Collision-free optimal motion planing algorithms

Cesar A. Ipanaque Zapata[†]
ICMC-USP, Brasil and CINVESTAV-IPN, Mexico
(Joint work with Jesús González).

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[†] cesarzapata@usp.br

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   - On the configuration space $F_r(\mathbb{R}^d, k)$
   - On the product $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$

Section 1

Motivation

Motivation

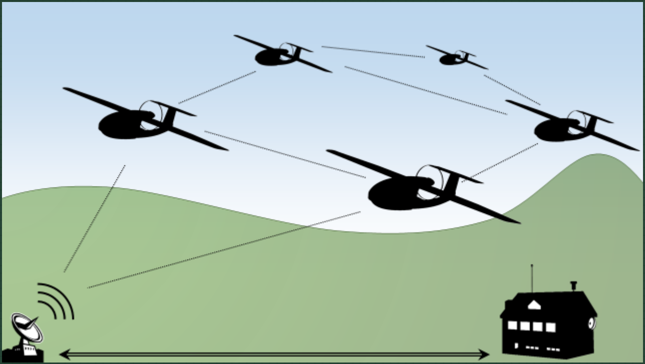
Figure: Asimo-$http : //robohub.org/morphological - computation - the - hidden - superpower - of - soft - bodied - robots/$

Motivation

Figure: $https : //www.drones.org/wp - content/uploads/Drones_vehicle_flight_aircraft_minimal_drone_flying_fly$

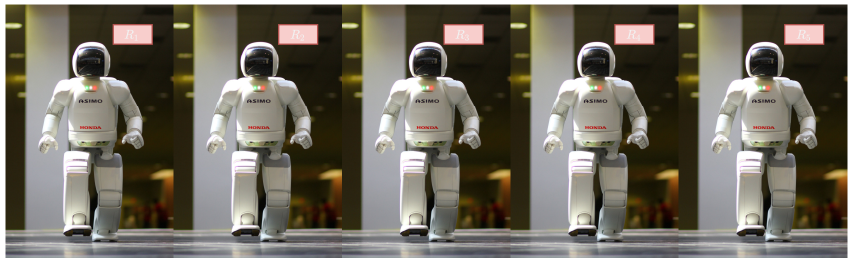
Motivation

Figure: $https://www.lsec.icmc.usp.br/images/wireless20communication_fanet.png$



Motivation

Figure: k robots (Asimos).



Motivation

We present optimal algorithms which can be used in designing practical systems controlling motion of many rigid bodies moving in space without collisions.

Motivation

- C. A. I. Zapata and J. González, *'Multitasking collision-free motion planning algorithms in Euclidean spaces'*, arXiv preprint arXiv:1906.03239 (2019).

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- T. Bajd, M. Mihelj, J. Lenarcic, A. Stanovnik and M. Munih, *’Robotics’*, International Series on Intelligent Systems, Control, and Automation: Science and Engineering, **43** (2010).

Section 2

Notations

Consider a multi-robot system consisting of k mobile robots $R_1, \ldots, R_k$, which are rigid bodies and we consider them as compact subsets of $\mathbb{R}^d$ $(d \geq 2)$, moving in Euclidean space $\mathbb{R}^d$ without collisions.

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We will suppose that the diameters of all robots are equal to $r > 0$, i.e., $\mathrm{diam}(R_i) = r > 0$, for any $i = 1, \dots, k$.

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The orientation-position determines the pose of a rigid object. The orientation of the local frame of the object and the position of the object are respect to the world frame.

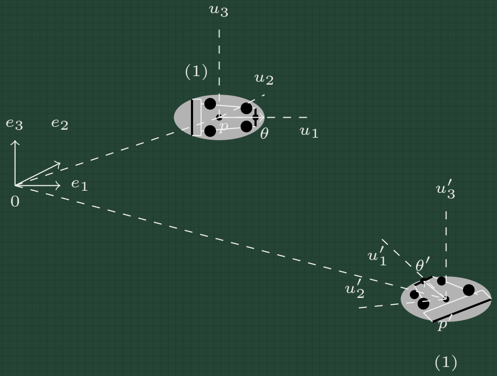

Figure: The Robot (1) has initial state $(\theta, p) = (id, p)$ and final state (θ', p').

