Combinatorics and invariants of toric arrangements

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Special Session on Arrangements and Related Topics

Luca Moci

Roma Tre

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Hyperplane arrangements

Notations:

- $\mathfrak{g}$ a simple Lie algebra of rank n over $\mathbb{C}$
- $\mathfrak{h}$ a Cartan subalgebra
- $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$
- $\Phi^\vee \subset \mathfrak{h}$ the coroot system
- W be the Weyl group of Φ

$\{\alpha(h) = 0\}_{\alpha \in \Phi}$ defines in $\mathfrak{h}$ a family $\mathcal{H}$ of intersecting hyperplanes.
$\mathcal{H}$ is called the hyperplane arrangement defined by Φ.
We call subspaces of $\mathcal{H}$ the intersections of elements of $\mathcal{H}$,
and $\mathcal{S}_d(\Phi)$ the set of $d-$dimensional subspaces of $\mathcal{H}$.
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The coroot system $\Phi^\vee$ spans a lattice $\langle \Phi^\vee \rangle$ in $\mathfrak{h}$.
$T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$ is a complex torus of rank n.
Each root α is a map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle \Phi^\vee \rangle$.
So it induces a map $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote e^α.

$\{e^\alpha(t) = 1\}_{\alpha \in \Phi}$ defines in T a finite family $\mathcal{T}$ of codimension 1 subtori.
$\mathcal{T}$ is called the toric (or toral) arrangement defined by Φ.
We call components of $\mathcal{T}$ all the connected components of all the intersections of elements of $\mathcal{T}$, and we denote by $\mathcal{C}_d(\Phi)$ the set of $d-$dimensional components.
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Kostant partition function

$\mathcal{T}$ provides a geometric way (De Concini and Procesi, 2005) to compute the values of the Kostant partition function, that counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots.
It is involved in:

- Kostant's formula for weight multiplicities
- Steinberg's formula for Littlewood-Richardson coefficients

These formulas are efficient and have been implemented in computer programs.
The values of Kostant partition functions are computed a sum of "residues" at some points, and the points giving nonzero contribution are the elements of $\mathcal{C}_0(\Phi)$, that we call points of the arrangement.

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The cohomology of the complement

One can try to generalize to $\mathcal{T}$ well-known results for $\mathcal{H}$.

Let C be the complement in $\mathfrak{h}$ of the union of the elements of $\mathcal{H}$.
The cohomology of C has been explicitly described.

Now, let R be the complement in T of the union of the elements of $\mathcal{T}$.
The cohomology of R can be expressed as sum of contributions of the components of $\mathcal{T}$.
What do we know about these components?

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Diagrams, subsystems and subgroups

Let $\alpha_1, \dots, \alpha_n$ be simple roots of Φ and α_0 the lowest root.
Let Φ_p be the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$.
Let Γ be the affine Dynkin diagram of Φ.
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$.
So the diagram Γ_p that we get removing from Γ its vertex p is the (genuine) Dynkin diagram of Φ_p.
Let W_p be the Weyl group of Φ_p, i.e. the subgroup of W generated by all the reflections $s_{\alpha_0}, \dots, s_{\alpha_n}$ except s_{α_p}.

For each $t \in \mathcal{C}_0(\Phi)$ let be

$$\Phi(t) \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1\}.$$

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Points theorem

Theorem

There is a bijection $V(\Gamma) \leftrightarrow \mathcal{C}_0(\Phi)/W$, *having the property that if* $\mathcal{O}_p$ *is the orbit corresponding to the vertex* p *and* $t \in \mathcal{O}_p$, *then:*

- $\Phi(t)$ *is* $W-$*conjugated to* Φ_p;
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Then we have:

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$

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Example: Case C_n

$$\Phi = \{h_i^* - h_j^*\} \cup \{ h_i^* + h_j^*\} \cup \{\pm 2h_i^*\}$$

Then on the torus $T = \{(t_1, \dots, t_n), t_i \in \mathbb{C}^*\}$ the equations $e^\alpha(t) = 1$ are:

$$\{t_i t_j^{-1} = 1\} \cup \{ t_i t_j = 1\} \cup \{ t_i^2 = 1\}.$$

There are are 2^n solutions: $(\pm 1, \dots, \pm 1)$

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Example: Case C_n, 2

$W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$ acts on these points permuting their coordinates.
Then orbits are given by the number of negative coordinates.
Let $\mathcal{O}_p$ be the set of points with p negative coordinates.

