Morita Invariance of Invariant Topological Complexity

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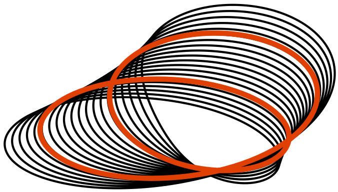
Joint work A. Angel, M. Grant and J. Oprea



Motion Planning Problem (MPP)

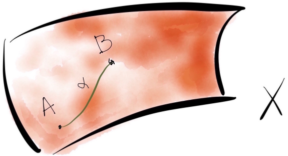
Robot: A mechanical system capable of moving autonomously.

Configuration space: The collection of all possible states of the system.

MPP: Given an initial state A and a final state B, find a path in X that moves the robot from A to B.

Motion Planning Algorithm (MPA)

A MPA is a section of the evaluation map



$$s : X \times X \to PX$$
$$(A, B) \mapsto \alpha_{A,B}$$

Definition

The topological complexity $\mathbf{TC}(X)$ is the least integer k such that $X \times X$ may be covered by k open sets $\{U_1, \ldots, U_k\}$, on each of which there is a continuous section $s_i : U_i \to PX$ such that

$$\mathrm{ev} \circ s_i = i_{U_i} : U_i \hookrightarrow X \times X.$$

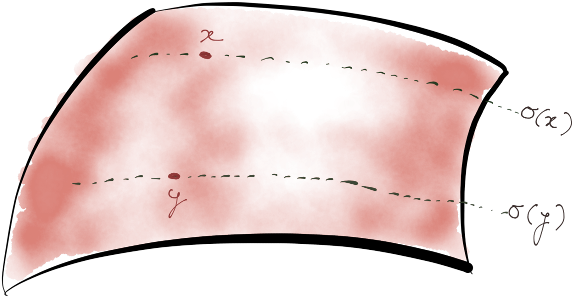
A group G acting on X

Equivariant Motion Planning Problem?

Equivariant MPP

Version 1

Given configurations x and y, find a path α between x and y, such that the path between configurations gx and gy is $g\alpha$.



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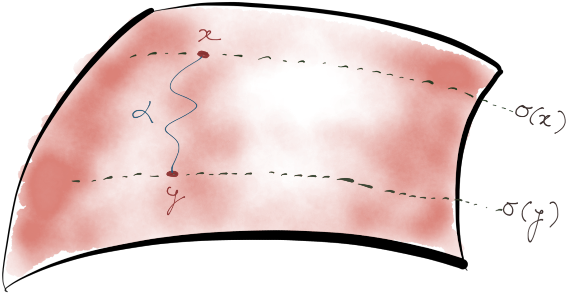

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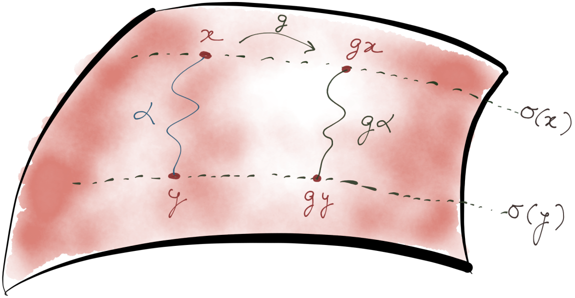

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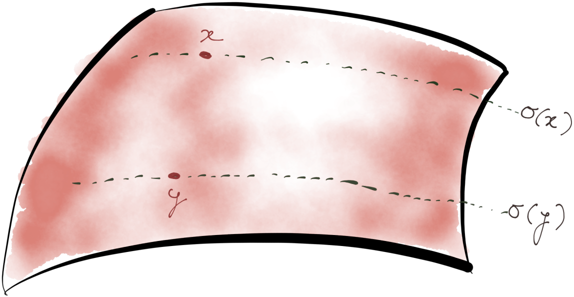
Version 2

Given configurations x and y, find a "broken path" (α, β) between them, such that the path between configurations hx and ky is $(h\alpha, k\beta)$.



