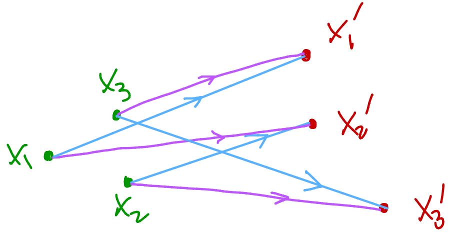
Linear Motion Planning with Controlled Collisions

joint with
José Luis León-Medina
Christopher Roque-Márquez

$$\mathrm{Conf}(X,n) = \{(x_1,..,x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$

$$\downarrow$$

$$\mathrm{Braid}(X,n) = \Sigma_n - \textit{orbit space}$$



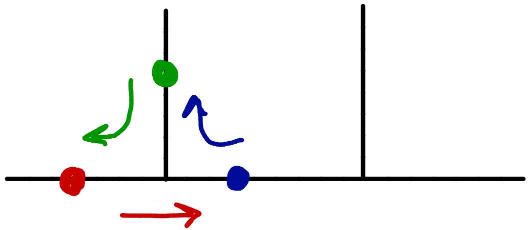
Collision-free MP

THEOREM. [Farber-Grant-Yuzvinsky 2009 (s=2)
González-Grant 2015 (s=3)]

For $d \geqslant 2$, $TC_s(\mathrm{Conf}(\mathbb{R}^d - Q_p, n)) = \begin{cases} s(n-1) - \mathrm{Even}(d), & p=0 \\ sn - \mathrm{Even}(d), & p=1 \\ sn, & p \geqslant 2 \end{cases}$

obstacles

- Quadratic function on s and n with ¨small¨ adjustments on d and p
- $TC_s(P_n) = s(n-1)-1$ — however $TC(B_n) = ??$
- Motion planning in dim > 1 has strong local flavor
- 1D - motion planning requires global info —plus different local input



$\mathrm{Conf}(\mathbb{R}, n)$-*model is meaningless and uninteresting*

- *Shopping carts in (not so) narrow aisles*
- *Sensors moving along a line with restricted interactions*
- *Digital microfluidics*

Configuration spaces with controlled collisions

Fat, thin, thinner...

$$\mathrm{Conf}(X,n) = X^n - \Delta_n$$

$$\Delta_n = \bigcup_{i\neq j} \Delta_{\{i,j\}}$$

$$\Delta_{\{i,j\}} = \{(x_1,\dots,x_n) : x_i = x_j\} \longrightarrow \Delta_\sigma = \{(x_1,\dots,x_n) : \underbrace{|\{x_i : i\in\sigma\}| = 1}_{\sigma\text{-multiple collision}}\}$$

$$\sigma \subseteq \{1,\dots,n\}$$

For $x = (x_1,\dots,x_n) : \{1,\dots,n\} \to X$,

$$x \in \mathrm{Conf}_K(X,n) \iff x^{-1}(x_0) \in K \text{ for all } x_0 \in X$$

Definition

Configuration space with collisions controlled by K

K = *abstract simplicial complex*

$$\mathrm{Conf}_K(X,n) := X^n - \Delta_K$$

$$\Delta_K = \bigcup_{\sigma\notin K} \Delta_\sigma$$

- $K \subseteq L \Rightarrow \mathrm{Conf}_K(X,n) \subseteq \mathrm{Conf}_L(X,n)$; $\quad X^n = \varinjlim_{K\ \mathrm{cx.}} \mathrm{Conf}_K(X,n)$
- $\mathrm{Conf}(X,n) = \mathrm{Conf}_1(X,n) \subseteq \mathrm{Conf}_2(X,n) \subseteq \cdots \subseteq \mathrm{Conf}_n(X,n) = X^n$

$$\left.\begin{array}{c} \mathrm{Conf}_e(X,n) := \mathrm{Conf}_{(\Delta^{n-1})^{e-1}}(X,n) \\ \text{"at-most-e-equal" conf. space} \end{array}\right\} \ \mathrm{Braid}_e(X,n)$$

- *Cohen-Lusk 1976.* *Blagojevic-Matchske-Ziegler 2011.*
- *Björner-Welker 1995.* *Baryshnikov 1997.* *Dobrinskaya-Turchin 2015.* $(X = \mathbb{R}^d)$
- *Kallel-Saihi 2016.* $(X = \mathrm{CW})$
- *Dobrinskaya 2008.*

$$X_1 \times \cdots \times X_n \supseteq Z(X_1, \dots, X_n ; K) = \left\{ (x_1, \dots, x_n) : \{ i : x_i \neq * \} \in K \right\}$$