Clearly the stabilizer of a such point is

$$\mathfrak{S}_p \times \mathfrak{S}_{n-p} \ltimes (\mathfrak{C}_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

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Example: Case C_n, 3

The previous choice is not canonical!
(we could define as well $\mathcal{O}_p$ as the set of points with p positive coordinates)

Observation:

- Γ has a symmetry exchanging the vertices p and $n-p$.
- Multiplication by -1 exchanges the corresponding orbits.

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The center

Given the coweight lattice

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi\}$$

we define the center

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle} = \{t \in T | \Phi(t) = \Phi\}.$$

Thus:

- $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$;
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Canonical bijection

We can make canonical the bijection between vertices and $W-$orbits by identifying:

- $Aut(\Gamma)-$conjugated vertices
- $Z(\Phi)-$conjugated orbits

Complete subsystems and subspaces

We define the completion of Θ:

$$\overline{\Theta} \doteq \langle\Theta\rangle_{\mathbb{R}} \cap \Phi$$

and we say that Θ is complete if $\Theta = \overline{\Theta}$.

Let S be a $d-$dimensional subspace of $\mathcal{H}$.
The set Θ_S of the elements of Φ vanishing on S
is a complete subsystem of Φ of rank $n-d$.

The map $S \to \Theta_S$ gives a bijection between $d-$dimensional subspaces and complete subsystems of rank $n-d$, whose inverse is

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A partition of the components

Given a component U of $\mathcal{T}$ let us consider

$$\Theta_U \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \forall t \in U\}.$$

In general Θ_U is not complete.
Then for each $\Theta \in \mathcal{K}_d$ let us define $\mathcal{C}_{\Theta}^{\Phi}$ as the set of components U such that $\overline{\Theta_U} = \Theta$.
This is clearly a partition of the set of $d-$dimensional components of $\mathcal{T}$, i.e.

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}_{\Theta}^{\Phi}$$

We may think of $S(\Theta)$ as the tangent space to each component in $\mathcal{C}_{\Theta}^{\Phi}$.
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Example: 1-dimensional components of C_3

There are 3 conjugation classes of 1-dimensional subspaces of $\mathcal{H}$, having representatives

$$(h, h, h)\ ,\ (h, h, 0)\ ,\ (h, 0, 0)\ ,\ h \in \mathbb{C}$$

tangent respectively to 1, 2, 4 components of $\mathcal{T}$:

$$(t, t, t)\ ,\ (t, t, \pm 1)\ ,\ (t, \pm 1, \pm 1)\ ,\ t \in \mathbb{C}^*$$

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Reduction theorem

Notations:

- Θ be a complete subsystem of Φ
- W^{Θ} its Weyl group
- $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle\Theta^{\vee}\rangle}$ the center
- $\mathcal{D}$ the toric arrangement defined by Θ on the torus D
- $\mathcal{C}_0(\Theta)$ the set of 0-dimensional components of $\mathcal{D}$

Theorem

There is a $W^{\Theta}-$equivariant surjective map

$$\varphi : \mathcal{C}^{\Phi}_{\Theta} \to \mathcal{C}_0(\Theta)/Z(\Theta)$$

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$$|\mathcal{C}_{\Theta}^{\Phi}| = n_{\Theta}^{-1} |\mathcal{C}_0(\Theta)|$$

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Then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1} |\mathcal{C}_0(\Theta)|.$$

Moreover the reduction theorem yields a description of the action of W on $\mathcal{C}(\Phi)$.
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By Orlik and Solomon's tables there are:

1. 12 subspaces of type B_3;
2. 12 subspaces of type C_3;
3. 96 subspaces of type $A_1 \times A_2$;

By our formula each subspace of type B_3 is tangent to:

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We work in the same way for the other types.

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The Poincaré polynomial of R

Our results about the components yield more explicit description of the cohomology.

Let $d_1, \ldots, d_n$ be the degrees of W.
It is well known that $d_1 \ldots d_n = |W|$.
We define $\mathcal{B}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1)$.

By the known formulas for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_U \mathcal{B}(\Theta_U)(q+1)^{d(U)} q^{n-d(U)}$$

where U varies on all the components of $\mathcal{T}$ and $d(U)$ is its dimension.

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Theorem

The Euler characteristic χ_Φ *is equal to* $(-1)^n|W|$

Proof.

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.
2. Applying points theorem we get

$$\chi_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{B}(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p - 1) \ldots (d_n^p - 1)}{d_1^p \ldots d_n^p} = 1$ (where $d_1^p, \ldots, d_n^p$ are the degrees of W_p).

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