Upper Bound for Monoidal Topological Complexity

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Joint with Mitsunobu Tsutaya

Introduction

M — the space of states of a robot.

$M \times M$ — initial and terminal states of a robot

$\mathcal{P}(M)$ — the space of robot motions (strings in M)

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$$\mathrm{tc}(M) := \min \left\{ n\geq 0 \,\middle|\, \begin{array}{l} \exists\{U_i; 0\leq i\leq n\} \text{ open cover of } M\times M \ \& \\ \exists\, s_i \text{ local section of } \pi \text{ on } U_i,\ 0\leq i\leq n \end{array} \right\}$$

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Prop. 1. $tc(M) = cat^{*}_{B}(d(M))$ the fibrewise L-S cat in $\underline{\underline{\mathcal{T}}}^{*}_{B}$.

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Results

By definition, we have $\mathrm{cat}(M) \le \mathrm{tc}(M) \le \mathrm{cat}(M \times M) \le 2\,\mathrm{cat}(M)$,
while we only have $\mathrm{tc}(M) \le \mathrm{tc}^{\mathcal{M}}(M) \le \mathrm{tc}(M)+1 \le 2\,\mathrm{cat}(M)+1$.

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Main Thm. (I-Tsutaya) $\mathrm{tc}^{\mathcal{M}}(M) \le 2\,\mathrm{cat}(M)$ for M a finite polyhedron.

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- For an odd sphere S^{odd}, we have $\mathrm{tc}(S^{\mathrm{odd}}) = 1$.
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- For an *even* sphere S^{ev}, we have $\mathrm{tc}(S^{\mathrm{ev}}) = 2$.
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By Main Thm, we have $\mathrm{tc}^{\mathcal{M}}(W) \le 2\,\mathrm{cat}(W) = 2$ for W a wedge of spheres.

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Main Thm. (I-Tsutaya) $\mathrm{tc}^{\mathcal{M}}(M) \leq 2\,\mathrm{cat}(M)$ for M a finite polyhedron.

(Farber)

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We construct a covering of $M \times M$ *by open sets on which there exists a (local) section satisfying the monoidal condition.*

1. *We choose a nice categorical covering of* M*.*
2. *We define an open covering of* $M \times M$ *using the covering obtained at the stage (1), which is due to R. Fox.*

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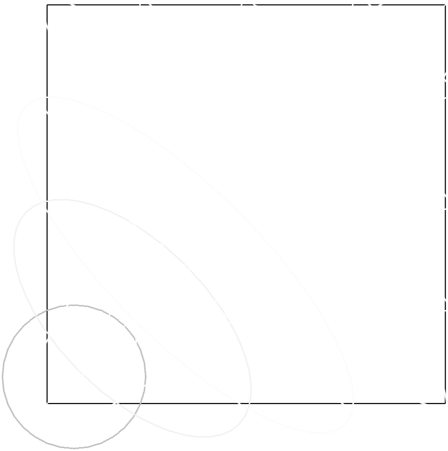

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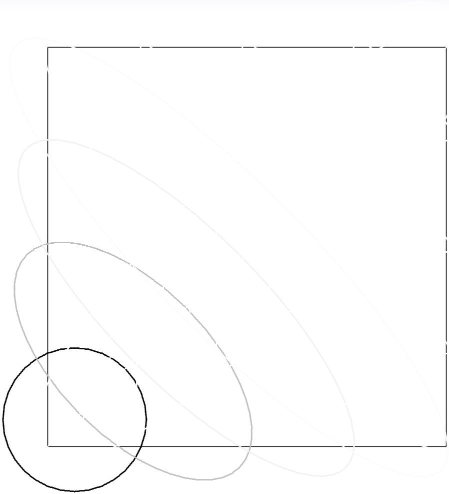

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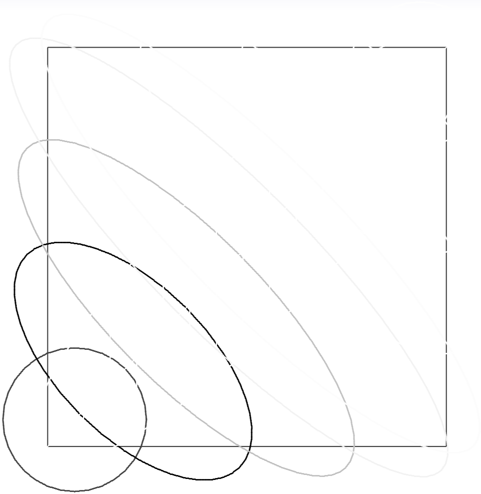

