Topological and geodesic complexity of n-dimensional Klein bottle

Don Davis

Gainesville Special Session, November, 2019

Planar polygon space

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$(n-3)$-manifold if generic (no straight-line polygons)

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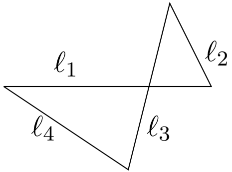

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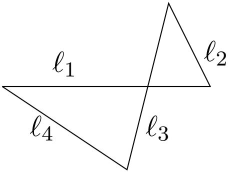

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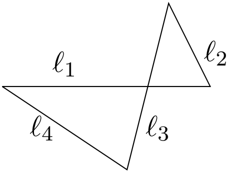

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For most ℓ, $2n-7 \leq \mathrm{TC}(\overline{M}(\ell)) \leq 2n-6$.

e.g., of the 2469 8-gon spaces, 2465 satisfy this, two do not, and for two, it is not known.

$\overline{M}(\frac{1}{n},\dots,\frac{1}{n},1,1,1) \approx T^{n-3}$. $\mathrm{TC} = n-3$.

$\overline{M}(1,\dots,1,n-2) \approx RP^{n-3}$. TC = imm dim, usually $< 2n-7$.

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$H^*(-) \Rightarrow \mathrm{TC} \geq n - 1$ for $n \geq 6$.

$\approx K_n = \mathbf{R}^n/(x + e_i \sim x),\ i < n,$
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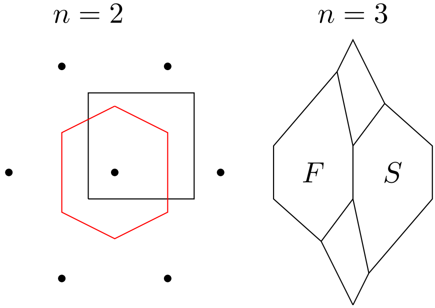

Here $n \le 5$. Similar but more complicated for $n \ge 6$.

Let $\mathcal{D}_\alpha = D_1 \times \cdots \times D_n$, with

$$D_i = \begin{cases} ((0,\frac12)\cup(\frac12,1))^{n-1} \text{ or } \{0,\frac12\} & i<n \\ (0,1) \text{ or } \{0\} & i=n. \end{cases}$$

On $\mathcal{D}_\alpha$, $\mathcal{R}(P)$ varies continuously bijectively with P, preserving $\sim$.

Let $R_j(P)$ be the set of equivalence classes of j-faces of $\mathcal{R}(P)$. Choose a representative of each equivalence class, and let $R'_j(P)$ denote their union.

Let $E_{\alpha,j} = \{(p(P), p(Q)) : P \in \mathcal{D}_\alpha,\ Q \in R'_j(P)\}$.

Geodesic motion planning rule on $E_{\alpha,j}$:
$s(p(P), p(Q)) = p(\sigma_{P,Q})$, where $\sigma_{P,Q}$ is linear path from P to Q.

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Proposition. If $\dim(\mathcal{D}_\alpha) + j = \dim(\mathcal{D}_{\alpha'}) + j'$, then $E_{\alpha,j}$ and $E_{\alpha',j'}$ are topologically disjoint.

Corollary. $GC(K_n) \le 2n$.

Proof. Have geodesic MP rules on $S_0, \ldots, S_{2n}$, where

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If $n = 6$, the domain $(0, \frac{1}{2})^5 \times (0, 1)$, must be split into parts separated by $\sum_{i=1}^{5}(a_i - \frac{1}{4})^2 = \frac{1}{16}$. When it is $< \frac{1}{16}$, the top and bottom pyramids intersect inside the walls, and the top and bottom pyramids need to be truncated above and below. So there is not a uniform way to choose geodesics in polytopes of the two types.