Topological complexity of basis-conjugating automorphisms of free groups

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M. Farber [2003, 2004] introduces the notion of *Topological Complexity*, which gives measure of the "‘navigational complexity’" of the system.

**Definition**

*Topological Complexity* of $X$, denoted by $TC(X)$, is the Schvarz genus (sectional category) of the path fibration $\pi: PX \to X \times X$, i.e. the smallest integer $k$ such that the Cartesian product $X \times X$ can be covered by $k$ open subsets $U_1, U_2, \ldots, U_k$, on each of which the restriction of the path fibration has a section.

$TC(X) = 1$ if and only if $X$ is contractible.
Farber [2006] poses the question:

- Find the $\text{TC}(X)$, where $X$ is an Eilenberg-Mac Lane space of type $K(G, 1)$, and describe it in terms of algebraic invariants of the group $G = \pi_1(X)$.

For right angled Artin groups, $\text{TC}$ was computed in [Cohen-P, 2008].

In [FY 2004], [FGY 2007], Farber, Grant, and Yuzvinsky solved problem for pure braid groups $P_n$, and configuration spaces $F(\mathbb{C}_m, n)$ of $n$ ordered points in $\mathbb{C} - \{m \text{ points}\}$.

**Objective:** Find the $\text{TC}$ for the basis-conjugating groups of $F_n$. 
Let $N \subset M$ be a compact subset in the interior of a manifold $M$. Dahm [1962] defines a motion of $N$ in $M$ as a path $h_t$ in $\mathcal{H}_c(M)$, the space of homeomorphisms of $M$ with compact support, satisfying $h_0 = \text{id}_M$ and $h_1(N) = N$. With an appropriate notion of equivalence, the set of equivalence classes of motions of $N$ in $M$ is a group, and, furthermore, there is a homomorphism from this group to the automorphism group of the fundamental group $\pi_1(M \setminus N)$.

Goldsmith [1981] gives an exposition of Dahm’s (unpublished) work, especially the case when $N = \mathcal{L}_n = C_1 \cup \cdots \cup C_n$ is a collection of $n$ unknotted, unlinked circles in $M = \mathbb{R}^3$.

Let $\mathcal{G}_n$ denote the corresponding motion group. Goldsmith shows that $\mathcal{G}_n$ is generated by three types of motions, flipping a single circle, interchanging two (adjacent) circles, and pulling one circle through another, and that the Dahm homomorphism $\phi: \mathcal{G}_n \to \text{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n))$ is an embedding.
\[ \pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n, e) = F_n, \] 
free group generated by \( x_1, \ldots, x_n \), so one has the embedding \( \phi : \mathcal{G}_n \rightarrow \text{Aut}(F_n) \).

Brownstein-Lee [1993] identified the group of "pure motions" of \( \mathcal{L}_n \), i.e. motions which bring each oriented circle back to its original position, with the group of basis-conjugating automorphisms of \( F_n \).
The group $P\Sigma_n$, \textit{basis-conjugating automorphism group} of $F_n$. McCool [1986] gave a presentation of $P\Sigma_n$:

- **Generators**
  
  \[ \alpha_{i,j}, \quad 1 \leq i, j \leq n, \quad i \neq j, \]

  acting on generators $x_k$ of $F_n$ via

  \[ \alpha_{i,j}(x_k) = \begin{cases} 
  x_j x_k x_j^{-1} & \text{if } k = i, \\
  x_k & \text{if } k \neq i. 
  \end{cases} \]

- **Relations**

  \[
  \begin{cases}
  [\alpha_{i,j}, \alpha_{k,l}] & \text{for } i, j, k, l \text{ distinct} \\
  [\alpha_{i,j}, \alpha_{k,j}] & \text{for } i, j, k \text{ distinct} \\
  [\alpha_{i,j}, \alpha_{i,k} \alpha_{j,k}] & \text{for } i, j, k \text{ distinct}
  \end{cases},
  \]

  where $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$ denotes the commutator.
Proposition

Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then

$$\text{geom dim}(P\Sigma_n) = \text{cd}(P\Sigma_n) = n - 1.$$
Let $k$ be a field, and let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded $k$-algebra. Define the \textit{cup length} of $A$, denoted by $\text{cl}(A)$, to be the largest integer $q$ for which there are homogeneous elements $a_1, \ldots, a_q$ of positive degree in $A$ such that $a_1 \cdots a_q \neq 0$.