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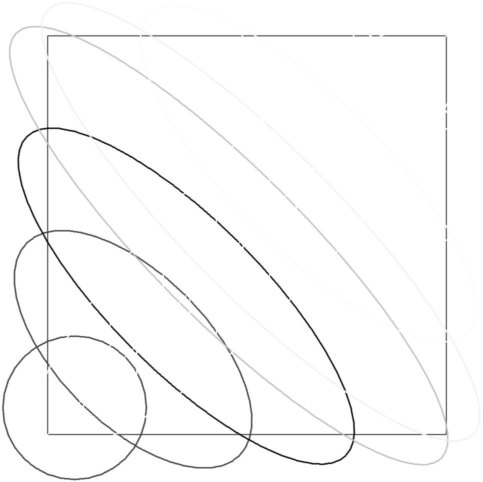

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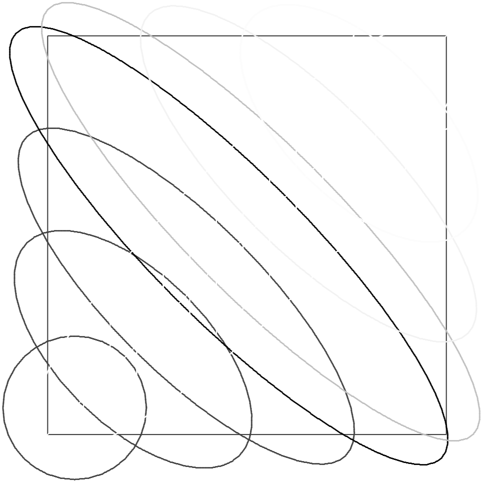

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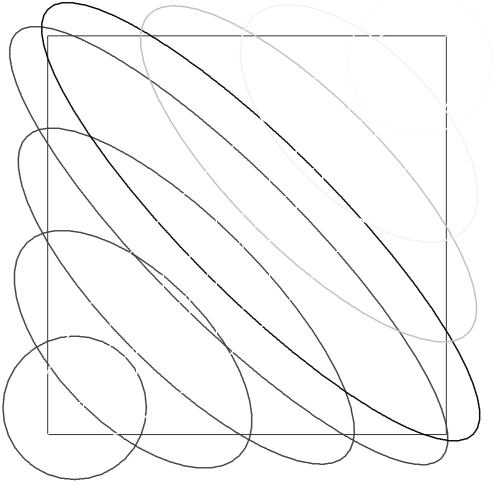

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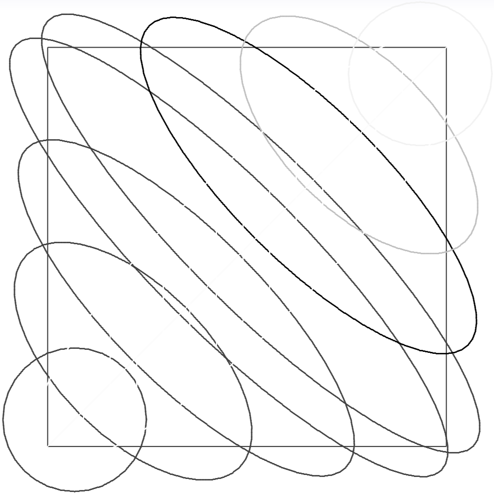

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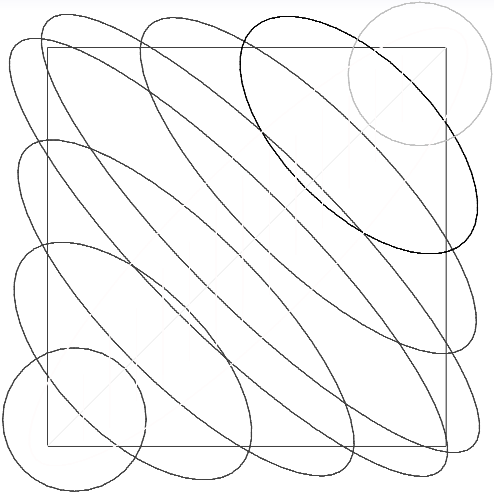

