. $C_G(t)$ is the subgroup of G of points fixed by the conjugacy by t, then its Lie algebra is $\mathfrak{g}^{Ad(t)}$, that is conjugated to $\mathfrak{g}^{\sigma_p}$ by the first part of Theorem 3. Therefore $W(t)$ is conjugated to W_p.

□

This completes the proof of Theorem 3 and also of Theorem 1, since by Remark 2 the map ψ defined in (7) can also be seen as the map

$$t \mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \Phi_p \text{ is conjugated to } \Phi(t)\}.$$

3 d-dimensional components

3.1 From hyperplane arrangements to toric arrangements

Let S be a $d-$dimensional subspace of $\mathcal{H}$. The set Θ_S of the elements of Φ vanishing on S is a complete subsystem of Φ of rank $n-d$. Hence the map $S \to \Theta_S$ gives a bijection between $\mathcal{S}_d$ and $\mathcal{K}_d$, whose inverse is

$$\Theta \to S(\Theta) \doteq \{h \in \mathfrak{h} | \alpha(h) = 0 \forall \alpha \in \Theta\}.$$

In [22, 6.4 and C] (following [21] and [4]) the subspaces of $\mathcal{H}$ are classified and counted, and the $W-$orbits of $\mathcal{S}_d$ are completely described. This is done case-by-case according to the type of Φ. We now show a case-free way to extend this analysis to the components of $\mathcal{T}$.

Given a component U of $\mathcal{T}$ let us consider

$$\Theta_U \doteq \{\alpha \in \Phi | e^\alpha(t) = 1 \forall t \in U\}.$$

In contrast with the case of linear arrangements, Θ_U in general is not complete. Then for each $\Theta \in \mathcal{K}_d$ let us define $\mathcal{C}_\Theta^\Phi$ as the set of components U such that $\overline{\Theta_U} = \Theta$. This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$, i.e.

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_\Theta^\Phi \tag{8}$$

We may think of $S(\Theta)$ as the tangent space in any point of each component of $\mathcal{C}_\Theta^\Phi$; then by [22] the problem of classifying and counting the components of $\mathcal{T}$ reduces to classify and count the components of $\mathcal{T}$ having a given tangent space, i.e. the elements of $\mathcal{C}_\Theta^\Phi$. We do this in the next section.

3.2 Theorems

Let Θ be a complete subsystem of Φ and W^Θ its Weyl group. Let $\mathfrak{k}$ and K be respectively the semisimple Lie algebra and the semisimple and simply connected algebraic group of root system Θ, $\mathfrak{d}$ a Cartan subalgebra of $\mathfrak{k}$, $\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)$ the coroot and coweight lattices, $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle \Theta^\vee \rangle}$ the center of K, D the maximal torus of K defined by $\mathfrak{d}/\langle \Theta^\vee \rangle$, $\mathcal{D}$ the toric arrangement defined by Θ on D and $\mathcal{C}_0(\Theta)$ the set of its 0-dimensional components.

We also consider the *adjoint group* $K_a \doteq K/Z(\Theta)$ and its maximal torus $D_a \doteq D/Z(\Theta) \simeq \mathfrak{d}/\Lambda(\Theta)$. We recall from [13] that K is the universal covering of K_a, and if D' is an algebraic torus having Lie algebra $\mathfrak{d}$, then $D' \simeq \mathfrak{d}/L$ for some lattice $\Lambda(\Theta) \supseteq L \supseteq \langle \Theta^\vee \rangle$; so there are natural covering projections $D \twoheadrightarrow D' \twoheadrightarrow D_a$ having kernel respectively $L/\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)/L$. Notice that Θ naturally defines an arrangement on each D', and that for $D' = D_a$ the set of the 0-dimensional components is $\mathcal{C}_0(\Theta)/Z(\Theta)$. Given a point t of some D' we set

$$\Theta(t) \doteq \{\alpha \in \Theta | e^\alpha(t) = 1\}.$$

Theorem 11. *There is a $W^\Theta-$equivariant surjective map*

$$\varphi : \mathcal{C}_\Theta^\Phi \twoheadrightarrow \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ *and* $\Theta_U = \Theta(\varphi(U))$.

