Generating sets, discriminantal bundles, and arrangement groups

Joint work with Daniel C. Cohen and Michael Falk

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Questions: Are arrangement groups linear? Are arrangement groups torsion-free?

The answers are "yes" for fiber-type (=supersolvable) arrangements.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement of complex hyperplanes, let M be the complement in $\mathbb{C}^\ell$ and let $G = \pi_1(M)$. Let X be a rank 2 lattice element, and as usual let $\mathcal{A}_X = \{H \in \mathcal{A} \,|X \subset H\}$, M_X and G_X the associated complement and group.

Note that $M \subset M_X$, and so inclusion induces a homomorphism $G \to G_X$. Our main object of study is the product homomorphism

$$\varphi : G \to \prod_{rkX=2} G_X$$

Now φ might be called the "Brieskorn" homomorphism: Brieskorn showed that the analogous homomorphism on second homology is an isomorphism. Clearly φ injects on first homology. Since second homology groups of arrangements are isomorphic to the second homology of the fundamental group, φ is also an isomorphism on the second homology of the groups. One reason for looking at rank two lattice elements is that these capture the relations in the fundamental group.

From the point of view of the two original questions, note that each G_X is the product of the integers with a free group, and that G_X embeds into a pure braid group. Thus if φ is injective in general, then arrangement groups would indeed be torsion-free and linear.

The observation that in fact M_X embeds into the complement of the pure braid space then perhaps raises a general question: When can one map an arrangement complement to M_{PB_μ}? Now the pure braid arrangement is defined by $z_i \neq z_j$, so for

$$f : M \to M_{PB_\mu}$$

one needs $f = (f_1, \cdots f_\mu)$ with $f_i \neq f_j$ on M. This is the notion of a *generating set.*

Definition 1 $\{f_1, \cdots f_\mu\}$ *is called a* generating set *for* $\mathcal{A}$ *provided that each* f_i *is holomorphic on* M*, and for all* $i \neq j$*,* $f_i - f_j \neq 0$ *on* M*. (if each* f_i *is polynomial, then the last condition is simply that* $f_i - f_j$ *divides the defining polynomial* Q *of the arrangement.)*

In this case, the map $f : M \to M_{PB_\mu}$ allows one to pull back the Fadell-Neuwirth bundle $M_{PB_{\mu+k}} \to M_{PB_\mu}$ with fiber the affine discriminantal arrangement $F_{k,\mu}$. Thus one gets discriminantal bundles

$$F_{k,\mu} \to E \to M$$

over M with structure group the pure braid group PB_μ and monodromy actions of $\pi_1(M)$. If the f_i happen to be linear, then E is actually an arrangement complement. Here are some interesting non-linear examples:

Example 2 *Let $\mathcal{A}$ be the D_ℓ arrangement, with defining polynomial $Q = \pi(z_i^2 - z_j^2)$, and let $f_i = z_i^2$.*

Example 3 *Let $\mathcal{A}$ be the B_ℓ arrangement, with defining polynomial $Q = z_1 \cdots z_\ell \pi(z_i^2 - z_j^2)$, and let $f_i = \frac{1}{z_i}$*

.Now let's get back to our main question, the injectivity of the map φ. In fact (see M. Falk's talk) the image of φ is normal, so there is an exact sequence

$$1 \to \mathrm{ker}(\varphi) \to G \to \prod_{rkX=2} G_X \to co\,\mathrm{ker}(\varphi) \to 1$$

Example 4 *Here is an example with $ker(\varphi)$ non-trivial. Consider the pure braid arrangement in four variables,* *so that the fundamental group is the pure braid group on four strands. One may choose generators* $A_{i,j}$ *,* $1 \leq i < j \leq 4$ *as in Burau's presentation (see e.g. Birman's book). Then* $A_{1,4}$*,* $A_{2,4}$*, and* $A_{3,4}$ *generate a free subgroup. (As may be seen by the Fadell-Neuwirth bundle* $M_{PB_4} \to M_{PB_3}$*, which has fiber a thrice-punctured copy of the complex numbers.). Let*

$$g = [A_{14}, [A_{24}, A_{34}]]$$

Then $g \neq 1$*, as a reduced word in a free group, but* $g \in \ker(\varphi)$*, since in any coordinate function of* φ *one of the generators involved in* g *goes to* 1*.*

An iterated product of commutators such as g is called a *monic commutator* by T. Stanford:

Definition 5 *Suppose* G *is a finitely generated group, with generating set* Y*. A monic commutator in* Y *is defined recursively by*

(i) 1 is a monic commutator

(ii) Each element of Y and its inverse is a monic commutator

(iii) if a and b are monic commutators, then $[a, b] = aba^{-1}b^{-1}$ is a monic commutator

The *support* of a monic commutator is the set of generators which appears.

Stanford considered this idea in the study of Brunnian braids, those which become trivial when some subset of the strings is deleted.

Theorem 6 *The kernel of* $\varphi : G \to \prod_{rkX=2} G_X$ *is the subgroup generated by monic commutators whose support non-trivially intersects* $\mathcal{A} - \mathcal{A}_X$ *for every rank two lattice element* X.

Proof. (Sketch, after Stanford). Any element of G can be written as a product of monic commutators with "increasing" support from left to right. Then use induction and the fact that each inclusion-induced homomorphism is a retraction homomorphism. ■

More generally, the result is true for any collection of lattice elements. For rank 1, φ amounts to just abelianization. Even if one takes all non-maximal elements of the lattice, however, the example of the pure braid arrangement in four variables still shows that φ has a non-trivial kernel.