Bounding coindices of function spaces via motion planning

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November 2, 2019

AMS Fall Southeastern Sectional Meeting, Gainesville

We set out to prove a result in functional analysis.

We succeeded, and along the way we had to prove a result about motion planning algorithms (mpa) in an equivariant setting.

We define a weak notion of (equivariant) mpa that instead of contractibility gives weaker bounds for the topology.

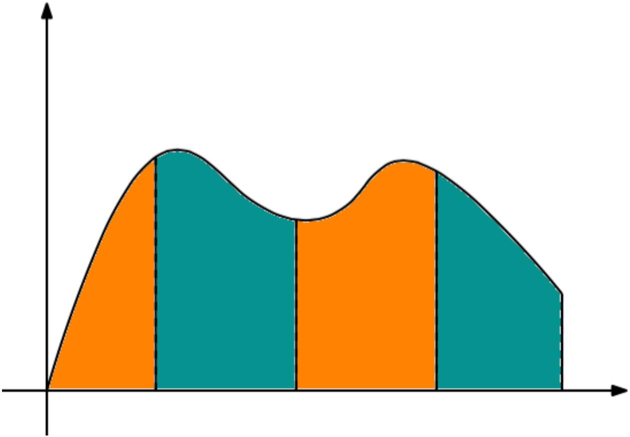
`https://arxiv.org/abs/1906.04417`

Hobby and Rice (1965)

Let $f_j\colon [0,1] \to \mathbb{R}$, $j = 1, 2, \ldots, n$ be integrable functions. Then there are $0 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{n+1} = 1$ such that

$$\sum \int_{\xi_i}^{\xi_{i+1}} (-1)^i f_j(t)\ dt = 0$$

for all j.



Let $f_j\colon [0,1] \to \mathbb{R}$, $j = 1, 2, \ldots, n$ be L^2 functions. Then there is $g\colon [0,1] \to \{-1, 1\}$ that changes sign at most n times such that $\langle f_j, g \rangle_{L^2} = 0$ for all j.

Lazarev and Lieb (2013)

Let $f_j \colon [0,1] \to \mathbb{C}$, $j = 1, 2, \ldots, n$ be L^2 functions.
Then there is smooth $g \colon [0,1] \to S^1 \subset \mathbb{C}$ such that $\langle f_j, g \rangle_{L^2} = 0$ for all j.

Lazarev and Lieb prove this with a technical approximation argument using the Hobby–Rice theorem and ask whether there is a more conceptual Borsuk–Ulam type result that implies their theorem.

Rutherfoord showed that g can be chosen with $\|g\|_{W^{1,1}} \leq 1 + 5\pi n$.

Lazarev and Lieb – restated

Let $\psi\colon C^\infty([0,1]; S^1) \to \mathbb{C}^n$ be continuous with respect to the L^2-norm and linear. Then there exists $g \in C^\infty([0,1]; S^1)$ with $\psi(g) = 0$.

Goal: Prove a Borsuk–Ulam type result for $C^\infty([0,1]; S^1)$ with the L^2-norm. What if ψ is not necessarily linear but only a $\mathbb{Z}/2$-map: $\psi(-h) = -\psi(h)$ for all $h \in C^\infty([0,1]; S^1)$?

F. and Superdock (2019)

Let $\psi\colon C^\infty([0,1];S^1) \to \mathbb{R}^n$ be continuous with respect to the L^p-norm, $p < \infty$, such that $\psi(-h) = -\psi(h)$ for all $h \in C^\infty([0,1];S^1)$. Then there exists $g \in C^\infty([0,1];S^1)$ with $\psi(g) = 0$ and $\|g\|_{W^{1,1}} \leq 1 + \pi n$.

In particular, this improves Rutherfoord's bound for the linear case from $\|g\|_{W^{1,1}} \leq 1 + 5\pi n$ to $\|g\|_{W^{1,1}} \leq 1 + 2\pi n$.

Show that there is a $\mathbb{Z}/2$-map $S^n \to C^\infty([0,1]; S^1)$ such that any element g in the image satisfies $\|g\|_{W^{1,1}} \leq 1 + \pi n$.

If X is a space with a $\mathbb{Z}/2$-action, the largest n such that there is a $\mathbb{Z}/2$-map $S^n \to X$ is called the coindex of X.