Proof. Let $S(\Theta)$ be the subspace of $\mathfrak{h}$ defined in Section 3.1 and H the corresponding subtorus of T. T/H is a torus with Lie algebra $\mathfrak{h}/S(\Theta) \simeq \mathfrak{d}$, then Θ defines an arrangement $\mathcal{D}'$ on $D' \doteq T/H$. The projection $\pi : T \twoheadrightarrow T/H$ induces a bijection between $\mathcal{C}^{\Phi}_{\Theta}$ and the set of 0-dimensional components of $\mathcal{D}'$, because $H \in \mathcal{C}^{\Phi}_{\Theta}$ and for each $U \in \mathcal{C}^{\Phi}_{\Theta}$, $\Theta_U = \Theta(\pi(U))$.

Moreover the restriction of the projection $d\pi : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/S(\Theta)$ to $\langle \Phi^{\vee} \rangle$ is simply the map that restricts the coroots of Φ to Θ. Set $R^{\Phi}(\Theta) \doteq d\pi(\langle \Phi^{\vee} \rangle)$; then $\Lambda(\Theta) \supseteq R^{\Phi}(\Theta) \supseteq \langle \Theta^{\vee} \rangle$ and $D' \simeq \mathfrak{d}/R^{\Phi}(\Theta)$. Denote by p the projection $\Lambda(\Phi) \twoheadrightarrow \frac{\Lambda(\Phi)}{\langle \Phi^{\vee} \rangle}$ and embed $\Lambda(\Theta)$ in $\Lambda(\Phi)$ in the natural way. Then the kernel of the covering projection of $D' \twoheadrightarrow D_a$ is isomorphic to

$$\frac{\Lambda(\Theta)}{R^{\Phi}(\Theta)} \simeq p(\Lambda(\Theta)) \simeq Z(\Phi) \cap Z(\Theta).$$

□

We set

$$n_{\Theta} \doteq \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}.$$

The following corollary is straightforward from Theorem 11.

Corollary 12.

$$|\mathcal{C}^{\Phi}_{\Theta}| = n_{\Theta}^{-1} |\mathcal{C}_0(\Theta)|$$

and then by (8)

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1} |\mathcal{C}_0(\Theta)|.$$

Notice that two components U, U' of $\mathcal{T}$ are $W-$conjugated if and only if are verified both the conditions:

1. their tangent spaces are $W-$conjugated , i.e. $\exists w \in W$ such that $\overline{\Theta_U} = w.\overline{\Theta_{U'}}$;

2. U and $w.U'$ are $W^{\overline{\Theta_U}}-$conjugated.

Then the action of W on $\mathcal{C}(\Phi)$ is described by the following remark.

Remark 13.

1. By Theorem 11, φ induces a surjective map $\overline{\varphi}$ from the set of the $W^{\Theta}-$orbits of $\mathcal{C}^{\Phi}_{\Theta}$ to the set of the $W^{\Theta} \times Z(\Theta)-$orbits of $\mathcal{C}_0(\Theta)$, that are described by Theorem 3.

2. In particular if Θ is irreducible, set Γ^{Θ} its affine Dynkin diagram, Q^{Θ} the set of the $Aut(\Gamma)-$orbits of its vertices, Γ^{Θ}_p the diagram that we

get from Γ^{Θ} removing the vertex p, and Θ_p the associated root system. Then there is a surjective map

$$\widehat{\varphi} : \mathcal{C}_{\Theta}^{\Phi} \twoheadrightarrow Q^{\Theta}$$

such that, if $\widehat{\varphi}(U) = q$ and p is a representative of q, then $\Theta_U \simeq \Theta_p$.

3.3 Examples

Case F_4. $Z(\Phi) = \{1\}$, thus $n_{\Theta} = |Z(\Theta)|$. Then in this case n_{Θ} does not depend on the conjugacy class, but only on the isomorphism class of Θ.