Let $X$ be an Eilenberg-Mac Lane complex of type $K(G, 1)$. Consider the multiplication homomorphism

$$H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \xrightarrow{\cup} H^*(X; \mathbb{Q}),$$

and call the kernel $Z = Z(H^*(X; \mathbb{Q}))$ of this map the \textit{ideal of zero divisors}. Define the \textit{zero-divisor cup length}, denoted by $\text{zcl}(H^*(X; \mathbb{Q}))$, to be the cup length of the ideal of zero divisors $\text{zcl}(H^*(X; \mathbb{Q})) = \text{cl}(Z)$. 
Structure of integral cohomology of $PΣ_n$ was conjectured by Brownstein-Lee [1993], and was proved by Jensen-McCammond-Meier [2006]:

**Theorem (Jensen-McCammond-Meier, 2006)**

Let $PΣ_n$ be the basis-conjugating automorphism group. Then the cohomology algebra $H^*(PΣ_n; \mathbb{Q}) = E/I$ is the quotient of the exterior algebra on generators $a_{i,j}$ in degree one with $i \neq j$, by the homogeneous ideal of degree two

\[ I = \langle a_{i,j}a_{j,i}, \ a_{k,j}a_{j,i} - (a_{k,j} - a_{i,j})a_{k,i} \mid i, j, k \text{ distinct} \rangle . \]
Theorem (Cohen-P)

Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then
\[
zcl(H^*(P\Sigma_n; \mathbb{Q})) = 2n - 2.
\]

Proof: Since $\text{cd}(P\Sigma_n) = n - 1$, $zcl(P\Sigma_n) \leq 2n - 2$.

$H^*(P\Sigma_n, \mathbb{Q})$ generated by the $q$-fold products of $\alpha_{i,j}$ that

- do not repeat any $a_{j,k}$,
- do not repeat the first index,
- do not contain any cyclic products $\alpha_{i,j}\alpha_{j,k} \cdots \alpha_{s,t}\alpha_{t,i}$.

Apply relations
\[
a_{k,j}a_{k,i} = a_{k,j}a_{j,i} + a_{i,j}a_{k,i}, \tag{1}
\]

eliminate repetition in the first index.
For each \( i < n \), consider the elements \( X_i = a_{i,i+1} \), and \( Y_i = a_{i+1,i} \), in the cohomology algebra \( H^*(P\Sigma_n; \mathbb{Q}) \), and the corresponding zero divisors \( \bar{X}_i = X_i \otimes 1 - 1 \otimes X_i \), and \( \bar{Y}_i = Y_i \otimes 1 - 1 \otimes Y_i \), in the tensor product algebra \( H^{n-1}(P\Sigma_n; \mathbb{Q}) \otimes H^{n-1}(P\Sigma_n; \mathbb{Q}) \).

