Periodicity of hyperplane arrangements with integral coefficients modulo positive integers

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References:

- **Periodicity of hyperplane arrangements with integral coefficients modulo positive integers.** *Journal of Algebraic Combinatorics*, **doi:10.1007/s10801-007-0091-2. H.Kamiya, A.Takemura and H.Terao. 2007.**

- **Periodicity of non-central integral arrangements modulo positive integers. H.Kamiya, A.Takemura and H.Terao.** `arXiv:0803.2755v1`, **2008.**

Introduction

- Consider an arrangement $\mathcal{A}$ of hyperplanes defined by linear forms with integral coefficients.

$$\mathcal{A} = \{H_1, \ldots, H_n\},\ H_j : c_{1j}x_1 + \cdots + c_{mj}x_m = 0,$$
$$c_{ij} \in \mathbb{Z}, \qquad (m : \text{dimension},\ \text{central case})$$

- **Finite field method**: consider $\mathcal{A}$ in $\mathbb{F}_q^m$, where q is a large prime. Write $\mathcal{A}_q$.
  - Complement of $\mathcal{A}_q$: $M(\mathcal{A}_q) = \mathbb{F}_q^m \setminus \cup_i H_i$

– For sufficiently large q, the characteristic polynomial of $\mathcal{A}$ coincides with the cardinality of $M(\mathcal{A}_q)$.

$$\chi(\mathcal{A}, q) = |M(\mathcal{A}_q)|$$

– For some problems, this relation is useful, because we can count $|M(\mathcal{A}_q)|$ by computer. This is "brute force", but numerical results may suggest theoretical results.

- **Question**: the characteristic polynomial $\chi(A, t)$ can be evaluated at a non-prime q. We can also define arrangement of "hyperplanes" in $\mathbb{Z}_q^m$, $\mathbb{Z}_q = \mathbb{Z}/(q\mathbb{Z})$, by

$$H_{j,q} : c_{1j}x_1 + \cdots + c_{mj}x_m \equiv 0 \pmod{q}$$

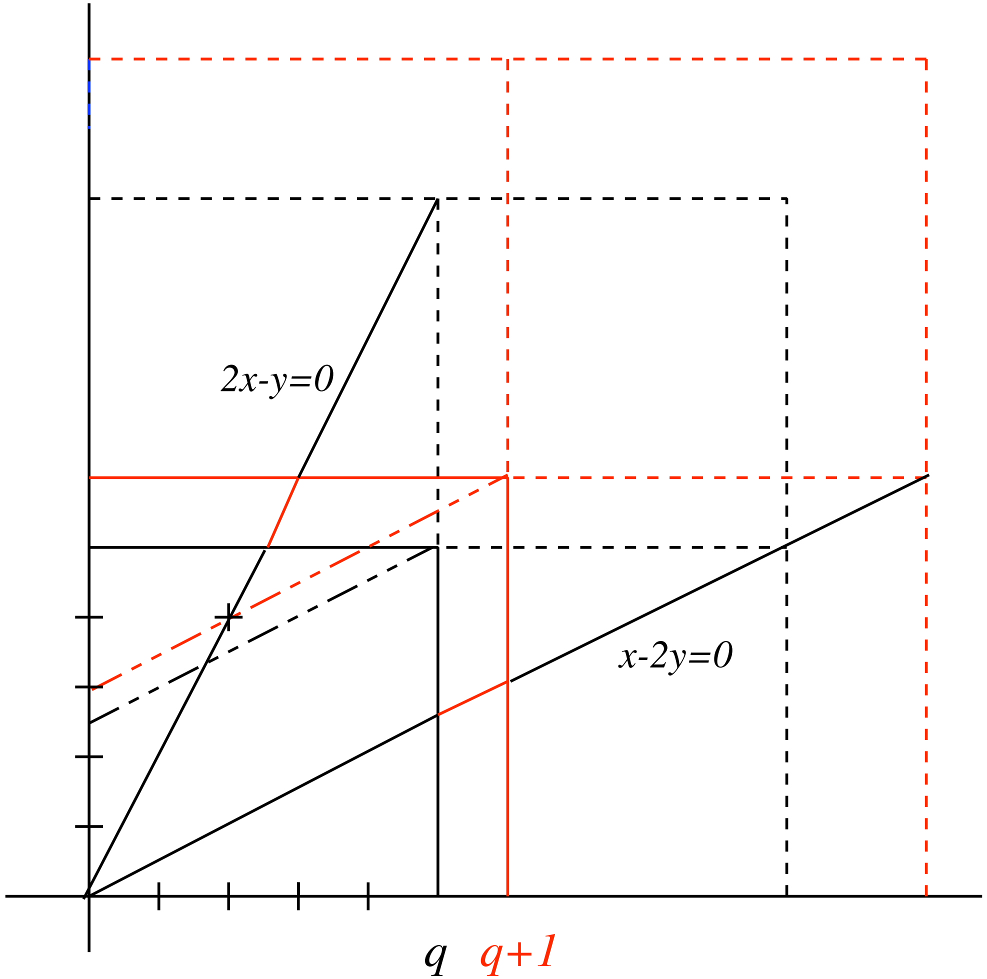
and count the number of points in the complement of $\mathcal{A}_q$. Are they the same?

