Jacobian ideals of hyperplane arrangements

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Supported by NSF # 0600893 and the NSF Japan program

Baton Rouge, Louisiana

March 29, 2008

Outline

- Lines and points in the plane
- Jacobian ideals of hyperplane arrangements
- Jacobian ideals of subspace arrangements
- Hilbert schemes

- Setting: the real plane, $\mathbb{R}^2$.

- Characters: (1) a collection of lines in $\mathbb{R}^2$ say $\mathcal{A} = \{H_1, \dots, H_n\}$ such that no two lines are parallel and (2) their intersection points $L(\mathcal{A})$.

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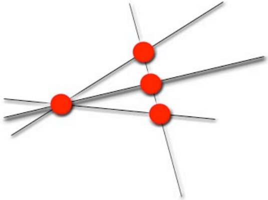

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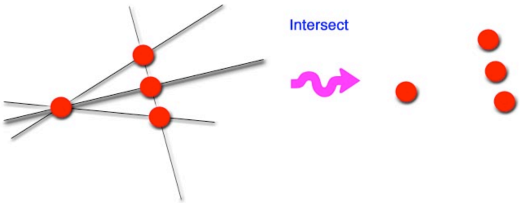

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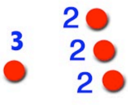

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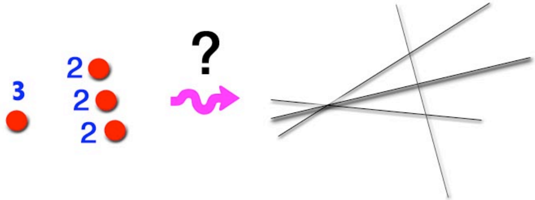

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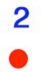

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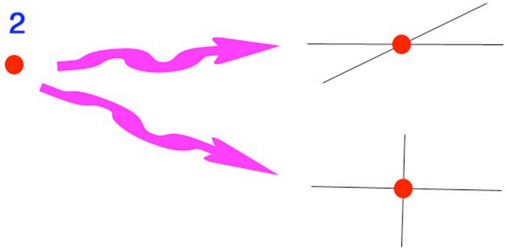

Yes! If there is more than one intersection point.

Solution

- Let $\mu : L(\mathcal{A}) \to \mathbb{Z}$ be defined by
$$\mu(p) = |\{\text{lines passing through } p\}| - 1$$
- Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$.
- Let $\mu_{\mathcal{A}} : \mathcal{L} \to \mathbb{Z}$ be defined by
$$\mu_{\mathcal{A}}(H) = \sum_{p \in H} \mu(p)$$
- $$\mathcal{A} = \{H \in \mathcal{L} | \mu_{\mathcal{A}}(H) = \max(\mu_{\mathcal{A}})\}$$

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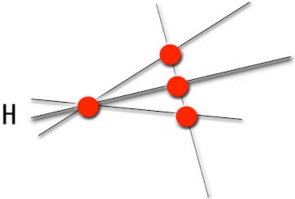

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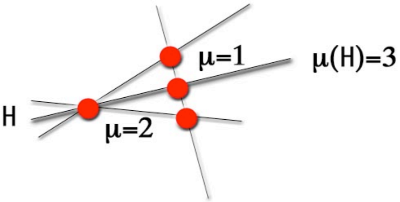

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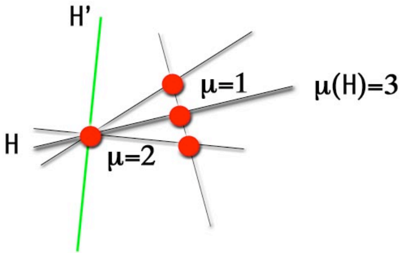

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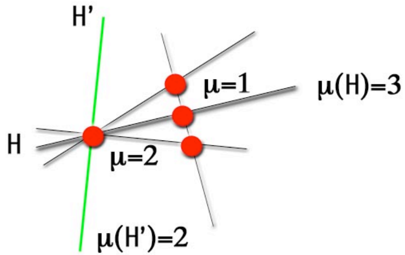

If you are bored...

Setting: $\mathbb{R}^3$

Characters: lines in $\mathbb{R}^3$ such that each line intersects with at least two other lines

Question: Is the same true here?

Jacobian ideals

Setting: $V \cong \mathbb{C}^\ell$

Characters:

- an arrangement of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_n\}$
- it's intersection lattice $L(\mathcal{A})$
- the Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$
- the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_\ell] \cong \mathcal{S}(V^*)$
- the Jacobian ideal of $\mathcal{A}$:

$$J(\mathcal{A}) = (\partial Q/\partial_{x_1}, \ldots, \partial Q/\partial_{x_1})$$

where Q is the product of the linear forms defining H_i

Why $J(\mathcal{A})$?

- Zero locus is the singular locus
  Thus, $\mathrm{codim}(J(\mathcal{A})) = 2$
- The module of logarithmic vector fields is a module over the polynomial ring S given by

$$D(\mathcal{A}) = \left\{ \theta \in \bigoplus_{i=1}^{\ell} S \frac{\partial}{\partial_{x_i}} \;:\; \theta(Q) \in QS \right\}$$

There is an exact sequence:

$$0 \to D(\mathcal{A}) \to S^{\ell+1} \to S \to S/J(\mathcal{A}) \to 0$$

Theorem (Terao, 1981)

$D(\mathcal{A})$ is a free S-module if and only if $S/J(\mathcal{A})$ is a Cohen-Macaulay ring.

Related results

Theorem (Dolgachev and Kapranov, 1993)

Let $\mathcal{A}$ be a generic arrangement. Then we can reconstruct $\mathcal{A}$ from $D(\mathcal{A})$.