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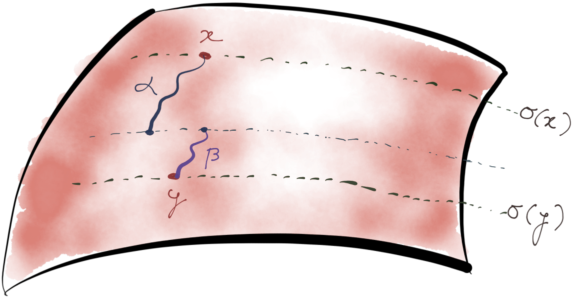

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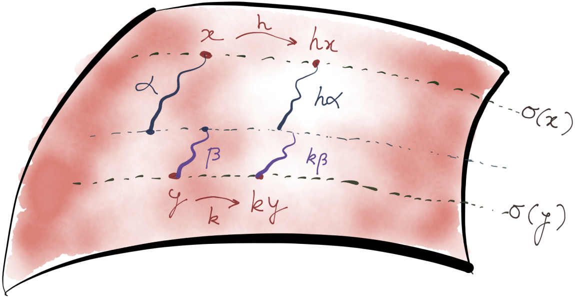
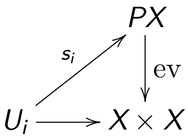


Equivariant TC (Colman-Grant)

$$G \times PX \to PX, \qquad G \times (X \times X) \to X \times X,$$

$$g(\gamma)(t) = g(\gamma(t)), \qquad g(x, y) = (gx, gy).$$

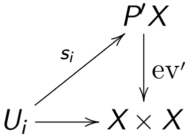
The equivariant topological complexity of X, $\mathrm{TC}_G(X)$, is the least integer k such that $X \times X$ may be covered by k G-invariant open sets $\{U_1, \dots, U_k\}$, on each of which there is a G-equivariant map $s_i : U_i \to PX$ such that the diagram commutes:



Invariant TC (Lubawski-Marzantowicz)

$P'X = PX \times_{X/G} PX = \big\{(\alpha,\beta) \in PX \times PX : G\alpha(1) = G\beta(0)\big\}$
$\mathrm{ev}' \colon P'X \to X \times X$ given by $\mathrm{ev}(\alpha,\beta) = \big(\alpha(0), \beta(1)\big)$ is a $(G \times G)$-fibration.

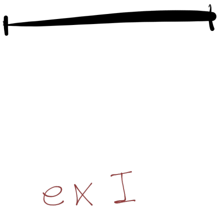
The invariant topological complexity of X, $\mathrm{TC}^G(X)$, is the least integer k such that $X \times X$ may be covered by k $(G \times G)$-invariant open sets $\{U_1, \dots, U_k\}$, on each of which there is a $(G \times G)$-equivariant section $s_i : U_i \to P'X$ such that the diagram commutes:

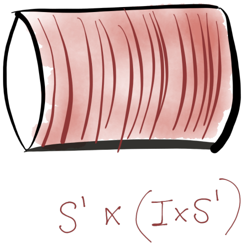


Notion of equivalence for group actions

- Let G be a Lie group acting continuously on a space X, notation $G \ltimes X$
- $\varphi \ltimes \epsilon : G \ltimes X \to K \ltimes Y$ equivariant map if $\epsilon(gx) = \varphi(g)\epsilon(x)$

Equivalent actions





Essential Equivalence $\varphi \ltimes \epsilon : G \ltimes X \to K \ltimes Y$

1. (essentially surjective) $\phi' \circ \pi$ is an open surjection:

$$\begin{array}{ccccc} X \times_Y (K \times Y) & \xrightarrow{\pi} & K \times Y & \xrightarrow{\phi'} & Y \\ \downarrow & & \downarrow{\scriptstyle p_2} & & \\ X & \xrightarrow{\epsilon} & Y & & \end{array}$$

2. (fully faithful) the following diagram is a pullback:

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi \times \epsilon} & K \times Y \\ \downarrow{\scriptstyle (p_2, \phi)} & & \downarrow{\scriptstyle (p_2, \phi')} \\ X \times X & \xrightarrow{\epsilon \times \epsilon} & Y \times Y \end{array}$$

$G \times X = \{((k, y), (x, x')) | y = \epsilon(x), ky = \epsilon(x')\}$.

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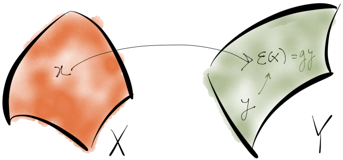

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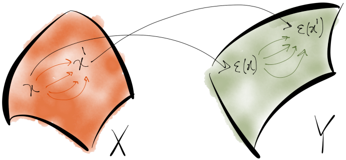

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$$G \times X = \{((k, y), (x, x')) | y = \epsilon(x), ky = \epsilon(x')\}.$$

An ee has to reach to all orbits and there is a bijection induced by φ: $\{g \in G | x' = gx\} \to \{k \in K | \epsilon(x') = k\epsilon(x)\}$.

