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Generic section of a hyperplane arrangement and twisted Hurewicz maps

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Notation

A *hyperplane arrangement* is a collection

$$\mathcal{A} = \{H_1, H_2, \dots, H_n\}$$

of affine hyperplanes $H_i \subset \mathbb{C}^\ell$. And denote

$$M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H.$$

0 Contents

1 Randell's result

Thm. (Randell) If $k \geq 2$, the Hurewicz map

$$h : \pi_k(M(\mathcal{A}), x_0) \longrightarrow H_k(M(\mathcal{A}), \mathbb{Z})$$

is the zero map.

(Proof) Let $f : S^k \to M(\mathcal{A})$. Consider

$$\begin{array}{ccc} H^k(S^k) & \xleftarrow{f^*} & H^k(M(\mathcal{A})) \\ \uparrow & & \uparrow \text{surj.} \\ \overset{k}{\wedge} H^1(S^k) = 0 & \xleftarrow{f^*} & \overset{k}{\wedge} H^1(M(\mathcal{A})). \end{array}$$

1 Randell's result

$$\begin{tikzcd} \pi_k(M) \arrow[r, bend left, "h=0"] \arrow[r, bend right] & H_k(M,\mathcal{L}) \end{tikzcd}$$

Goal:

Twisted version detects!
(Sometimes)

2 Twisted Hurewicz maps

Generalities:

Let $\mathcal{L}$ be a local system on X,

C be a closed manifold of $\dim_{\mathbb{R}} C = k$,

$$f : (C, *) \to (X, x_0)$$

a continuous map. The map f and a section

$$t \in \Gamma(C, f^*\mathcal{L})$$

determines a twisted cycle $[f] \otimes t \in H_k(X, \mathcal{L})$.

2 Twisted Hurewicz maps

Let $\mathcal{L}$ be a rank one local system on $M(\mathcal{A})$,

$$f : (S^k, *) \longrightarrow (M(\mathcal{A}), x_0)$$

a continuous map. $k \geq 2$. Since S^k is simply connected, $f^*\mathcal{L}$ on S^k is trivial and hence

$$\Gamma(S^k, f^*\mathcal{L}) \cong \mathcal{L}_{x_0}.$$

2 Twisted Hurewicz maps

We have

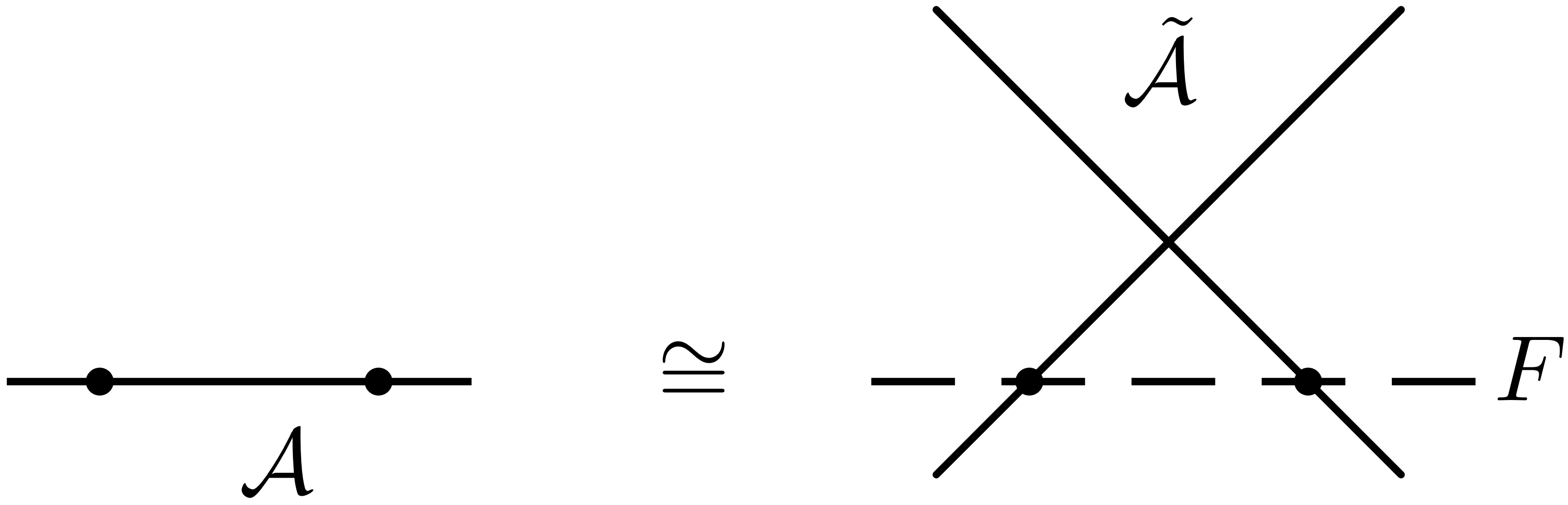
$$h : \pi_k(S^k, *) \otimes \mathcal{L}_{x_0} \longrightarrow H_k(M(\mathcal{A}), \mathcal{L})$$