Configuration space

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A more common task for mobile robots is to request them to navigate in an indoor environment, as shown in Figure above.

In this work the task of each robot consists of the point that can be reached by the pose of the robot, that is, a robot might be asked to perform tasks such as arriving at a particular place with a particular orientation. Thus, the workspace of this k robots coincides with the configuration space $(SO(d))^k \times F_r(\mathbb{R}^d, k)$ and the work map is the identity map.

Configuration space

The configuration space to the multi-robot system is the product $(SO(d))^k \times F_r(\mathbb{R}^d, k)$,

$$\{(\theta_1, \dots, \theta_k; p_1, \dots, p_k) \mid \ (\theta_1, \dots, \theta_k) \in (SO(d))^k \\ \text{and } (p_1, \dots, p_k) \in F_r(\mathbb{R}^d, k)\}$$

where $F_r(\mathbb{R}^d, k) = \{(p_1, \dots, p_k) \in (\mathbb{R}^d)^k \mid \ \| p_i - p_j \| > 2r \text{ for } i \neq j\}$ is the configuration space of all possible arrangements of k nonoverlapping disks of radius r in $\mathbb{R}^d$, equipped with subspace topology of the Cartesian power $(\mathbb{R}^d)^k$.

To give collision-free optimal algorithms we need to know the smallest possible number of regions of continuity for any collision-free motion planning algorithm, that is, the value of the *topological complexity* a la Farber TC$((SO(d))^k \times F_r(\mathbb{R}^d, k))$.

Section 3

Principal results: Topological complexity

In this paper we compute the value of $\mathrm{TC}((SO(d))^k \times F_r(\mathbb{R}^d, k))$ for $d = 2, 3$.

Theorem (Principal theorem)

Let $k \geq 2$, we have

1. $\mathrm{TC}((\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)) = 3k - 2$.
2. $\mathrm{TC}((\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)) = 5k - 1$.

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Furthermore, we present optimal tame motion planning algorithms in $(SO(d))^k \times F_r(\mathbb{R}^d, k)$ with $3k - 2$ (for $d = 2$) and $5k - 1$ (for $d = 3$) regions of continuity, respectively. These algorithms work for any $k \geq 2$ and they are easily implementable in practice.

Tame motion planner in a product

In general, to get a motion planning algorithm in the product $X \times Y$ requires partitions of unity subordinate to covers from motion planning algorithms to X and Y, respectively (M. Farber, 2003). However, we will recall here (see M. Farber, 2004) a simple explicit construction of a tame motion planning algorithm in $X \times Y$ with $\mathrm{TC}(X) + \mathrm{TC}(Y) - 1$ domains of continuity, under an additional assumption. This of course suits best our implementation-oriented objectives.

Tame motion planner in a product

Let $s := \{s_i : F_i \to PX\}_{i=1}^{n}$ be an optimal tame motion planner in X and let $\sigma := \{\sigma_j : G_j \to PY\}_{j=1}^{m}$ be an optimal tame motion planner in Y. Assume that the motion planner s, satisfies the following condition:

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'*Topologically disjoint condition*'- the closure of each set F_i is contained in the union $F_1 \cup \cdots \cup F_i$, in other words, it require that all sets of the form $F_1 \cup \cdots \cup F_i$ be closed.

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Similarly, we will assume that σ is a tame motion planner in Y such that all sets of the form $G_1 \cup \cdots \cup G_j$ are closed.