$\Rightarrow$ generally NO!

- However $|M(\mathcal{A}_q)|$ is a quasi-polynomial (i.e., coefficients are periodic) in $q \in \mathbb{Z}_{>0}$.

$\Rightarrow$ "characteristic quasi-polynomial"

- **Intuitively, the hyperplanes have more chances to meet at integer points, if q has many divisors.**



- **NOTE: for a non-prime q, the set**

$$H : c_1x_1 + \cdots + c_mx_m \equiv 0 \pmod{q}$$

depends on the choice of normalization of the coefficient vector.

- H **defined in terms of** $c \times (c_1, \ldots, c_m)$ **is generally different if** $\gcd(c, q) > 1$**. (Even for prime** q, H **obviously depends on** $q|c$ **or not.)**

- **When q is not a prime, $\mathbb{Z}_q^m$ is not a vector space. In this case it may not be appropriate to call H a hyperplane. However abusing the terminology we still call H a "hyperplane".**

Results on characteristic quasi-polynomials

- **Coefficient matrix $C = (c_{ij}) : m \times n$. Each column determines a hyperplane.**

- **Let $J \subseteq \{1, \dots, n\}$ be a subset of hyperplanes and let C_J denote the submatrix of C consisting of columns $j \in J$.**

- **Let $e(J)$ denote the largest elementary divisor of C_J.**

- **Let**

$$\rho_0 = \mathrm{lcm}\{e(J) \mid J \subseteq \{1, \ldots, n\}, J \neq \emptyset\}.$$

Theorem 1 The function $|M(\mathcal{A}_q)|$ is a monic quasi-polynomial in $q \in \mathbb{Z}_{>0}$ of degree m with a period ρ_0. Furthermore the coefficients of the (constituents of the) quasi-polynomial depend only on $\gcd\{\rho_0, q\}$.

An example

- Let
$$C = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

- Corresponding hyperplanes in $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ is $\mathcal{A} = \{H_1, H_2, H_3\}$:
$$H_1 : x - y = 0, \quad H_2 : x + y = 0, \quad H_3 : -2x + y = 0.$$

- $\rho_0 = 6$.

- **Characteristic quasi-polynomial:**

$$|M(\mathcal{A}_q)| = \begin{cases} q^2 - 3q + 2 & \text{when } \gcd\{6, q\} = 1, \\ q^2 - 3q + 3 & \text{when } \gcd\{6, q\} = 2, \\ q^2 - 3q + 4 & \text{when } \gcd\{6, q\} = 3, \\ q^2 - 3q + 5 & \text{when } \gcd\{6, q\} = 6. \end{cases}$$

- **Relation to the characteristic polynomial (already stated by Athanasiadis).**

  Theorem 2 Let ρ be a period of the quasi-polynomial $|M(\mathcal{A}_q)|$ and q be a positive integer relatively prime to ρ. Then $|M(\mathcal{A}_q)| = \chi(\mathcal{A}, q)$.

- **This theorem shows that we can apply the "finite field method" with a composite q relatively prime to ρ for obtaining the characteristic polynomial of $\mathcal{A}$.**

Periodicity of intersection posets

- **The intersection posets of $\mathcal{A}_q$ are also periodic.**
- **Periodicity of $|M(\mathcal{A}_q)|$ and that of the intersection poset are not equivalent.**
- **Our example:**
  $H_1 : x - y = 0, H_2 : x + y = 0, H_3 : -2x + y = 0.$

- **"Hyperplanes"** (all modulo q)