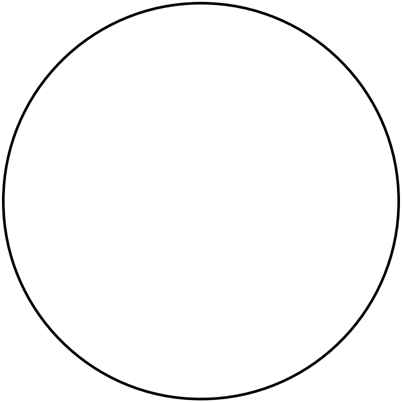
Borsuk–Ulam theorem

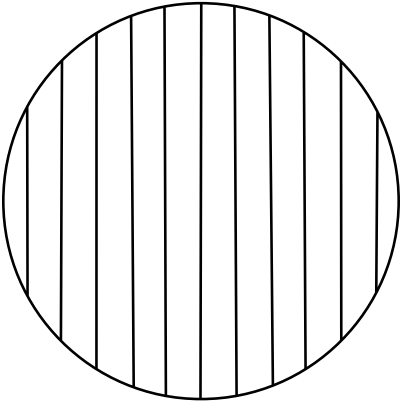
If X has coindex n then any $\mathbb{Z}/2$-map $X \to \mathbb{R}^n$ has a zero.

Maps $S^n \to X$ can be constructed using obstruction theory. If a $\mathbb{Z}/2$-map $f\colon S^{n-1} \to X$ is homotopically trivial, it can be extended to the upper hemisphere of S^n, and–by symmetry–to the lower hemisphere.

Inductive construction

Shoot rays from the upper hemisphere straight down to the lower hemisphere and extend the map along those paths.





Let PX be the space of paths in X with the compact-open topology.

If $s\colon X \times X \to PX$ satisfies $s(x, x) = c_x$ then this can be used to extend a map $S^{n-1} \to X$ to S^n (for any n). But for s to exist, X has to be contractible.

Is there a weaker notion, perhaps a partially defined s, that gives lower bounds for the topology that are weaker than contractibility?

We will also use the map $\phi\colon C^\infty([0,1];\mathbb{R}) \to C^\infty([0,1];S^1)$ given by $\phi(g)(x) = e^{ig(x)}$.

This map is continuous but not a covering space if the topology is induced by an L^p-norm, $p < \infty$.

Lastly, we will introduce a time parameter as another degree of freedom.

Let $\phi\colon Y \to Z$ be continuous, $(\preceq)$ a preorder on Y, and let
$Y^2_\preceq = \{(y_0, y_1) \in Y^2 : y_0 \preceq y_1\}$.

A **lifted motion planning algorithm** (or **lifted mpa**) for $(Y, Z, \phi, \preceq)$ is a family of maps $s_w\colon Y^2_\preceq \to PY$ for $w \in (0, 1]$ with $s_w(y_0, y_1)(0) = y_0$ and $s_w(y_0, y_1)(1) = y_1$, assembling into a continuous map $s\colon (0, 1] \times Y^2_\preceq \to PY$, with the following continuity property:

For all $y \in Y$ and all neighborhoods V of $\phi(y) \in Z$,

there exists a neighborhood U of $\phi(y) \in Z$ and $\delta > 0$ such that:

if $\quad \phi(y_0), \phi(y_1) \in U, \quad w < \delta,$

then $\quad \phi(s_w(y_0, y_1)(t)) \in V$ for all $t \in [0, 1]$.

Note that an mpa $s\colon Z \times Z \to PZ$ satisfying $s(z,z) = c_z$ for all $z \in Z$ extends to a lifted mpa for $(Z, Z, 1_Z)$ by taking $s_w = s$ for all w; the continuity property just restates the continuity of s at diagonal points $(z,z) \in Z \times Z$.

The definition of lifted mpa gives us three degrees of freedom:

1. Preorder $\preceq$ to restrict the domain
2. s has additional time parameter $(0,1]$
3. Can choose $Z \neq Y$ and argue about continuous image Z of Y

(Y,ρ) a $\mathbb{Z}$-space, (Z,σ) a $\mathbb{Z}/2$-space, $\phi\colon Y \to Z$ continuous and equivariant. Let $(\preceq)$ be a preorder on Y and $s\colon (0,1] \times Y^2_{\preceq} \to PY$ a lifted mpa for $(Y, Z, \phi, \preceq)$ such that:

1. $y \preceq \rho(y)$.
2. $\rho(y_0) \preceq \rho(y_1)$ if and only if $y_0 \preceq y_1$.
3. $y_0 \preceq y_1$ implies $y_0 \preceq s_w(y_0, y_1)(t) \preceq y_1$, for all $w \in (0,1]$, $t \in [0,1]$.

Then for each integer $n \geq 0$, there exists a $\mathbb{Z}/2$-map $\beta_n\colon S^n \to Z$. Moreover, for any choice of initial point $y^* \in Y$, the maps β_n can be chosen such that β_n maps each positive point of S^n to a point in Z of the form $\phi(y)$, with $y^* \preceq y \preceq \rho^n(y^*)$, that is, the subspace of these points $\phi(y)$ and their antipodes $\sigma(\phi(y))$ in Z has coindex at least n.