Used the sheafification $\widetilde{D(\mathcal{A})}$ and its set of jumping lines to reconstruct $\mathcal{A}$.
Generalized to a larger class of arrangements by Dolgachev in 2007.

Theorem (Donagi, 1983)

Let f be a homogeneous polynomial. Then f can be recovered from $J(f)$ up to a projective linear transformation.

Main result

Since $J(\mathcal{A})$ is a homogeneous ideal we have an associated projective scheme

$$\mathrm{Proj} S/J(\mathcal{A})$$

Theorem

Suppose that $\mathcal{A}$ is a central and essential arrangement in dimension $\ell \geq 3$. Then we can reconstruct $\mathcal{A}$ from the scheme $\mathrm{Proj} S/J(\mathcal{A})$.

Idea of proof: (very elementary)

- Intersect $\mathrm{Proj} S/J(\mathcal{A})$ with arbitrary hyperplanes
- Calculate degree of this intersection
- Show that this degree is given by summing up the Möbius function along the hyperplane

General Jacobian ideals

- Assume $I = (f_1, \ldots, f_s)$ is a pure c codimensional radical ideal in the polynomial ring S
- let $J(I)$ be the ideal generated by all $c \times c$ minors of the Jacobian matrix
$$\left(\frac{\partial f_i}{\partial x_j}\right)$$
- Then the singular locus of I is the zero locus of $J(I)$

In order to write down generators, one would like to 'iterate' Jacobian ideals.

Unfortunately, $J(\mathcal{A})$ is far from being radical!

Fact: there exists an $\mathcal{A}$ such that $J(\mathcal{A})$ has no embedded associated primes and $\operatorname{pdim}(S/J(\mathcal{A})) > 2$.

Generic arrangements

Theorem

If $\mathcal{A}$ is a generic arrangement then $\mathrm{Sat}(J(\mathcal{A}))$ is radical and of pure codimension 2.

Theorem

If $\mathcal{A}$ is a generic essential arrangement then $\mathrm{Sat}(J(\mathcal{A}))$ can be reconstructed from $J(\mathrm{Sat}(J(\mathcal{A})))$.

Idea of proof: Basically the same.

What about general subspaces?

Computing degree's of certain schemes is difficult. Plus.......

Example

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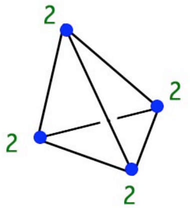

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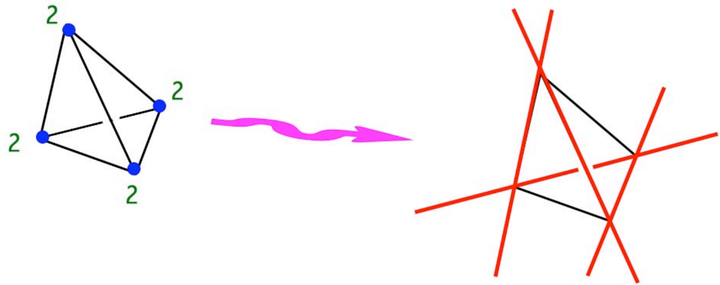

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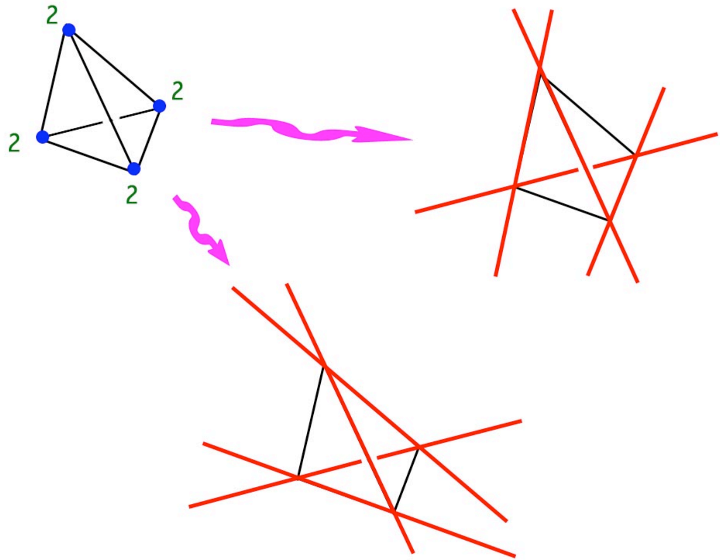

Corollary's?

- Let $\mathcal{H}(\mathbb{CP}^{\ell-1}, k)$ be the Hilbert scheme of zero dimensional schemes in $\mathbb{CP}^{\ell-1}$ of degree k
- Let $\mathcal{M}(\ell, k)$ be the moduli space of all essential and central arrangements $\mathcal{A}$ in dimension ℓ such that

$$\deg J(\mathcal{A}) = \sum_{X \in L(\mathcal{A})_2} \mu(X)^2 = k$$

Corollary

The map given by taking the Jacobian

$$\mathcal{M}(3, k) \to \mathcal{H}(\mathbb{CP}^2, k)$$

is an injection.

Corollary's

- Let $\mathcal{GM}(4,k)$ be the moduli space of all essential, central, generic arrangements $\mathcal{A}$ in dimension 4 such that
$$\deg J(\mathcal{A}) = \binom{n}{2} = k$$
here $n = |\mathcal{A}|$

Corollary

The map given by taking the Jacobian of the saturation of the Jacobian
$$\mathcal{GM}(4,k) \to \mathcal{H}(\mathbb{CP}^3, k)$$
is an injection.

Wakefield, M., and Yoshinaga, M. The Jacobian ideal of a hyperplane arrangement. to appear in Math. Res. Lett., arXiv:0707.2672.

THANK YOU!!