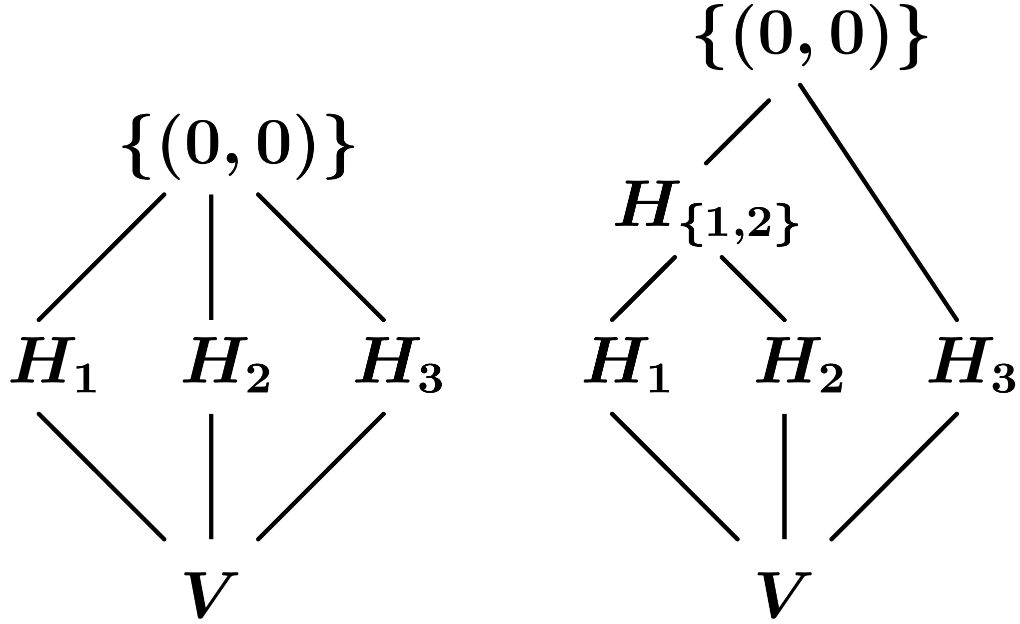
$$H_{1,q} = \{(0,0),(1,1),\dots,(q-1,q-1)\}$$
$$H_{2,q} = \{(0,0),(1,q-1),\dots,(q-1,1)\}$$
$$H_{3,q} = \{(0,0),(1,2),(2,4),\dots,(q-1,q-2)\}$$

- **Intersections for $q \geq 4$,**

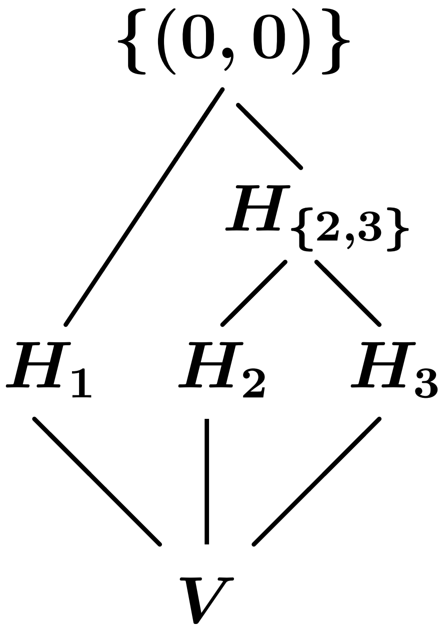
$$
\begin{aligned}
H_{\{1,2\},q} &= \begin{cases} \{(0,0)\}, & q : \text{odd}, \\ \{(0,0), (\frac{q}{2}, \frac{q}{2})\}, & q : \text{even}, \end{cases} \\
H_{\{2,3\},q} &= \begin{cases} \{(0,0)\}, & 3 \not| \, q, \\ \{(0,0), (\frac{q}{3}, \frac{2q}{3}), (\frac{2q}{3}, \frac{q}{3})\}, & 3 \mid q, \end{cases} \\
H_{\{1,3\},q} &= H_{\{1,2,3\},q} = \{(0,0)\}
\end{aligned}
$$

(all modulo q)

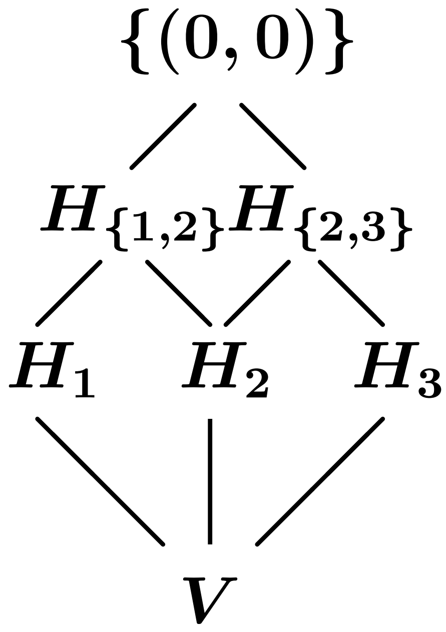
- **Hasse diagrams of intersection lattices for $q \geq 4$**



$\gcd\{6, q\} = 1$ $\qquad$ $\gcd\{6, q\} = 2$



$\gcd\{6, q\} = 3$



$\gcd\{6, q\} = 6$

Theorem 3 **The intersection lattices $L(\mathcal{A}_q)$ are periodic for all sufficiently large q with a period ρ_0.**

- **NOTE: $|M(\mathcal{A}_q)|$ is periodic for all $q > 0$. On the other hand $L(\mathcal{A}_q)$ are periodic from some q on.**

Proof via elementary divisor theory (sketch)

- Let $V = \mathbb{Z}_q^m$.
- Let $I_Y(\cdot)$, $Y \subseteq V$: the characteristic function (indicator function) of Y : $I_Y(x) = 1$, $x \in Y$ and $I_Y(x) = 0$, $x \in V \setminus Y$.