Morita Equivalence $\sim$

Two actions $G \times X \to X$ and $K \times Y \to Y$ are Morita equivalent if there is a third action $J \times Z \to Z$ and two essential equivalences

$$G \ltimes X \xleftarrow{\psi \ltimes \sigma} J \ltimes Z \xrightarrow{\varphi \ltimes \epsilon} K \ltimes Y.$$

We write $G \ltimes X \sim K \ltimes Y$.

Any notion relevant to the geometric object defined by the action, should be invariant under Morita equivalence.

Examples

1. Let G be a topological group, then

$$e \ltimes X \sim G \ltimes (G \times X)$$

2. If H is a subgroup of G acting on X, then

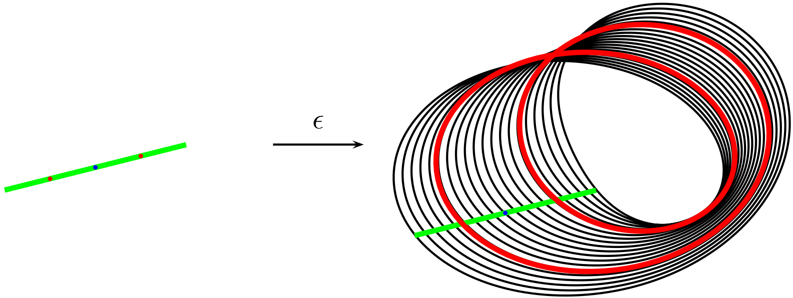
$$H \ltimes X \sim G \ltimes (G \times_H X)$$

where $[gh, x] = [g, hx]$.

(G acting trivially on X and by multiplication on G)

Example $\mathbb{Z}_2 \ltimes I \sim_\epsilon S^1 \ltimes M$

There is an essential equivalence between the mirror action of $\mathbb{Z}_2$ on the interval $I = (-1, 1)$ and the action of S^1 on the Moebius band M.



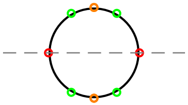
Examples

1. If G acts freely on X, then $G \ltimes X \sim e \ltimes X/G$
2. If $H \trianglelefteq G$ acts freely on X, then $G \ltimes X \sim G/H \ltimes X/H$

Example $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes S^1 \sim \mathbb{Z}_2 \ltimes S^1$

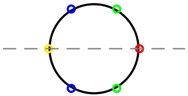
There is an essential equivalence between the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the circle by rotation+reflection and the action of $\mathbb{Z}_2$ on S^1 by just reflection.

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, \rho, \sigma, \rho\sigma\}$$



acting on S^1

$$\mathbb{Z}_2\times\mathbb{Z}_2/_{<\rho>} =< \sigma >= \mathbb{Z}_2$$



acting on $S^1/_{<\rho>} = S^1$

Pronk-Scull characterization

Any essential equivalence is a composite of maps as below:

1. (quotient map) $G \ltimes X \to G/K \ltimes X/K$
   where $K \trianglelefteq G$ and K acts freely on X.

2. (inclusion map) $K \ltimes Z \to H \ltimes (H \times_K Z)$
   where $K \leq H$ acting on Z and $H \times_K Z = H \times Z/\sim$ with $[hk, z] \sim [h, kz]$ for any $k \in K$.

Equivariant LS-category

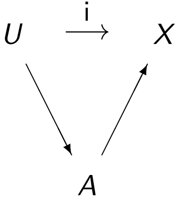
The equivariant category of a G-space X, $\mathrm{cat}_G(X)$, is the least integer k such that X may be covered by k invariant open sets $\{U_1, \ldots, U_k\}$, each of which is G-compressible into a single orbit.

That is, inclusion map $i : U \to X$ is G-homotopic to a G-map $c : U \to X$ with $c(U) \subseteq \mathrm{orb}_G(z)$ for some $z \in X$.

$$\begin{array}{ccc} U & \xrightarrow{\text{i}} & X \\ & \searrow \qquad \nearrow & \\ & \mathrm{orb}_G(z) & \end{array}$$

Equivariant Clapp-Puppe $\mathcal{A}$-category

Let $\mathcal{A}$ be a class of G-invariant subsets of X. The equivariant $\mathcal{A}$-category, ${}_{\mathcal{A}}cat_G(X)$, is the least integer k such that X may be covered by k G-invariant open sets $\{U_1, \ldots, U_k\}$, each G-compressible into some space $A \in \mathcal{A}$.