the *twisted Hurewicz map*. (Note that it is defined only when $k \geq 2$.)

e.g. If $\mathcal{L}$ is a trivial local system, then h is the classical one.

3 Main result

<u>Def.</u> An arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is called *generic-section type* if there is another arrangement $\tilde{\mathcal{A}}$ of rank $(\ell+1)$ in $\mathbb{C}^{\ell+1}$ and a generic hyperplane $F \subset \mathbb{C}^{\ell+1}$ such that $\mathcal{A}$ is isomorphic to $F \cap \tilde{\mathcal{A}}$.



3 Main result

Thm. Assume $\ell \geq 2$. If $\mathcal{A}$ is generic-section type and $\mathcal{L}$ is nonresonant, then the top twisted Hurewicz map

$$h : \pi_\ell(M(\mathcal{A}), x_0) \otimes \mathcal{L}_{x_0} \longrightarrow H_\ell(M(\mathcal{A}), \mathcal{L})$$

is surjective.

Note: $H_\ell(M(\mathcal{A}), \mathcal{L}) \cong \mathbb{C}^{|\chi(M)|}$. Hence $\pi_\ell(M) \neq 0$ (Randell).

4 Proof

Proof is based on two results:

– Lefschetz Theorem on hyperplane section, (or minimality of $M(\mathcal{A})$).
– Nonresonance theorem for local system homology groups.

4 Proof

$\tilde{M} = M(\tilde{\mathcal{A}})$,

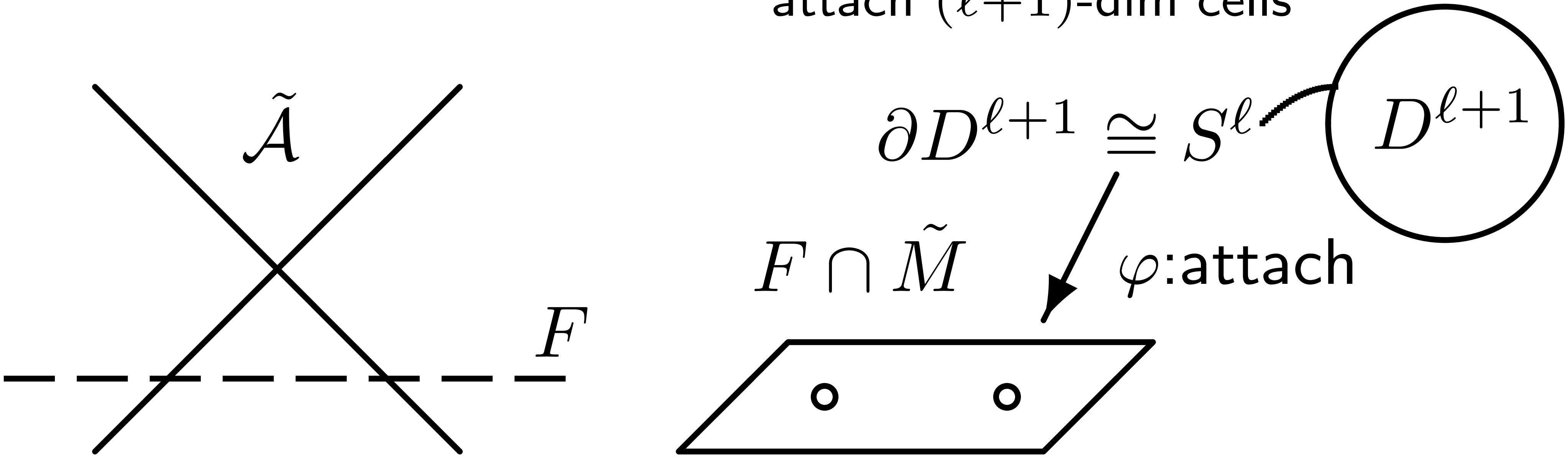
$F \subset \mathbb{C}^{\ell+1}$: a generic. hyperplane.

