Hyperplane arrangements

Notations:

- $\mathfrak{g}$ a simple Lie algebra of rank $n$ over $\mathbb{C}$
- $\mathfrak{h}$ a Cartan subalgebra
- $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$
- $\Phi^\vee \subset \mathfrak{h}$ the coroot system
- $W$ be the Weyl group of $\Phi$

$\{\alpha(h) = 0\}_{\alpha \in \Phi}$ defines in $\mathfrak{h}$ a family $\mathcal{H}$ of intersecting hyperplanes. $\mathcal{H}$ is called the hyperplane arrangement defined by $\Phi$.

We call subspaces of $\mathcal{H}$ the intersections of elements of $\mathcal{H}$, and $S_d(\Phi)$ the set of $d$–dimensional subspaces of $\mathcal{H}$. $W$ acts naturally on $\mathcal{H}$ and on $S_d(\Phi)$. 

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The coroot system $\Phi^\vee$ spans a lattice $\langle \Phi^\vee \rangle$ in $\mathfrak{h}$.

$T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$ is a complex torus of rank $n$.

Each root $\alpha$ is a map $\mathfrak{h} \to \mathbb{C}$ taking integer values on $\langle \Phi^\vee \rangle$.

So it induces a map $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote $e^\alpha$.

$\{e^\alpha(t) = 1\}_{\alpha \in \Phi}$ defines in $T$ a finite family $\mathcal{T}$ of codimension 1 subtori.

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We call components of $\mathcal{T}$ all the connected components of all the intersections of elements of $\mathcal{T}$, and we denote by $C_d(\Phi)$ the set of $d$–dimensional components.

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- Kostant’s formula for weight multiplicities
- Steinberg’s formula for Littlewood-Richardson coefficients

These formulas are efficient and have been implemented in computer programs. The values of Kostant partition functions are computed as a sum of “residues” at some points, and the points giving nonzero contribution are the elements of $C_0(\Phi)$, that we call points of the arrangement.
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One can try to generalize to $\mathcal{T}$ well-known results for $\mathcal{H}$.

Let $C$ be the complement in $\mathcal{H}$ of the union of the elements of $\mathcal{H}$. The cohomology of $C$ has been explicitly described.

Now, let $R$ be the complement in $\mathcal{T}$ of the union of the elements of $\mathcal{T}$. The cohomology of $R$ can be expressed as sum of contributions of the components of $\mathcal{T}$. What do we know about these components?
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Abstract

Today we will:

1. count the components of $\mathcal{T}$, describing the action of $W$ on them.

2. compute the Euler characteristic and the Poincaré polynomial of the complement $R$. 
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1. count the components of $\mathcal{T}$, describing the action of $\mathcal{W}$ on them.

2. compute the Euler characteristic and the Poincaré polynomial of the complement $R$. 
Let $\alpha_1, \ldots, \alpha_n$ be simple roots of $\Phi$ and $\alpha_0$ the lowest root.
Let $\Phi_p$ be the subsystem of $\Phi$ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$.
Let $\Gamma$ be the affine Dynkin diagram of $\Phi$.
The set of its vertices $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.
So the diagram $\Gamma_p$ that we get removing from $\Gamma$ its vertex $p$ is the (genuine) Dynkin diagram of $\Phi_p$.
Let $W_p$ be the Weyl group of $\Phi_p$, i.e. the subgroup of $W$ generated by all the reflections $s_{\alpha_0}, \ldots, s_{\alpha_n}$ except $s_{\alpha_p}$.

For each $t \in C_0(\Phi)$ let be

$$\Phi(t) \doteq \{\alpha \in \Phi | e^\alpha(t) = 1\}.$$ 

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$$\Phi(t) \triangleq \{ \alpha \in \Phi \mid e^\alpha(t) = 1 \}.$$ 

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Points theorem

**Theorem**

There is a **bijection** $V(\Gamma) \leftrightarrow C_0(\Phi)/W$, having the property that if $O_p$ is the orbit corresponding to the vertex $p$ and $t \in O_p$, then:

- $\Phi(t)$ is $W$–conjugated to $\Phi_p$;
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Then we have:

$$|C_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}.$$
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Example: Case $C_n$

\[\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\}\]

Then on the torus $T = \{(t_1, \ldots, t_n), t_i \in \mathbb{C}^*\}$ the equations $e^\alpha(t) = 1$ are:

\[\{t_it_j^{-1} = 1\} \cup \{t_it_j = 1\} \cup \{t_i^2 = 1\}\]

There are $2^n$ solutions: $(\pm 1, \ldots, \pm 1)$
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$\mathcal{W} \cong S_n \times (C_2)^n$ acts on these points permuting their coordinates. Then orbits are given by the number of negative coordinates. Let $\mathcal{O}_p$ be the set of points with $p$ negative coordinates.

Clearly the stabilizer of a such point is

$$S_p \times S_{n-p} \times (C_2)^n$$

thus $|\mathcal{O}_p| = \binom{n}{p}$ and our formula is checked:

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- $\Gamma$ has a symmetry exchanging the vertices $p$ and $n - p$.
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The center

Given the coweight lattice

$$\Lambda(\Phi) \doteq \{ h \in h | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi \}$$

we define the center

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle} = \{ t \in T | \Phi(t) = \Phi \}.$$ 

Thus:

- $Z(\Phi) \subseteq C_0(\Phi)$;
- $Z(\Phi)$ acts by multiplication on $C_0(\Phi)$. 
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We can make canonical the bijection between vertices and $W$—orbits by identifying:

- $Aut(\Gamma)$—conjugated vertices
- $Z(\Phi)$—conjugated orbits
Complete subsystems and subspaces

We define the completion of $\Theta$:

$$\overline{\Theta} \doteq \langle \Theta \rangle_R \cap \Phi$$

and we say that $\Theta$ is complete if $\Theta = \overline{\Theta}$.