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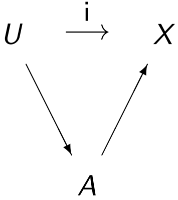


In particular, ${}_{\mathcal{A}}cat_G(X) = \mathrm{cat}_G(X)$ when $\mathcal{A} =$ orbits.

Equivariant TC as $\mathcal{A}$-category

Theorem

For a G-space X, the following statements are equivalent:

1. $\mathrm{TC}_G(X) \leq n$.
2. ${}_{\Delta(X)}\mathrm{cat}_G(X \times X) \leq n$: *there exist G-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *which are G-compressible into* $\Delta(X)$.

$\mathrm{TC}_G(X)$ is NOT invariant under Morita equivalence.

Counterexample: S^1 acting on S^1 by rotation
$\mathrm{TC}_{S^1}(S^1) \geq 2$ and $\mathrm{TC}(*) = 1$

Invariant TC as $\mathcal{A}$-category

Let $\Delta^{G\times G}(X)$ be the saturation of the diagonal $\Delta(X)$ with respect to the $(G \times G)$-action.

Theorem

For a G-space X the following are equivalent:

1. $\mathrm{TC}^G(X) \leq n$.
2. ${}_{\Delta^{G\times G}(X)}\mathrm{cat}_{G\times G}(X \times X) \leq n$*: there exist* $(G \times G)$*-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *which are* $(G \times G)$*-compressible into* $\Delta^{G\times G}(X)$.

Equivariant $\mathcal{A}$-cat is Morita invariant

Theorem (Angel, Colman, Grant, Oprea)

Let G be a compact Lie group acting on a metrizable space X, $H \leq G$ and $K \lhd G$ acting freely on X. If $\mathcal{A}$ is a class of G-invariant subsets of X and $\mathcal{B}$ is a class of H-invariant subsets of X, let $\mathcal{A}/K = \{A/K \mid A \in \mathcal{A}\}$ and $G \times_H \mathcal{B} = \{G \times_H B \mid A \in \mathcal{B}\}$. Then

1. ${}_{\mathcal{A}}\mathrm{cat}_G X = {}_{\mathcal{A}/K}\,\mathrm{cat}_{G/K}(X/K)$
2. ${}_{\mathcal{B}}\mathrm{cat}_H X = {}_{G \times_H \mathcal{B}}\,\mathrm{cat}_G(G \times_H X)$.

Invariance under Morita equivalence

Corollary

Let G and H be compact Lie groups. If $G \ltimes X \sim H \ltimes Y$, then

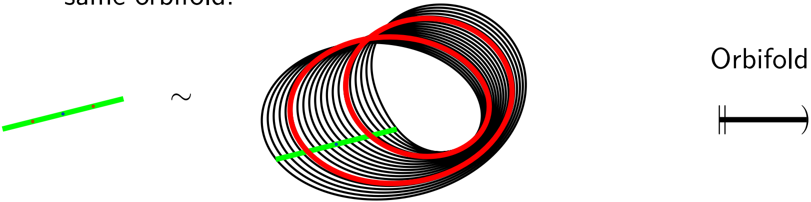
1. $\mathrm{cat}_G X = \mathrm{cat}_H Y$
2. $\mathrm{TC}^G X = \mathrm{TC}^H Y$

TC for orbifolds

Definition

A representable orbifold $\mathcal{X}$ is a space X equipped with a Morita equivalence class of orbifold structures. A specific such structure is given by a G compact group acting on X with finite isotropy.

If two group actions are Morita equivalent, then they define the same orbifold.



TC for orbifolds

Let $\mathcal{X}$ be a representable orbifold presented by the action $G \ltimes X$ where G is a compact Lie group and X a metrizable space. The orbifold invariant topological complexity of $\mathcal{X}$, $\mathrm{TC}_{\mathcal{O}}(\mathcal{X})$, is the invariant topological complexity of the group action $G \ltimes X$; that is $\mathrm{TC}_{\mathcal{O}}(\mathcal{X}) = \mathrm{TC}^{G}(X)$.

For instance, if $\mathcal{X} =$  then $\mathrm{TC}_{\mathcal{O}}(\mathcal{X}) = 1$.

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