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Then we will set

$$W_\ell = \bigsqcup_{i+j=\ell} F_i \times G_j, \;\; \ell = 2, \ldots, n+m. \tag{1}$$

Tame motion planner in a product

The sets W_ℓ are ENRs and form a partition of $(X \times X) \times (Y \times Y) = (X \times Y) \times (X \times Y)$. Our assumptions guarantee that each product $F_i \times G_j$ is closed in W_ℓ, where $l = i + j$. Since different products in the union 1 are disjoint, we see that the maps $s_i \times \sigma_j$, where $i + j = \ell$, determine a continuous motion planning strategy over each set W_ℓ.

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Furthermore, we note that the motion planner in $X \times Y$ as above also satisfies the 'Topologically disjoint condition', i.e., all sets of the form $W_2 \cup \cdots \cup W_\ell$ be closed.

Lemma (TC for products)

Let $\mathbb{K}$ be a field and X and Y be any path-connected finite CW complexes. If $\mathsf{TC}(X) = zcl_{\mathbb{K}}(X) + 1$ and $\mathsf{TC}(Y) = zcl_{\mathbb{K}}(Y) + 1$, then

$$\mathsf{TC}(X \times Y) = \mathsf{TC}(X) + \mathsf{TC}(Y) - 1.$$

Furthermore, $\mathsf{TC}(X \times Y) = zcl_{\mathbb{K}}(X \times Y) + 1.$
In particular, for any $k \geq 2$, $\mathsf{TC}(\underbrace{X \times \cdots \times X}_{k \text{ times}}) = k\mathsf{TC}(X) - (k-1)$.

Spheres

By [M. Farber, 2003], we have

$$\mathsf{TC}(\mathbb{S}^n) = zcl_{\mathbb{Z}_2}(\mathbb{S}^n) + 1 = \begin{cases} 2, & \text{for } n \text{ odd;} \\ 3, & \text{for } n \text{ even.} \end{cases}$$

Moreover, it is easy to see $\mathsf{TC}(\mathbb{RP}^3) = zcl_{\mathbb{Z}_2}(\mathbb{RP}^3) + 1 = 4$. Hence, we have the following statement.

Lemma

For any $k \geq 2$, one has

1. $TC(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k \text{ times}}) = zcl_{\mathbb{Z}_2}(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k \text{ times}}) + 1 = k + 1.$
2. $TC(\underbrace{\mathbb{RP}^3 \times \cdots \times \mathbb{RP}^3}_{k \text{ times}}) = zcl_{\mathbb{Z}_2}(\underbrace{\mathbb{RP}^3 \times \cdots \times \mathbb{RP}^3}_{k \text{ times}}) + 1 = 3k + 1.$

Lemma (Homotopy type of $F_r(\mathbb{R}^d, k)$)

For any $r > 0$ and $k \geq 2$, one has $F_r(\mathbb{R}^d, k)$ and $F(\mathbb{R}^d, k)$ are homotopy equivalent.

Proof of Theorem 1

We recall that TC is a homotopy invariant, so by Lemma 4, it is sufficient to calculate the topological complexity TC$((SO(d))^k \times F(\mathbb{R}^d, k))$. By [M. Farber and S. Yuzvinsky, 2004], we have

$$\text{TC}(F(\mathbb{R}^d, k)) = zcl_{\mathbb{Z}_2}(F(\mathbb{R}^d, k)) + 1 = \begin{cases} 2k-2, & \text{if } d = 2; \\ 2k-1, & \text{if } d = 3. \end{cases}$$

Then by Lemmas 2 and 3 we obtain our principal theorem. □

Section 4

Algorithms

In this section, we present optimal tame motion planning algorithms in:

(1) product of odd-dimensional spheres,
(2) product of 3-dimensional real projective spaces,
(3) the configuration space $F_r(\mathbb{R}^d, k)$. Here, the algorithms will be induce from the algorithms given by [C. A. I. Zapata and J. González, 2019],
(4) the product $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$.

All the algorithms are easily implementable in practice.

