Topological complexity of basis-conjugating automorphisms of free groups

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AMS Southeastern Sectional Meeting
LSU, Baton Rouge, LA
March 28-30, 2008

Introduction

M. Farber [2003, 2004] introduces the notion of *Topological Complexity*, which gives measure of the "‘navigational complexity"’ of the sysyem.

Definition

Topological Complexity of X, denoted by $\mathrm{TC}(X)$, is the Schvarz genus (sectional category) of the path fibration $\pi \colon PX \to X \times X$, i.e. the smallest integer k such that the Cartesian product $X \times X$ can be covered by k open subsets $U_1, U_2, ..., U_k$, on each of which the restriction of the path fibration has a section.

$\mathrm{TC}(X) = 1$ if and only if X is contractible.

Introduction

Farber [2006] poses the question:

- Find the $\mathrm{TC}(X)$, where X is an Eilenberg-Mac Lane space of type $K(G, 1)$, and describe it in terms of algebraic invariants of the group $G = \pi_1(X)$.

For right angled Artin groups, TC was computed in [Cohen-P, 2008].

In [FY 2004], [FGY 2007], Farber, Grant, and Yuzvinsky solved problem for pure braid groups P_n, and configuration spaces $F(\mathbb{C}_m, n)$ of n ordered points in $\mathbb{C} - \{m \text{ points}\}$.

Objective: Find the TC for the basis-conjugating groups of F_n.

Group of motions

Let $N \subset M$ be a compact subset in the interior of a manifold M. Dahm [1962] defines a motion of N in M as a path h_t in $\mathcal{H}_c(M)$, the space of homeomorphisms of M with compact support, satisfying $h_0 = \mathrm{id}_M$ and $h_1(N) = N$. With an appropriate notion of equivalence, the set of equivalence classes of motions of N in M is a group, and, furthermore, there is a homomorphism from this group to the automorphism group of the fundamental group $\pi_1(M \setminus N)$.

Goldsmith [1981] gives an exposition of Dahm's (unpublished) work, especially the case when $N = \mathcal{L}_n = C_1 \cup \cdots \cup C_n$ is a collection of n unknotted, unlinked circles in $M = \mathbb{R}^3$.
Let $\mathcal{G}_n$ denote the corresponding motion group. Goldsmith shows that $\mathcal{G}_n$ is generated by three types of motions, flipping a single circle, interchanging two (adjacent) circles, and pulling one circle through another, and that the Dahm homomorphism $\phi\colon \mathcal{G}_n \to \mathrm{Aut}(\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n))$ is an embedding.

Group of Motions

$\pi_1(\mathbb{R}^3 \setminus \mathcal{L}_n, e) = F_n$, free group generated by $x_1, \ldots, x_n$, so one has the embedding $\phi : \mathcal{G}_n \to \mathrm{Aut}(F_n)$.

Brownstein-Lee [1993] identified the group of "‘pure motions"’ of $\mathcal{L}_n$, i.e. motions which bring each oriented circle back to its original position, with the group of basis-conjugating automorphisms of F_n.

Basis-conjugating group of automorphisms of F_n

The group $P\Sigma_n$, *basis-conjugating automorphism group* of F_n.
McCool [1986] gave a presentation of $P\Sigma_n$:

- Generators

  $$\alpha_{i,j}, \quad 1 \leq i, j \leq n, \quad i \neq j,$$

  acting on generators x_k of F_n via

  $$\alpha_{i,j}(x_k) = \begin{cases} x_j x_k x_j^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

- relations

  $$\left\{ \begin{array}{ll} [\alpha_{i,j}, \alpha_{k,l}] & \text{for } i,j,k,l \text{ distinct} \\ [\alpha_{i,j}, \alpha_{k,j}] & \text{for } i,j,k \text{ distinct} \\ [\alpha_{i,j}, \alpha_{i,k}\alpha_{j,k}] & \text{for } i,j,k \text{ distinct} \end{array} \right\},$$

  where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ denotes the commutator.