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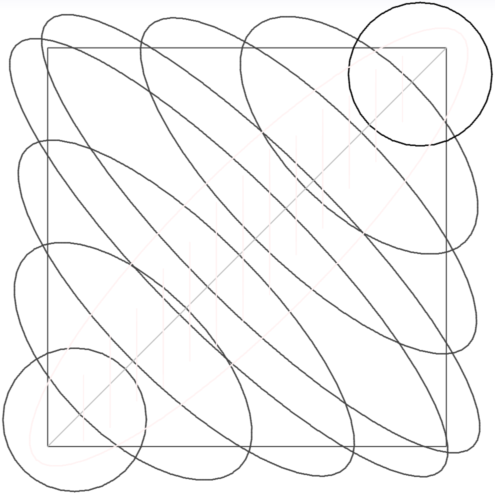

Prob. Can we remove the finiteness condition from Main Thm?

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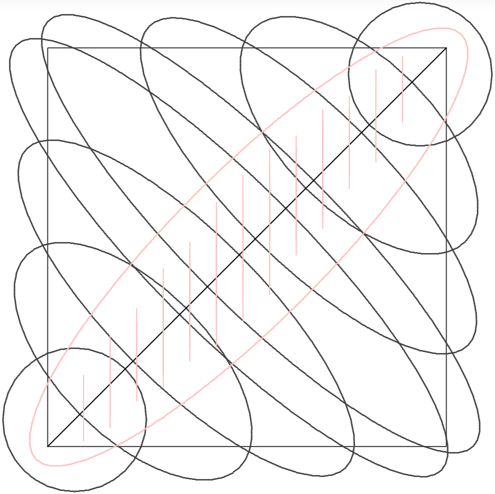

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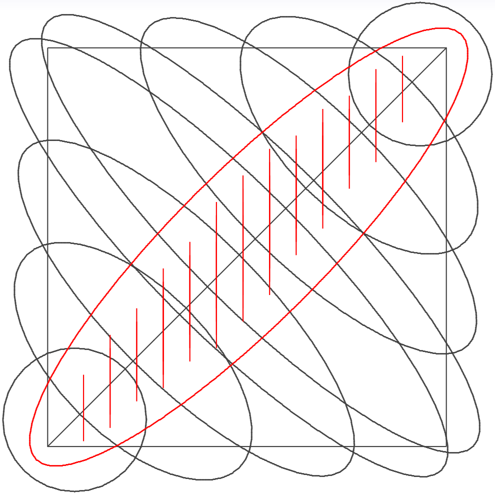

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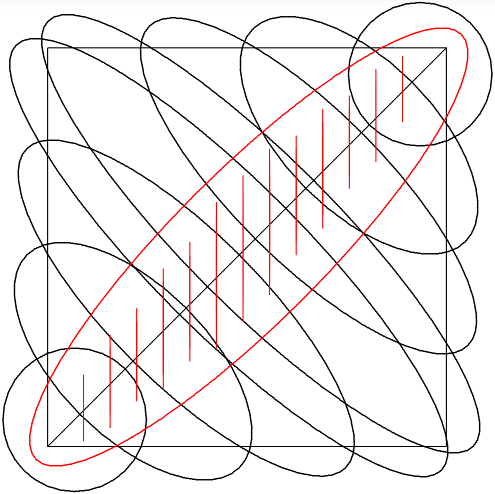

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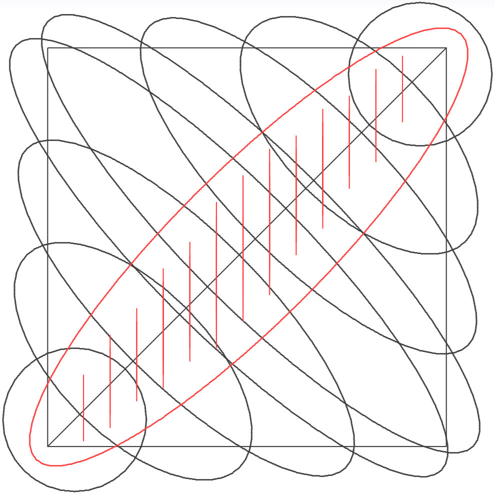
That's all. Thank you for your attention!



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