On product of odd-dimensional spheres

Let v denote a fixed unitary tangent vector field on S^m, say $v(x_1, y_1, \ldots, x_\ell, y_\ell) = (-y_1, x_1, \ldots, -y_\ell, x_\ell)$ with $m + 1 = 2\ell$.

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A tame motion planning algorithm to S^m is given by $s := \{s_i : U_i \to PS^m\}_{i=1}^2$, where

$$\begin{aligned} F_1 &= \{(\theta_1, \theta_2) \in S^m \times S^m \mid \ \theta_1 = -\theta_2\}, \\ F_2 &= \{(\theta_1, \theta_2) \in S^m \times S^m \mid \ \theta_1 \neq -\theta_2\}, \end{aligned}$$

On product of odd-dimensional spheres

For all $(\theta_1, \theta_2) \in F_1$,

$$s_1(\theta_1, \theta_2)(t) = \begin{cases} \dfrac{(1-2t)\theta_1 + 2tv(\theta_1)}{\| (1-2t)\theta_1 + 2tv(\theta_1) \|}, & 0 \le t \le \dfrac{1}{2}; \\ \dfrac{(2-2t)v(\theta_1) + (2t-1)\theta_2}{\| (2-2t)v(\theta_1) + (2t-1)\theta_2 \|}, & \dfrac{1}{2} \le t \le 1, \end{cases}$$

and

$$s_2(\theta_1, \theta_2)(t) = \frac{(1-t)\theta_1 + t\theta_2}{\| (1-t)\theta_1 + t\theta_2 \|} \text{ for all } (\theta_1, \theta_2) \in F_2.$$

We note that

$$F_1 \cap F_2 = \emptyset, \overline{F_1} = F_1 \text{ and } \overline{F_2} = S^m. \tag{2}$$

On product of odd-dimensional spheres

Let $k \geq 2$ and for each $\ell = k, \ldots, 2k$ define a tame optimal motion planning algorithm $\rho = \{\rho_\ell : W_\ell \to P(S^m)^k\}$ where

$$W_\ell = \bigsqcup_{i_1+\cdots+i_k=l} F_{i_1} \times \cdots \times F_{i_k}.$$

On product of 3-dimensional projective spaces

We recall that the topological complexity $\mathsf{TC}(\mathbb{R}P^3) = 4$ and for any $k \geq 2$, $\mathsf{TC}(\underbrace{\mathbb{R}P^3 \times \cdots \times \mathbb{R}P^3}_{k \text{ times}}) = 3k + 1$.

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Will give an optimal tame motion planning algorithm on $\underbrace{\mathbb{R}P^3 \times \cdots \times \mathbb{R}P^3}_{k \text{ times}}$ having $3k + 1$ domains of continuity $X_k, \ldots, X_{4k}$ such that each X_ℓ satisfies the *'Topological disjoint condition'*, i.e., $\overline{X_\ell} \subset \bigcup_{j \leq \ell} X_j$.

On product of 3-dimensional projective spaces

For our purposes, using the idea from [M. Farber, 2004], we give an optimal tame motion planning algorithm on $\mathbb{R}P^3$ having 4 domains of continuity E_1, E_2, E_3, E_4 such that each E_i satisfies the 'Topological disjoint condition'.

On product of 3-dimensional projective spaces

Here we consider the real projective space $\mathbb{R}P^d = \dfrac{S^d}{x \sim -x}$ as the quotient space from S^d under the antipodal action.

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Consider the open covering

$$U_1 \cup \cdots \cup U_{d+1} = \mathbb{R}P^d,$$

where for each $i = 1, \ldots, d+1$, $U_i = \{[x_1, \ldots, x_{d+1}] \in \mathbb{R}P^d : \ x_i \neq 0\}$.

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where for each $i = 1, \ldots, d+1$, $U_i = \{[x_1, \ldots, x_{d+1}] \in \mathbb{R}P^d : \;\; x_i \neq 0\}$.