- **For every $x \in V$,**

$$\begin{aligned}\prod_{j=1}^{n}\big(1 - I_{H_{j,q}}(x)\big) &= \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} I_{H_{J,q}}(x) \\ &= I_V(x) + \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|} I_{H_{J,q}}(x),\end{aligned}$$

where

$$H_{J,q} = \cap_{j \in J} H_{j,q} = \{x \in \mathbb{Z}_q^m \mid xC_J = 0\}.$$

- **From the relation**

$$x \in M(\mathcal{A}_q) \Leftrightarrow 1 = \prod_{j=1}^{n}(1 - I_{H_{j,q}}(x)),$$

we have

$$\begin{aligned} |M(\mathcal{A}_q)| &= \sum_{x \in V} \prod_{j=1}^{n} \left(1 - I_{H_{j,q}}(x)\right) \\ &= q^m + \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \left|H_{J,q}\right|, \end{aligned}$$

- **It suffices to verify that for each $J \neq \emptyset$ the cardinality of $H_{J,q}$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$.**

- **The Smith normal form and elementary divisors.**
  - **Let**

$$\begin{aligned} \boldsymbol{S C_J T} &= \begin{pmatrix} \boldsymbol{E_J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathrm{Mat}_{m\times |J|}(\mathbb{Z}), \\ \boldsymbol{E_J} &= \mathrm{diag}(\boldsymbol{e_1}, \ldots, \boldsymbol{e_{\ell(J)}}), \quad \boldsymbol{\ell(J)} = \mathrm{rank}\boldsymbol{C_J}, \\ & \qquad \boldsymbol{e_1}, \ldots, \boldsymbol{e_{\ell(J)}} \in \mathbb{Z}_{>0}, \quad \boldsymbol{e_1}|\boldsymbol{e_2}|\cdots|\boldsymbol{e_{\ell(J)}}. \end{aligned}$$

  - $\boldsymbol{S} \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$ **and** $\boldsymbol{T} \in \mathrm{Mat}_{|J|\times |J|}(\mathbb{Z})$ **are unimodular matrices.**

- **Take the q-reduction of the Smith normal form:**

$$[S]_q[C_J]_q[T]_q = \mathrm{diag}([e_1]_q, \ldots, [e_{\ell(J)}]_q, 0, \ldots, 0).$$

- **$[S]_q$ and $[T]_q$ remain unimodular.**
- **The cardinality of the kernel**

$$H_{J,q} = \{x \in \mathbb{Z}_q^m \mid xC_J = 0\}$$

is described by the behavior of the q-reduction of the elementary divisors $[e_1]_q, \ldots, [e_{\ell(J)}]_q$ for each J.

- $[e_1]_q, \ldots, [e_{\ell(J)}]_q$ **are periodic in** q **for each** J.
- **As a period we can use**

$$\rho_0 = \mathrm{lcm}\{e_{\ell(J)} \mid J \subseteq \{1, \ldots, n\}, J \neq \emptyset\}.$$

Relation to Ehrhart polynomial theory

- **As suggested in the figure on p.6, $\mathbb{Z}_q^m$ are cut into chambers by the hyperplanes.**
- **We can apply the Ehrhart polynomial theory to each chamber.**
- **We get the periodicity result for $|M(\mathcal{A}_q)|$ by this argument.**

- **However the period guaranteed by the Ehrhart polynomial theory is generally larger than $\boldsymbol{\rho_0}$.**
- **In our problem, the quasi-polynomial depends only on gcd$\{\boldsymbol{\rho_0}, \boldsymbol{q}\}$. This fact can not be proved by the Ehrhart polynomial theory.**

Summary and concluding remarks

- **We have extended the finite field method to non-prime q and defined a characteristic quasi-polynomial.**
- **Properties of the characteristic quasi-polynomial have been derived by the theory of elementary divisors.**

- **We have recently extended our results to non-central cases. In the non-central case, $|M(\mathcal{A}_q)|$ is again a quasi-polynomial with the same period as the central case. However unlike the central case, the periodicity holds from some q on.**

- **ρ_0 may not be the actual minimum period for $|M(\mathcal{A}_q)|$. The question of "period collapse" seems to be a hard question.**