Thm. (Lefschetz)

$$\tilde{M} \simeq (\tilde{M} \cap F) \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1}}_{\text{attach } (\ell+1)\text{-dim cells}}$$

4 Proof

$$\tilde{M} \simeq (\tilde{M} \cap F) \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1}}_{\text{attach } (\ell+1)\text{-dim cells}}$$



Hyperplane section

4 Proof

$$\tilde{M} \simeq (\tilde{M} \cap F) \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1}}_{\text{attach } (\ell+1)\text{-dim cells}}$$

How many $(\ell + 1)$-dim cells to attach?
Minimality (Dimca-Papadima-Randell-Suciu)

$$\Longrightarrow b = b_{\ell+1}(\tilde{M}).$$

4 Proof

$\tilde{M} = M(\tilde{\mathcal{A}})$ and $M = M(\mathcal{A}) = \tilde{M} \cap F$.

$$\tilde{M} \simeq M \quad \underbrace{\cup_{\varphi} \bigcup_{i=1}^{b_{\ell+1}} D^{\ell+1}}_{\text{attach } (\ell+1)\text{-dim cells}}$$

Associated twisted chain complexes:

$$C_{\bullet}(\tilde{M}) = C_{\bullet}(M) \oplus \mathbb{C}^{b_{\ell+1}}.$$

4 Proof

$$\begin{array}{ccccccccc} C_{\ell+1}(\tilde{M}) & \stackrel{\partial_{\mathcal{L}}}{\longrightarrow} & C_\ell(\tilde{M}) & \longrightarrow & \cdots & \longrightarrow & C_0(\tilde{M}) \\ & & \| & & & & \| \\ 0 & \longrightarrow & C_\ell(M) & \longrightarrow & \cdots & \longrightarrow & C_0(M) \end{array}$$

Nonresonance Theorem:

Suppose $\mathcal{L}$ is a generic local system. Then only $H_{\ell+1}(C_\bullet(\tilde{M}))$ and $H_\ell(C_\bullet(M))$ survive.

4 Proof

$$\begin{tikzcd} C_{\ell+1}(\tilde{M}) \arrow[r, "\partial_{\mathcal{L}}"] & C_\ell(\tilde{M}) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & C_0(\tilde{M}) \arrow[d, equal] \\ 0 \arrow[r] & C_\ell(M) \arrow[r] & \cdots \arrow[r] & C_0(M) \end{tikzcd}$$

Only $H_{\ell+1}(C_\bullet(\tilde{M}))$ and $H_\ell(C_\bullet(M))$ survive.