For each $i = 1, \ldots, d+1$ define a map $\varphi_i : U_i \to \mathbb{R}^d$ by

$$\varphi_i[x_1, \ldots, x_{d+1}] = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{d+1}}{x_i} \right)$$

One has φ_i is a homeomorphism, because it has a continuous inverse given by

$$\psi_i(x_1, \ldots, x_d) = \left[\frac{1}{(x_1^2 + \cdots + x_d^2 + 1)^{1/2}} (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_d) \right].$$

On product of 3-dimensional projective spaces

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Now, for each $i = 1, \ldots, d+1$, U_i is contractible. In fact, we can define the homotopy $H^i : U_i \times [0,1] \to \mathbb{R}P^d$ by

$$H^i([x_1, \ldots, x_{d+1}], t) = \psi_i \left(H \left(\varphi_i[x_1, \ldots, x_{d+1}], t \right) \right).$$

On product of 3-dimensional projective spaces

On the other hand, for each $i = 1, \ldots, d+1$, set

$$f_i : \mathbb{R}P^d \to [0,1], \quad f_i([x_1, \ldots, x_{k+1}]) = x_i^2.$$

On has f_i are well-defined smooth functions.

On product of 3-dimensional projective spaces

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On has f_i are well-defined smooth functions.

The support of f_i being the closure of U_i. Indeed the set $\{[x_1, \ldots, x_{k+1}] \in \mathbb{R}P^d : \;\; f_i([x_1, \ldots, x_{k+1}]) \neq 0\}$ is the subset U_i. Moreover, for any $[x] \in \mathbb{R}P^d$,

$$f_1[x] + \cdots + f_{d+1}[x] = 1.$$

On product of 3-dimensional projective spaces

Let a subset $V_i \subset \mathbb{R}P^d$, where $i = 1, \ldots, d+1$, be defined by the following system of inequalities

$$\begin{cases} f_j[x] < \dfrac{2j}{(d+1)(d+2)}, & \text{for all } j < i, \\ f_i[x] \geq \dfrac{2i}{(d+1)(d+2)}. \end{cases}$$

Note that each $\dfrac{i}{(d+1)(d+2)}$ is a regular value of the function f_i, so each V_i is a manifold with boundary and hence an ENR.

On product of 3-dimensional projective spaces

Furthermore, one easily checks that:

- V_i is contained in U_i; therefore, the homotopy $H^i : U_i \times [0,1] \to \mathbb{R}P^d$ restricts onto V_i and defines a homotopy H^i over V_i;
- the sets V_i are pairwise disjoint, $V_i \cap V_j = \varnothing$ for $i \neq j$;
- $V_1 \cup \cdots \cup V_{d+1} = \mathbb{R}P^d$.
- each V_i satisfies the 'Topological disjoint condition', i.e., $\overline{V_i} \subset \bigcup_{j \leq i} V_j$.

On product of 3-dimensional projective spaces

Now, recall that $\mathbb{R}P^3$ is a Lie group under the quaternionic product

$$\begin{aligned}[x_1, x_2, x_3, x_4] \cdot [y_i, y_2, y_3, y_4] \quad = \quad & [\langle x, (y_1, -y_2, -y_3, -y_4)\rangle, \\ & \langle x, (y_2, y_1, y_4, -y_3)\rangle, \\ & \langle x, (y_3, -y_4, y_1, y_2)\rangle, \\ & \langle x, (y_4, y_3, -y_2, y_1)\rangle],\end{aligned}$$

with unit $[1, 0, 0, 0]$ and inverse (given by the quaternionic conjugation) $[x_1, x_2, x_3, x_4]^{-1} = [x_1, -x_2, -x_3, -x_4]$.