Claim: The product of these 2\( n - 2 \) zero divisors

\[
M = \prod_{i=1}^{n-1} \bar{X}_i \cdot \prod_{i=1}^{n-1} \bar{Y}_i = \sum_{I \subseteq [1,n-1]} (-1)^{|I|} U_I \otimes V_I \neq 0
\]

where \([1, n - 1]\) denotes the set of integers \( \{1, 2, \ldots, n - 1\} \), \(|I|\) is the number of elements in \( I \), \( U_I = z_1 \cdots z_{n-1} \), and \( V_I = \hat{z}_1 \cdots \hat{z}_{n-1} \) with

\[
z_i = \begin{cases} Y_i, & \text{if } i \notin I, \\ X_i, & \text{if } i \in I \end{cases}
\]

and

\[
\hat{z}_i = \begin{cases} Y_i, & \text{if } i \in I, \\ X_i, & \text{if } i \notin I. \end{cases}
\]

In order to prove this claim, we apply relations (1) to \( M \), and reduce it into the induced basis of the tensor product, and identify at least one monomial that stays unaffected throughout the reduction process.
When $I$ is the empty set $I = \emptyset$, then up to the sign, the summand $U_I \otimes U_I$ in the expression (??) is

$$Y_{n-1} Y_{n-2} \cdots Y_1 \otimes X_1 X_2 \cdots X_{n-1} = a_{n,n-1} a_{n-1,n-2} \cdots a_{2,1} \otimes a_{1,2} a_{2,3} \cdots a_{n-1,n}.$$  

This monomial is already a basis element of $H^{n-1}(P \Sigma_n; \mathbb{Q}) \otimes H^{n-1}(P \Sigma_n; \mathbb{Q})$. We claim that any other $U_I$, after reducing into the basis, will avoid the specified basis element $U_\emptyset$. Namely, if the monomial $U_I$ is already a basis element, there is nothing to prove. Otherwise, $U_I$ contains a factor $a_{k,j} a_{k,i}$ for at least one $k$ with $1 < k < n$, and these are the only generators in the product, having subindex $k$. Applying the relation (1) to the product $a_{k,j} a_{k,i}$, we obtain (up to the sign)

$$U_I = (a_{k,j} a_{j,i} + a_{i,j} a_{k,i}) \cdot \{\text{other factors}\} = a_{k,j} P + a_{k,i} Q,$$

where none of the factors of $P$ and $Q$ has subindex $k$. 

Further application of reductive relations to $P$ and $Q$ will result in no further appearance of $k$ in indices. This means that multiplication by the generator with the first subindex $k$ will produce a basis element of $H^{n-1}(P\Sigma_n; \mathbb{Q})$. As a result, such a reducible monomial $U_I$ after reduction in the basis elements has exactly one factor with subindex $k$, while our fixed monomial $U_\emptyset = a_{n,n-1} \cdots a_{k+1,k} a_{k,k-1} \cdots a_{2,1}$ contains two factors with subindex $k$. Hence, the basis monomial $U_\emptyset \otimes V_\emptyset$ is different from any other possible basis summand coming from $U_I \otimes V_I$ with $I \neq \emptyset$, and our claim holds.
Farber [2003, 2004] developed the estimates for the $TC(X)$:

Let $X$ and $Y$ be path-connected paracompact locally contractible topological spaces (or in particular, CW-complexes). Then

$$zcl(X) + 1 \leq TC(X) \leq 2 \text{cat}(X) - 1 \leq 2 \text{dim}(X) + 1,$$

and

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$
Let $\Sigma_n$ be the basis-conjugating automorphism group of the free group $F_n$. Then the topological complexity is $\text{TC}(\Sigma_n) = 2n - 1$. 

Theorem (Cohen-P)
Cohen-Pakianathan-Vershinin-Wu [2008] consideres the "upper triangular" version of $P\Sigma_n$.

*Upper triangular McCool group* $P\Sigma_n^+$ is a subgroup of $P\Sigma_n$, generated by $\alpha_{i,j}$ with $i < j$.

**Theorem (Cohen-P, 2008)**

$$\text{TC}(P\Sigma_n^+) = \text{zcl}(H^*(P\Sigma_n^+; \mathbb{Q})) + 1 = 2n - 2.$$
Let $G$ be the upper triangular McCool group $P\Sigma_n^+$. Then the Eilenberg-Mac Lane space of type $K(G, 1)$ is a formal space.