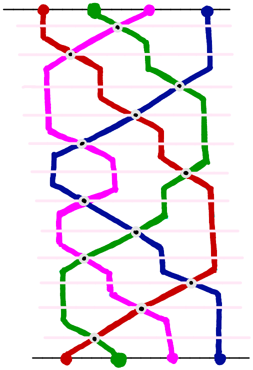
$$H_*\left(\Omega^1 Z(X_1, \dots, X_n; K)\right) = F\left[H_*(\Omega X_i),\ H_*\left(\mathrm{Conf}_K(\mathbb{R}^1, n)\right) \right]$$

$$\left[\begin{array}{c} \Omega^d \longleftrightarrow \mathbb{R}^d \\ \text{stable splitting, if } X_i = \Sigma Y_i \end{array} \right]$$

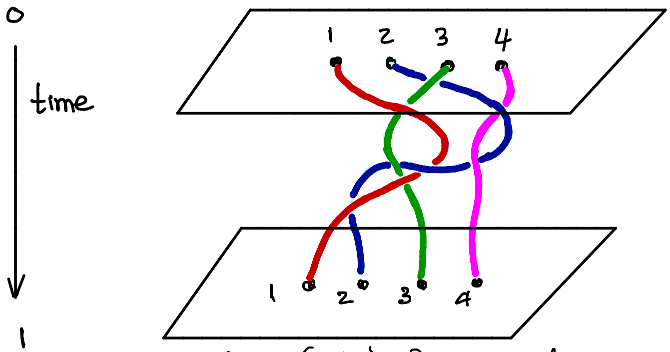
Khovanov :

$\mathrm{Conf}_2(\mathbb{R}^1, n)$ *is a real analogue of Artin's* $\mathrm{Conf}_1(\mathbb{R}^2, n)$

$$\mathrm{Conf}_2(\mathbb{R}^1, n) = K(PP_n, 1) \qquad \mathrm{Conf}_1(\mathbb{R}^2, n) = K(P_n, 1)$$



Planar Pure Braid



Usual (Artin) Pure Braid

Q: Topology of $\mathrm{Conf}_k(G, n)$ $\begin{cases} \text{(a) for } 0 < k \le n \\ \text{(b) for } K \text{ complex} \end{cases}$?   Eg: TCs ?

or Braid

graph (1D-MP)

Example - Motivation (Cohen-Pruidze)

$$TC\big(\mathcal{Z}(S^{d_1},\dots,S^{d_n};K)\big) = \max\Big\{ |\sigma|+|\tau| - \big|\sigma\cap\tau\cap\{i : d_i \text{ is odd}\}\big| \;:\; \sigma,\tau \text{ facet of } K \Big\}$$

complex

Theorem (G - León - Roque, 2018)

For $2\leq e<n$, $$TC_s(\mathrm{Conf}_e(\mathbb{R}^1,n)) = \begin{cases} s\left\lfloor \frac{n}{e+1} \right\rfloor, & e<n-1 \\ s-\mathrm{odd}(n), & e=n-1 \end{cases}$$

$$\mathrm{Conf}_e(\mathbb{R},n) \;\big\downarrow\; e\to n-1 \quad \mathrm{Conf}_{n-1}(\mathbb{R},n)\simeq S^{n-2}$$

[a la Kallel-Saihi : stable value for $e\geq n/2$]

Corollary (e=2)

$$TC_s(PP_n) = \begin{cases} s-1 & n=3 \\ s\left\lfloor \frac{n}{3} \right\rfloor & n\geq 4 \end{cases}$$

Mostovoy- Roque :

$$PP_3 = \mathbb{Z}, \quad PP_4 = \mathop{\ast}_{7}\mathbb{Z}, \quad PP_5 = \mathop{\ast}_{31}\mathbb{Z}, \quad PP_6 = \Big(\mathop{\ast}_{71}\mathbb{Z}\Big) \ast \Big(\mathop{\ast}_{20}(\mathbb{Z}\times\mathbb{Z})\Big), \dots$$

[Dranishnikov's $TC(G\ast H)$-formula: PP_n would be a free prod of $\mathbb{Z}^{\ell}$'s with $\ell\leq\left\lceil \frac{n}{3}\right\rceil$]

Proof ideas

homotopy *upper estimate (Severs-White 2012)* : $\mathrm{Conf}_e(\mathbb{R},n)$ —— $\left[\frac{n}{e+1}\right](e-1)$ —— $e-1$ [parabolic arrangmts]

cohomology *lower estimate (Baryshnikov 1997)* : $H^*(\mathrm{Conf}_e(\mathbb{R},n))$ is an algebra of preorders :

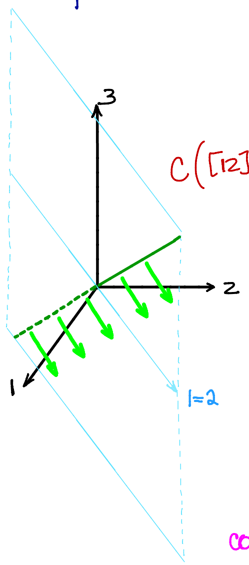
- Preorder on $\{1,\dots,n\}$: reflexivity + transitivity ($-$ antisymmetry)
  Eg : Empty $(1,\dots,n)$ Full $[1,\dots,n]$
- String preorder : height function with "fibers" either empty or full
  Eg : $[3,7](2,8)(1,6)[4,10](5,9)$ in $\{1,\dots,10\}$
- e-basic string preorder of dim $=(e-1)d$:
  $(I_0)\underbrace{[J_1](I_1)}\cdots(I_{d-1})\underbrace{[J_d](I_d)}$ with $|J_i|=e$ & $\max\{J_i, I_i\}\in I_i$
  normalization condition