On product of 3-dimensional projective spaces

For $i = 1, 2, 3, 4$ set

$$E_i = \{([x], [y]) \in \mathbb{R}P^3 \times \mathbb{R}P^3 : \ [x][y]^{-1} \in V_i\}.$$

It is clear that $E_1 \cup E_2 \cup E_3 \cup E_4 = \mathbb{R}P^3 \times \mathbb{R}P^3$, the sets E_i are pairwise disjoint, each E_i is an ENR and each E_i satisfies the 'Topological disjoint condition'.

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Then we may define $\sigma_i : E_i \to P\left(\mathbb{R}P^3\right)$ by the formula

$$\sigma_i([x], [y]) = H^i([x][y]^{-1}, t) \cdot [y]. \tag{3}$$

It is a continuous motion planning over E_i. Hence, $\sigma = \{s_i : E_i \to P\left(\mathbb{R}P^3\right)\}_{i=1}^4$ is an optimal tame motion planner on $\mathbb{R}P^3$ and each E_i satisfies $\overline{E_i} \subset \bigcup_{j \leq i} E_i$.

On product of 3-dimensional projective spaces

Let $k \geq 2$ and for each $\ell = k, \ldots, 4k$ define

$$X_\ell = \bigsqcup_{i_1+\cdots+i_k=l} E_{i_1} \times \cdots \times E_{i_k}.$$

One has that each X_ℓ is an ENR and $X_k, \ldots, X_{4k}$ form a partition of

$$\Big(\underbrace{\mathbb{R}P^3 \times \cdots \times \mathbb{R}P^3}_{k \text{ times}}\Big) \times \Big(\underbrace{\mathbb{R}P^3 \times \cdots \times \mathbb{R}P^3}_{k \text{ times}}\Big).$$

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We have thus constructed a tame motion planning algorithm (say σ) in $\underbrace{\mathbb{R}P^3 \times \cdots \times \mathbb{R}P^3}_{k \text{ times}}$ having $3k+1$ regions of continuity $X_k, \dots, X_{4k}$.

Furthermore, each X_ℓ satisfies $\overline{X_\ell} \subset \bigcup_{i \leq \ell} X_i$.

Algorithms on $F(\mathbb{R}^d, k)$ for any $d \geq 2$

Section
$\Gamma : F(\mathbb{R} - Q_m, k) \times F(\mathbb{R} - Q_m, k) \to F(\mathbb{R}^d - Q_m, k),\ (C, C') \mapsto \Gamma^{C,C'}$.

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$\Gamma : F(\mathbb{R} - Q_m, k) \times F(\mathbb{R} - Q_m, k) \to F(\mathbb{R}^d - Q_m, k), \ (C, C') \mapsto \Gamma^{C,C'}$.

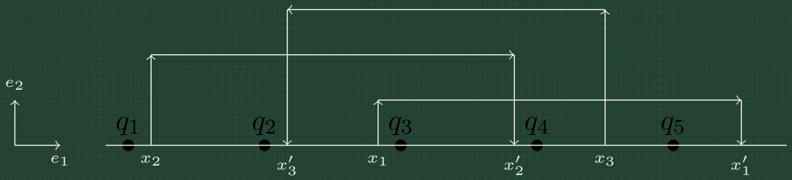

Figure: Section over $F(\mathbb{R} - Q_r, k) \times F(\mathbb{R} - Q_r, k)$. Vertical arrows pointing upwards (downwards) describe the first (last) third of the path $\Gamma^{C,C'}$, whereas horizontal arrows describe the middle third of $\Gamma^{C,C'}$.

Algorithms on $F(\mathbb{R}^d, k)$ for any $d \geq 2$

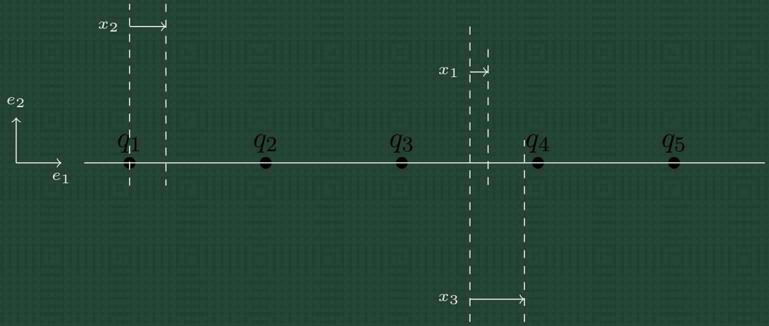

Figure: Desingularization deformation.