Observation 1:

$$\partial_{\mathcal{L}} : C_{\ell+1}(\tilde{M}) \to H_\ell(M, \mathcal{L}) \text{ is surjective.}$$

4 Proof

Recall the decomposition

$$\tilde{M} \simeq M \cup_{\varphi} \bigcup_{i=1}^{b_{\ell+1}} D^{\ell+1}$$

is defined by attaching maps

$$\varphi_i : \partial(D^{\ell+1}) = S^{\ell} \longrightarrow M.$$

4 Proof

Observation 2: The twisted boundary map splits

$$\begin{array}{ccc} C_{\ell+1}(\tilde{M}) & \xrightarrow{\partial_{\mathcal{L}}} & C_\ell(M) \\ \downarrow & \searrow & \cup \\ \pi_\ell(M) \otimes \mathcal{L}_{x_0} & \xrightarrow{h} & H_\ell(M, \mathcal{L}) \end{array}$$

to the twisted Hurewicz map h. Thus

$$h : \pi_\ell(M) \otimes \mathcal{L}_{x_0} \longrightarrow H_\ell(M, \mathcal{L})$$

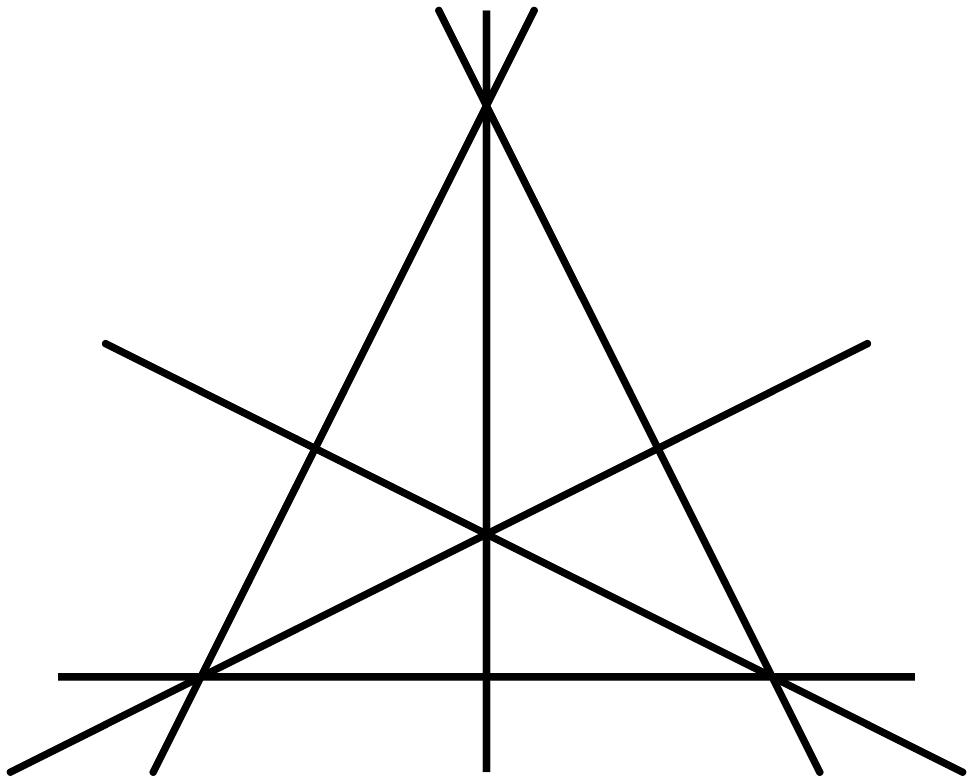
is surjective. $\square$

5 Corollary

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ of generic section type and $\mathcal{L}$ be a nonresonant rank one local system. Then every twisted ℓ-cycle is represented by an ℓ-dimensional sphere.

5 Corollary

(Falk) Let $\mathcal{A}$ be a line arrangement in $\mathbb{C}^2$ with $|\mathcal{A}| \geq 3$. Suppose no two lines are parallel. Then $\pi_2(M(\mathcal{A})) \neq 0$.



5 Corollary

(Falk) Let $\mathcal{A}$ be a line arrangement in $\mathbb{C}^2$.
Let F be a generic line.
Then $\mathcal{A} \cup \{F\}$ is not $K(\pi, 1)$.

6 Reference

M. Falk, $K(\pi, 1)$ arrangements. *Topology*, **34** (1995) 141–154.

R. Randell, Homotopy and group cohomology of arrangements. *Topology and its applications*, **78** (1997) 201–213.

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