Proposition

Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then

$$\operatorname{geom\,dim}(P\Sigma_n) = \operatorname{cd}(P\Sigma_n) = n - 1.$$

Zero-divisor cup length

Let $\mathbb{k}$ be a field, and let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded $\mathbb{k}$-algebra. Define the *cup length* of A, denoted by $\mathrm{cl}(A)$, to be the largest integer q for which there are homogeneous elements $a_1, \ldots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$.

Let X be an Eilenberg-Mac Lane complex of type $K(G, 1)$. Consider the multiplication homomorphism

$$H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \xrightarrow{\cup} H^*(X; \mathbb{Q}),$$

and call the kernel $Z = Z(H^*(X; \mathbb{Q}))$ of this map the *ideal of zero divisors*. Define the *zero-divisor cup length*, denoted by $\mathrm{zcl}(H^*(X; \mathbb{Q}))$, to be the cup length of the ideal of zero divisors $\mathrm{zcl}(H^*(X; \mathbb{Q})) = \mathrm{cl}(Z)$.

Cohomology of $P\Sigma$

Structure of integral cohomology of $P\Sigma_n$ was conjectured by Brownstein-Lee [1993], and was proved by Jensen-McCammond-Meier [2006]:

Theorem (Jensen-McCammond-Meier, 2006)

Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then the cohomology algebra $H^*(P\Sigma_n;\mathbb{Q}) = E/I$ is the quotient of the exterior algebra on generators $a_{i,j}$ in degree one with $i \neq j$, by the homogeneous ideal of degree two
$I = \langle a_{i,j}a_{j,i},\ \ a_{k,j}a_{j,i} - (a_{k,j} - a_{i,j})a_{k,i} \mid i,j,k \text{ distinct}\rangle$.

Zero-divisor cup length

Theorem (Cohen-P)

Let $P\Sigma_n$ be the basis-conjugating automorphism group. Then $\mathrm{zcl}(H^*(P\Sigma_n;\mathbb{Q})) = 2n-2$.

Proof: Since $\mathrm{cd}(P\Sigma_n) = n-1$, $\mathrm{zcl}(P\Sigma_n) \leq 2n-2$.

$H^*(P\Sigma_n, \mathbb{Q})$ generated by the q-fold products of $\alpha_{i,j}$ that

- do not repeat any $a_{j,k}$,
- do not repeat the first index,
- do not contain any cyclic products $\alpha_{i,j}\alpha_{j,k}\cdots\alpha_{s,t}\alpha_{t,i}$.

Apply relations

$$a_{k,j}a_{k,i} = a_{k,j}a_{j,i} + a_{i,j}a_{k,i}, \tag{1}$$

eliminate repetition in the first index.

For each $i < n$, consider the elements $X_i = a_{i,i+1}$, and $Y_i = a_{i+1,i}$, in the cohomology algebra $H^*(P\Sigma_n; \mathbb{Q})$, and the corresponding zero divisors $\bar{X}_i = X_i \otimes 1 - 1 \otimes X_i$, and $\bar{Y}_i = Y_i \otimes 1 - 1 \otimes Y_i$, in the tensor product algebra $H^{n-1}(P\Sigma_n; \mathbb{Q}) \otimes H^{n-1}(P\Sigma_n; \mathbb{Q})$.

Claim: The product of these $2n - 2$ zero divisors

$$M = \prod_{i=1}^{n-1} \bar{X}_i \cdot \prod_{i=1}^{n-1} \bar{Y}_i = \sum_{I \subseteq [1,n-1]} (-1)^{|I|} U_I \otimes V_I \neq 0$$

where $[1, n-1]$ denotes the set of integers $\{1, 2, ..., n-1\}$, $|I|$ is the number of elements in I, $U_I = z_1 \cdots z_{n-1}$, and $V_I = \hat{z}_1 \cdots \hat{z}_{n-1}$ with

$$z_i = \begin{cases} Y_i, & \text{if } i \notin I, \\ X_i, & \text{if } i \in I \end{cases} \quad \text{and } \hat{z}_i = \begin{cases} Y_i, & \text{if } i \in I, \\ X_i, & \text{if } i \notin I. \end{cases}$$

In order to prove this claim, we apply relations (1) to M, and reduce it into the induced basis of the tensor product, and identify at least one monomial that stays unaffected throughout the reduction process.