Thm (Baryshnikov) : $H^*(\mathrm{Conf}_e(\mathbb{R},n))$ *is free on the e-basic string preorders*

cup product $\longleftrightarrow$ transitive closure

up to normalization

Example:



$C = C\big((I)[J](K)\big) = \{(x_1,\dots,x_n)\in\Delta_J : x_i > x_j > x_k \ \ \forall (i,j,k)\in I\times J\times K\}$

$\underbrace{(I)[J](K)}_{\substack{e\text{-basic} \\ \dim = e-1}}$

— a closed unbounded cell in $\mathbb{R}^n$ having:

- codimension $= e-1$
- boundary contained in $\mathbb{R}^n - \mathrm{Conf}_e(\mathbb{R},n)$

$C\big([12](3)\big)$

So $C \in H^{BM}_{n-k+1}\big(\mathbb{R}^n, \mathbb{R}^n - \mathrm{Conf}_k(\mathbb{R},n)\big) \underset{\text{Poincaré}}{\cong} H^{k-1}\big(\mathrm{Conf}_k(\mathbb{R},n)\big)$

intersection product $\longleftrightarrow$ cup products

$e = k-1$

cohomology *lower estimate (upshot) :* $TC\big(\mathrm{Conf}_e(\mathbb{R},n)\big) \geq 2\left[\frac{n}{e+1}\right]$

basic for $m+e \leq n$ $\begin{cases} x_m = (1,\dots,m-1)[m,\dots,m+e-1](m+e,\dots,n), & x_m \in H^{e-1}\big(\mathrm{Conf}_e(\mathbb{R},n)\big) \\ y_m = x_m\otimes 1 - 1\otimes x_m, & y_m \in H^{e-1}\big(\mathrm{Conf}_e(\mathbb{R},n)\big)^{\otimes 2} \end{cases}$

Set $n = i(e+1)+r$ with $0 \leq r \leq e$. Then

$$0 \neq \underbrace{(y_1 y_2)(y_{(e+1)+1}\, y_{(e+1)+2})\cdots(y_{(i-1)(e+1)+1}}_{\text{basic}}\ \underbrace{y_{(i-1)(e+1)+2}}_{\text{non-basic}}) \in H^*\big(\mathrm{Conf}_e(\mathbb{R},n)\big)^{\otimes 2}$$

Questions

① $TC_s(\mathrm{Braid}_e(\mathbb{R},n))$ $\left\{\begin{array}{l}\Sigma_n\text{-equivariant model ?}\\ = TC_s(\mathrm{Conf}_e(\mathbb{R},n))\text{ ?}\end{array}\right.$ ... maybe for $e=2$

② $TC_s(\mathrm{Conf}_e(\mathbb{R}^d,n))$ (... in progress)

③ $TC_s(\mathrm{Conf}_K(\mathbb{R},n))$ and the role of the complex K

①

$K=1$

classical case: $TC(\mathrm{Conf}(\mathbb{R}^2,n)) \leq TC(\mathrm{Braid}(\mathbb{R}^2,n))$

1-dim'l: $TC(\mathrm{Conf}(G,n)) > TC(\mathrm{Braid}(G,n))$

same homotopy dimension, but some chunks in Conf become tori in Braid

$K=2$ 1-dim'l:

$$TC(\mathrm{Conf}_2(\mathbb{R},n)) \overset{?}{\sim} TC\Big((\underset{q_1}{*}\mathbb{Z}) * (\underset{q_2}{*}\mathbb{Z}\times\mathbb{Z}) * \cdots * (\underset{q_n}{*}\mathbb{Z}\times\cdots\times\mathbb{Z})\Big)$$

$n = 3q_n + r$
$0 \leq r \leq 2$

??

$$TC(\mathrm{Conf}_2(\mathbb{R},n)) = TC(\mathrm{Braid}_2(\mathbb{R},n))$$

Questions

① $TC_s(\mathrm{Braid}_e(\mathbb{R},n))$ $\left\{\begin{array}{l}\Sigma_n\text{-equivariant model ?}\\ = TC_s(\mathrm{Conf}_e(\mathbb{R},n))\ ?\end{array}\right.$ ... maybe for $e=2$

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②

$TC(\mathrm{Conf}(\mathbb{R}^d,n))$: upper htpy bound − lower cohmlgy bound ≤ 1
actual answer depends on n and on (the parity) of d

$TC(\mathrm{Conf}_K(\mathbb{R}^d,n))$: upper htpy bound − lower cohmlgy bound can be > 1 if $d \geq 2$