We say that a subspace S of $\mathcal{H}$ (respectively a component U of $\mathcal{T}$) is of a given type if the corresponding subsystem Θ_S (respectively Θ_U) is of such type. Then by [22, Tab. C.9] and Corollary 12 there are:

1. 1 subspace of type "A_0", tangent to 1 component of the same type (the whole spaces);
2. 24 subspaces of type A_1, each tangent to 1 component of the same type;
3. 72 subspaces of type $\mathsf{A}_1 \times \mathsf{A}_1$, each tangent to 1 component of the same type;
4. 32 subspaces of type A_2, each tangent to 1 component of the same type;
5. 18 subspaces of type B_2, each tangent to 1 component of the same type and 1 component of type $\mathsf{A}_1 \times \mathsf{A}_1$;
6. 12 subspaces of type C_3, each tangent to 1 component of the same type and 3 of type $\mathsf{A}_2 \times \mathsf{A}_1$;
7. 12 subspaces of type B_3, each tangent to 1 component of the same type, 1 of type A_3 and 3 of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$;
8. 96 subspaces of type $\mathsf{A}_1 \times \mathsf{A}_2$, each tangent to 1 component of the same type;
9. 1 subspace of type F_4 (the origin), tangent to: 1 component of the same type, 12 of type $\mathsf{A}_1 \times \mathsf{C}_3$, 32 of type $\mathsf{A}_2 \times \mathsf{A}_2$, 24 of type $\mathsf{A}_3 \times \mathsf{A}_1$, and 3 of type C_4.

Case A_{n-1}. It is easily seen that each subsystem Θ of Φ is complete and is product of irreducible factors $\Theta_1, \dots, \Theta_k$, with Θ_i of type $A_{\lambda_i - 1}$ for some positive integers λ_i such that $\lambda_1 + \cdots + \lambda_k = n$ and $n - k$ is the rank of Θ. In other words, as it is well known, the $W-$conjugacy classes of subspaces of $\mathcal{H}$ are in bijection with the partitions λ of n, and if a subspace has dimension d then corresponding partition has length $|\lambda| \doteq k$ equal to $d+1$. The number of subspaces of partition λ is easily seen to be equal to $n!/b_\lambda$, where b_i is the number of λ_j that are equal to i and $b_\lambda \doteq \prod i!^{b_i} b_i!$ (see [22, 6.72]). Now let g_λ be the greatest common divisor of $\lambda_1, \dots, \lambda_k$. By example 4 in Section 2.2 we have that $|Z(\Theta)| = \lambda_1 \dots \lambda_k = |\mathcal{C}_0(\Theta)|$ and $|Z(\Phi) \cap Z(\Theta)| = g_\lambda$. Then by Corollary 12 $|\mathcal{C}_\Theta^\Phi| = g_\lambda$ and

$$|\mathcal{C}_d(\Phi)| = \sum_{|\lambda|=d+1} \frac{n! g_\lambda}{b_\lambda}.$$

This could also be seen directly as follows. We can think T as the subgroup of $(\mathbb{C}^*)^n$ given by the equation $t_1 \dots t_n - 1 = 0$. Θ imposes the equations

$$t_1 = \dots = t_{\lambda_1},\ \dots\ , t_{\lambda_1 + \dots + \lambda_{k-1}+1} = \dots = t_n.$$

Then we have the relation $x_1^{\lambda_1} \dots x_k^{\lambda_k} - 1 = 0$. If $g_\lambda = 1$ this polynomial is irreducible, because the vector $(\lambda_1, \dots, \lambda_k)$ can be completed to a basis of the lattice $\mathbb{Z}^k$. If $g_\lambda > 1$ this polynomial has exactly g_λ irreducible factors over $\mathbb{C}$. Then in every case it defines an affine variety having exactly g_λ irreducible components, that are precisely the elements of $\mathcal{C}_\Theta^\Phi$.

4 Topological invariants

4.1 Theorems

Let R be the complement in T of the union of all the subtori of the toric arrangement $\mathcal{T}$. In this section we prove that the Euler characteristic of R, that we denote by χ_Φ, is equal to $(-1)^n|W|$. We also give a formula for the Poincaré polynomial of R, that we denote by $P_\Phi(q)$.

Let $d_1, \dots, d_n$ be the *degrees* of W, i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$; it is well known that $d_1 \dots d_n = |W|$. Moreover by [2] $\mathcal{B}(\Phi) \doteq (d_1 - 1) \dots (d_n - 1)$ is equal to the leading coefficient of the Poincaré polynomial of the complement of $\mathcal{H}$ in $\mathfrak{h}$, and then to the number of *unbroken basis* of Φ (see for example [9, 2.2.8 and 10.1.6]).