On the configuration space $F_r(\mathbb{R}^d, k)$

In this section we present a tame motion planning algorithm on $F_r(\mathbb{R}^d, k)$ having $2k - 1$ domains of continuity. The algorithm works for any $r > 0$, $d \geq 2$ and $k \geq 2$; this algorithm is optimal when d is odd.

On the configuration space $F_r(\mathbb{R}^d, k)$

In this section we present a tame motion planning algorithm on $F_r(\mathbb{R}^d, k)$ having $2k-1$ domains of continuity. The algorithm works for any $r>0$, $d\geq 2$ and $k\geq 2$; this algorithm is optimal when d is odd.

Note that the optimal tame motion planning algorithm $\omega=\{\omega_\ell : Y_\ell \to PF(\mathbb{R}^d,k)\}_{\ell=2}^{2k}$ in $F(\mathbb{R}^d,k)$ induces an optimal tame motion planning algorithm in $F_r(\mathbb{R}^d,k)$, say $\hat{\omega}=\{\hat{\omega}_\ell : Z_\ell \to PF_r(\mathbb{R}^d,k)\}_{\ell=2}^{2k}$, where each Z_ℓ is given by

$$Z_\ell = (i\times i)^{-1}\,(Y_\ell)$$

and each local motion planner $\hat{\omega}_\ell$ by

$$\hat{\omega}_\ell(p,q)=\begin{cases}\hat{H}_{3t}(p), & 0\leq t\leq \frac{1}{3};\\ \rho\left(\omega_\ell(p,q)(3t-1)\right), & \frac{1}{3}\leq t\leq \frac{2}{3};\\ \hat{H}_{3-3t}(q), & \frac{2}{3}\leq t\leq 1.\end{cases}$$

On the configuration space $F_r(\mathbb{R}^d, k)$

Similarly, the optimal tame motion planning algorithm $\Omega = \{\Omega_\ell : M_\ell \to PF(\mathbb{R}^d, k)\}_{\ell=3}^{2k}$ in $F(\mathbb{R}^d, k)$ (for d even) induces an optimal tame motion planning algorithm in $F_r(\mathbb{R}^d, k)$ (for d even), say $\hat{\Omega} = \{\hat{\Omega}_\ell : N_\ell \to PF_r(\mathbb{R}^d, k)\}_{\ell=3}^{2k}$, where each N_ℓ is given by

$$N_\ell = (i \times i)^{-1} (M_\ell)$$

and each local motion planner $\hat{\Omega}_\ell$ by

$$\hat{\Omega}_\ell(p, q) = \begin{cases} \hat{H}_{3t}(p), & 0 \leq t \leq \frac{1}{3}; \\ \rho\left(\Omega_\ell(p, q)(3t-1)\right), & \frac{1}{3} \leq t \leq \frac{2}{3}; \\ \hat{H}_{3-3t}(q), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

On the product $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$

The optimal tame motion planning algorithms in $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$ are given, one more time, by the construction given assembling the algorithms above.

On the product $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$

The optimal tame motion planning algorithms in $(\mathbb{S}^1)^k \times F_r(\mathbb{R}^2, k)$ and $(\mathbb{RP}^3)^k \times F_r(\mathbb{R}^3, k)$ are given, one more time, by the construction given assembling the algorithms above.

We note that the results and motion planning algorithms described in this work can also be extended to the case of higher topological complexity (in the sense of Rudyak) and obtain multitasking collision-free optimal motion planning algorithms for rigid bodies.

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