Let $S$ be a $d$–dimensional subspace of $H$. The set $\Theta_S$ of the elements of $\Phi$ vanishing on $S$ is a complete subsystem of $\Phi$ of rank $n - d$.

The map $S \rightarrow \Theta_S$ gives a bijection between $d$–dimensional subspaces and complete subsystems of rank $n - d$, whose inverse is

$$\Theta \rightarrow S(\Theta) \doteq \{ h \in H | \alpha(h) = 0 \forall \alpha \in \Theta \}.$$
Complete subsystems and subspaces

We define the completion of $\Theta$:

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We now show how to extend this analysis to the components of $\mathcal{T}$. 
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The subspaces of $H$ have been classified and counted, and the $W$—orbits of $S_d$ completely described. This was done in 1980 by Orlik and Solomon case-by-case according to the type of $\Phi$.

We now show how to extend this analysis to the components of $\mathcal{T}$. 
A partition of the components

Given a component $U$ of $\mathcal{T}$ let us consider

$$\Theta_U \triangleq \{ \alpha \in \Phi | e^\alpha(t) = 1 \forall t \in U \}.$$

In general $\Theta_U$ is not complete.

Then for each $\Theta \in \mathcal{K}_d$ let us define $C^\Theta$ as the set of components $U$ such that $\overline{\Theta_U} = \Theta$.

This is clearly a partition of the set of $d$–dimensional components of $\mathcal{T}$, i.e.

$$C_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} C^\Theta.$$

We may think of $S(\Theta)$ as the tangent space to each component in $C^\Theta$.

Then the problem of classifying and counting the components of $\mathcal{T}$ reduces to classify and count the components of $\mathcal{T}$ having a given tangent space, i.e. to describe $C^\Theta$ for each $\Theta$. 
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Example: 1-dimensional components of $C_3$

There are 3 conjugation classes of 1-dimensional subspaces of $H$, having representatives

$$(h, h, h), (h, h, 0), (h, 0, 0), \ h \in \mathbb{C}$$

tangent respectively to 1, 2, 4 components of $T$:

$$(t, t, t), (t, t, \pm 1), (t, \pm 1, \pm 1), \ t \in \mathbb{C}^*$$

This suggests to relate the components of $C_\emptyset$ to the 0-dimensional components of a smaller toric arrangement.
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This suggests to relate the components of $C_\mathcal{G}$ to the 0-dimensional components of a smaller toric arrangement.
Reduction theorem

Notations:

- $\Theta$ be a complete subsystem of $\Phi$
- $W^\Theta$ its Weyl group
- $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle \Theta^\vee \rangle}$ the center
- $\mathcal{D}$ the toric arrangement defined by $\Theta$ on the torus $D$
- $\mathcal{C}_0(\Theta)$ the set of 0-dimensional components of $\mathcal{D}$

Theorem

There is a $W^\Theta$—equivariant surjective map

$$\varphi : \mathcal{C}_0^\Phi \to \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ and $\Theta_U = \Theta(\varphi(U))$. 
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The number of the components

**Corollary**

\[ |C_\Theta^\Phi| = n_\Theta^{-1} |C_0(\Theta)| \]

where \( n_\Theta = \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|} \).

Then

\[ |C_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |C_0(\Theta)|. \]

Moreover the reduction theorem yields a description of the action of \( \mathcal{W} \) on \( C(\Phi) \).

Then we get a \( \mathcal{W} \)–equivariant decomposition of the cohomology of \( R \).
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Example: 1-dimensional components of $F_4$

By Orlik and Solomon’s tables there are:

1. 12 subspaces of type $B_3$;
2. 12 subspaces of type $C_3$;
3. 96 subspaces of type $A_1 \times A_2$;

By our formula each subspace of type $B_3$ is tangent to:

- 1 component of the same type;
- 1 components of type $A_3$;
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We work in the same way for the other types.
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The Poincaré polynomial of $R$

Our results about the components yield more explicit description of the cohomology.

Let $d_1, \ldots, d_n$ be the degrees of $W$. It is well known that $d_1 \ldots d_n = |W|$. We define $B(\Phi) = (d_1 - 1) \ldots (d_n - 1)$.

By the known formulas for cohomology, the Poincaré polynomial is

$$P_\Phi(q) = \sum_U B(\Theta_U)(q + 1)^{d(U)} q^{n-d(U)}$$

where $U$ varies on all the components of $\mathcal{T}$ and $d(U)$ is its dimension.
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The Euler characteristic

**Theorem**

The **Euler characteristic** $\chi_{\Phi}$ is equal to $(-1)^n |W|$

**Proof.**

1. When we evaluate the Poincaré polynomial in $q = -1$ all the contributions vanish except for those of the points.

2. Applying points theorem we get

$$\chi_{\Phi} = (-1)^n \sum_{p=0}^{n} \frac{|W_p|}{|W_p|} B(\Phi_p).$$

3. The equivalence between this expression and the claimed one is the "curious identity" $\sum_{p=0}^{n} \frac{(d_1^p-1)(d_n^p-1)}{d_1^p \ldots d_n^p} = 1$

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\[ P_\Phi(q) = \sum_{d=0}^{n} (q + 1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1} |W^{\Theta}| \]

Notice that this formula allows to compute explicitly \( P_\Phi(q) \).
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In dimension 1 there are:

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We do the same in the other dimensions and we find that

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