The cohomology of R can be expressed as a direct sum of contributions given by the components of $\mathcal{T}$ (see for example [7, Theor. 4.2] or [9, 14.1.5]). In terms of Poincaré polynomial this expression is:

Theorem 14.

$$P_\Phi(q) = \sum_{U \in \mathcal{C}(\Phi)} \mathcal{B}(\Theta_U)(q+1)^{d(U)} q^{n-d(U)}$$

where $d(U)$ is the dimension of the component U.

Now we use this expression to compute χ_Φ.

Lemma 15.

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p)$$

Proof.

$$\chi_\Phi = P_\Phi(-1) = (-1)^n \sum_{t \in \mathcal{C}_0(\Phi)} \mathcal{B}(\Phi(t)) \tag{9}$$

because the contributions of all components of positive dimension vanish at -1. Obviously isomorphic subsystems have the same degrees, so Theorem 1 yields the statement. □

Theorem 16.

$$\chi_\Phi = (-1)^n |W|$$

Proof. By the previous lemma we must prove that

$$\sum_{p=0}^{n} \frac{\mathcal{B}(\Phi_p)}{|W_p|} = 1$$

If we write $d_1^p, \dots, d_n^p$ for the degrees of W_p, the previous identity becomes

$$\sum_{p=0}^{n} \frac{(d_1^p - 1) \dots (d_n^p - 1)}{d_1^p \dots d_n^p} = 1.$$

This identity has been proved in [8], and later with different methods in [11]. □

Notice that W acts on R and then on its cohomology. So we can consider the *equivariant Euler characteristic* of R, that is, for each $w \in W$,

$$\widetilde{\chi}_\Phi(w) \doteq \sum_{i=0}^{n} (-1)^i \, Tr(w, H^i(R, \mathbb{C})).$$

Let ϱ_W be the character of the regular representation of W. From Theorem 16 we get the following

Corollary 17.

$$\widetilde{\chi}_\Phi = (-1)^n \varrho_W$$

Proof. Since W is finite and acts freely on R, it is well known that $\widetilde{\chi}_\Phi = k\varrho_W$ for some $k \in \mathbb{Z}$. Then to compute k we just have to look at $\widetilde{\chi}_\Phi(1) = \chi_\Phi$. □

Finally we give a formula for $P_\Phi(q)$ that, together with the mentioned results in [22], allows its explicit computation.

Theorem 18.

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|$$

Proof. By formula (8) we can restate Theorem 14 as

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} \sum_{U \in \mathcal{C}_\Theta^\Phi} \mathcal{B}(\Theta_U)$$

Moreover by Theorem 11 and Corollary 12 we get

$$\sum_{U \in \mathcal{C}_\Theta^\Phi} \mathcal{B}(\Theta_U) = n_\Theta^{-1} \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{B}(\Theta(t)).$$

Finally the claim follows by formula (9) and Theorem 16 applied to Θ:

$$\sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{B}(\Theta(t)) = (-1)^d \chi_\Theta = |W^\Theta|.$$

□

4.2 Examples

Case F_4. In Section 3.3 we have given a list of the 9 occurring types of complete subsystems, together with the multiplicity of each one. So we just have to compute the coefficient $n_\Theta^{-1}|W^\Theta|$ for each type. This is equal to:

- 1 for types 1., 2. and 3.
- 2 for types 4. and 8.
- 4 for type 5.
- 24 for types 6. and 7.
- 1152 for type 9.

Then

$$P_\Phi(q) = 2153q^4 + 1260q^3 + 286q^2 + 28q + 1.$$

Case A_{n-1}. By Section 3.3, $n_\Theta^{-1} = \frac{g_\lambda}{\lambda_1 \dots \lambda_k}$ and $|W^\Theta| = \lambda_1! \dots \lambda_k!$. Then by Theorem 17

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{|\lambda|=d+1} n! b_\lambda^{-1} g_\lambda (\lambda_1 - 1)! \dots (\lambda_k - 1)!.$$

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