Corollary

Let Y be a $\mathbb{Z}$-space, Z a $\mathbb{Z}/2$-space. Let $\phi\colon Y \to Z$ be continuous and equivariant. If there is a lifted mpa for (Y, Z, ϕ) for the full preorder, then there exists a $\mathbb{Z}/2$-map $\beta_n\colon S^n \to Z$ for all integers $n \geq 0$.

Proof of the Hobby–Rice theorem

Let $f_j\colon [0,1] \to \mathbb{R}$, $j = 1, 2, \ldots, n$ be L^2 functions.
Then there is $g\colon [0,1] \to \{-1, 1\}$ that changes sign at most n times such that $\langle f_j, g\rangle_{L^2} = 0$ for all j.

The idea is to lift the space of functions with range in $\{\pm 1\}$ to nondecreasing functions with range in $\mathbb{Z}$. By describing a continuous map from pairs of such functions to paths between them, we will produce a lifted mpa.

Let Y be the space of nondecreasing functions $g\colon [0,1] \to \mathbb{Z}$ with finite range, and let Z be the space of functions $h\colon [0,1] \to \{\pm 1\}$. Define $\rho(g) = g + 1$, $\sigma(h) = -h$, and

$$\phi(g)(x) = \begin{cases} 1 & g(x) \text{ even} \\ -1 & g(x) \text{ odd} \end{cases}$$

Let $g_0 \preceq g_1$ if $g_0(x) \leq g_1(x)$ for all $x \in [0,1]$. Finally, for $g_0 \preceq g_1$ define $s_w(g_0, g_1)$ to be the path (in t) of functions following g_0 on $[0, 1-t)$ and g_1 on $[1-t, 1]$:

$$s_w(g_0, g_1)(t)(x) = \begin{cases} g_0(x) & x < 1 - t \\ g_1(x) & x \geq 1 - t \end{cases}$$

We obtain a $\mathbb{Z}/2$-map $\beta_n \colon S^n \to Z$. Applying the Borsuk–Ulam theorem to $\psi \circ \beta_n \colon S^n \to \mathbb{R}^n$, where $\psi \colon h \mapsto (\int_0^1 f_j(x)h(x)dx)_j$, we obtain $x \in S^n$ with $\psi(\beta_n(x)) = 0$. Hence also $\psi(\beta_n(-x)) = 0$, so we may assume x is positive. Taking $y^* = 0$, we may ensure that β_n maps each positive point of S^n to a point in Z of the form $\phi(g)$ with $0 \le g \le n$, so that $\phi(g)$ has at most n sign changes. This completes the proof.

Nonlinear Lazarev–Lieb theorem

Let $\psi\colon C^\infty([0,1];S^1) \to \mathbb{R}^n$ be continuous with respect to the L^p-norm, $p < \infty$, such that $\psi(-h) = -\psi(h)$ for all $h \in C^\infty([0,1];S^1)$. Then there exists $g \in C^\infty([0,1];S^1)$ with $\psi(g) = 0$ and $\|g\|_{W^{1,1}} \leq 1 + \pi n$.

Consider the space $C^\infty([0,1];\mathbb{R})$ with the L^2-norm, and let Y be the subspace of nondecreasing functions in $C^\infty([0,1];\mathbb{R})$, equipped with the action $\rho\colon g \mapsto g + \pi$. Let Z be $C^\infty([0,1];S^1)$ with the L^2-norm, equipped with the action $\sigma\colon h \mapsto -h$.

Define $\phi\colon Y \to Z$ by $\phi(g)(x) = e^{ig(x)}$;

Define $(\preceq)$ on Y as $(\leq)$ pointwise.

Let $\tau \colon \mathbb{R} \to [0,1]$ be a smooth, nondecreasing function with $\tau(x) = 0$ for $x \leq -1$, and $\tau(x) = 1$ for $x \geq 1$. (For example, take an integral of a mollifier.) Then define $s_w \colon Y^2_{\preceq} \to PY$ by

$$s_w(g_0, g_1)(t)(x) =$$

$$\left(1 - \tau\left(\frac{x-(1-t)}{w}\right)\right) g_0(x) + \tau\left(\frac{x-(1-t)}{w}\right) g_1(x).$$

There exist $f_1, \ldots, f_n \in L^1([0,1];\mathbb{R})$, such that for any $h \in C^1([0,1]; S^1)$ with

$$\langle f_j, h \rangle_{L^2} = 0 \qquad j = 1, \ldots, n$$

we have $\|h\|_{W^{1,1}} > \pi n + 1$.

For integer $n \geq 1$ let Y_n denote the space of C^∞-functions $f\colon [0,1] \to S^1$ with $\|f\|_{W^{1,1}} \leq 1 + \pi n$. Then

$$n \leq \operatorname{coind} Y_n \leq 2n - 1.$$