When I is the empty set $I = \emptyset$, then up to the sign, the summand $U_\emptyset \otimes U_\emptyset$ in the expression (**??**) is

$$Y_{n-1} Y_{n-2} \cdots Y_1 \otimes X_1 X_2 \cdots X_{n-1} = a_{n,n-1} a_{n-1,n-2} \cdots a_{2,1} \otimes a_{1,2} a_{2,3} \cdots a_{n-1,n}.$$

This monomial is already a basis element of $H^{n-1}(P\Sigma_n; \mathbb{Q}) \otimes H^{n-1}(P\Sigma_n; \mathbb{Q})$. We claim that any other U_I, after reducing into the basis, will avoid the specified basis element $U_\emptyset$. Namely, if the monomial U_I is already a basis element, there is nothing to prove. Otherwise, U_I contains a factor $a_{k,j} a_{k,i}$ for at least one k with $1 < k < n$, and these are the only generators in the product, having subindex k. Applying the relation (1) to the product $a_{k,j} a_{k,i}$, we obtain (up to the sign)

$$U_I = (a_{k,j} a_{j,i} + a_{i,j} a_{k,i}) \cdot \{\text{other factors}\} = a_{k,j} P + a_{k,i} Q,$$

where none of the factors of P and Q has subindex k.

Further application of reductive relations to P and Q will result in no further appearance of k in indices. This means that multiplication by the generator with the first subindex k will produce a basis element of $H^{n-1}(P\Sigma_n;\mathbb{Q})$. As a result, such a reducible monomial U_I after reduction in the basis elements has exactly one factor with subindex k, while our fixed monomial $U_\emptyset = a_{n,n-1}\cdots a_{k+1,k}a_{k,k-1}\cdots a_{2,1}$ contains two factors with subindex k. Hence, the basis monomial $U_\emptyset \otimes V_\emptyset$ is different from any other possible basis summand coming from $U_I \otimes V_I$ with $I \neq \emptyset$, and our claim holds.

Estimates of topological complexity

Farber [2003, 2004] developed the estimates for the $\mathrm{TC}(X)$:

Let X and Y be path-connected paracompact locally contractible topological spaces (or in particular, CW-complexes). Then

$$\mathrm{zcl}(X) + 1 \leq \mathrm{TC}(X) \leq 2\,\mathrm{cat}(X) - 1 \leq 2\dim(X) + 1,$$

and

$$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1.$$

Topological complexity of $P\Sigma_n$

Theorem (Cohen-P)

Let $P\Sigma_n$ be the basis-conjugating automorphism group of the free group F_n. Then the topological complexity is $\mathrm{TC}(P\Sigma_n) = 2n - 1$.

Groups $P\Sigma_n^+$

Cohen-Pakianathan-Vershinin-Wu [2008] consideres the "upper triangular" version of $P\Sigma_n$.

Upper triangular McCool group $P\Sigma_n^+$ is a subgroup of $P\Sigma_n$, generated by $\alpha_{i,j}$ with $i < j$.

Theorem (Cohen-P, 2008)

$\mathrm{TC}(P\Sigma_n^+) = \mathrm{zcl}(H^*(P\Sigma_n^+; \mathbb{Q})) + 1 = 2n - 2.$

Formality of $P\Sigma_n^+$

Theorem

Let G be the upper triangular McCool group $P\Sigma_n^+$. Then the Eilenberg-Mac Lane space of type $K(G,1)$ is a formal space.