Topological estimates of the number of vertices of minimal triangulations

by Dejan Govc **Wacław Marzantowicz** Petar Pavešić

from to at

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The research gives new results on the notion of covering type introduced and studied in M. Karoubi, C. Weibel, *On the covering type of a space*, arXiv 1612.00532v1, L'Enseignement Math. (2016), 62 (2016), p. 457–474.

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- $\{U_i\}$ is a good cover of X if $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_n} \neq \emptyset \implies U_{i_1} \cap U_{i_2} \cdots \cap U_{i_n}$ is contractible.

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- The *strict covering type* of a given space X, is the minimal cardinality of a good cover for X.
- We define the *covering type* of X as the minimal size of a good cover of spaces that are homotopy equivalent to X:

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Examples: $\mathrm{ct}(X) = 1 \Leftrightarrow X$ contractible.
$\mathrm{ct}(X) = 2 \Leftrightarrow X$ disjoint union of two contractible sets.

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- [14]: $n + 2 \leq \mathrm{ct}(\mathbb{R}P^n) \leq 2m + 3$ wrong,
  we show right estimate from below.
- the Hawaiian earring X does not admit any good covers, i.e $\mathrm{sct}(X) = \infty$.

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gives $X \sim |N(\mathcal{U})|$.

Theorem (Karoubi-Weibel)

X finite CW complex $\Rightarrow \mathrm{ct}(X) =$ min. elements of a good closed cover of some complex Y, $Y \sim X$.

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- For a PL-manifold M we define $\Delta^{PL}(M) := \min\left\{\mathrm{card}(K^{(0)}) \,\middle|\, K \text{ is a PL-triangulation of } M\right\}$.
- If X has the homotopy type of compact polyhedron we introduce a homotopy analogue of $\Delta(P)$ as $\Delta^{\simeq}(X) := \min\{\Delta(P) \mid P \simeq X\}$.

Computing $\Delta(P)$ and its variants is a hard and intensively studied problem of combinatorial topology - see Datta [7] and Lutz [17] for surveys of the vast body of work related to this question.
Clearly, $\Delta^{\simeq}(P)$ is a lower bound for other invariants, since

$$\Delta^{\simeq}(P) \leq \Delta(P), \quad \text{and if } M \text{ is a PL-manifold } \Delta^{\simeq}(M) \leq \Delta^{PL}(M).$$

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Conjecture (GMP - 2017)

If M is a closed PL-manifold, then $\mathrm{ct}(M) = \Delta^{PL}(M)$.

$\mathrm{ct}(X)$ versus $\mathrm{cat}(X)$ - the Lusternik-Schnirelman category.

Definition (Lusternik-Schnirelman category, Geometric category)

- $\mathrm{cat}(X) :=$ min. elements of a cover $\mathcal{U} = \{U_i\}$ such that $U_i \rightsquigarrow * $ in X.

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  The geometric category is not a homotopy invariant of X, so one defines the *strong category*, $\mathrm{Cat}(X)$ as the min of geometric categories of $Y \simeq X$.

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$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1$ (see [6, Proposition 3.15])

Example

$\mathrm{ct}(S^n) = n + 2, \quad \mathrm{cat}(S^n) = 2$ - the difference arbitrary large.

Examples

- For the wedge on n circles W_n we have $\mathrm{sct}(W_n) = n + 2$, while $\mathrm{ct}(W_n) = \left\lceil \frac{3+\sqrt{1+8n}}{2} \right\rceil$ (see [14, Proposition 4.1])
- $\mathsf{cat}(X) = n > 1 \Rightarrow \dim X = n - 1$
  If X admits a good cover $\mathcal{U}$ of order $\leq n$ (i.e., at most n different sets have non-empty intersection), then X is homotopy equivalent to a simplicial complex of dimension $n - 1$.

Theorem (GMP)

$$\mathrm{ct}(X) \geq \frac{1}{2}\,\mathsf{cat}(X)\,(\mathsf{cat}(X)+1)$$

Estimates by L-S category

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For real and complex projective spaces
$\mathrm{cat}(\mathbb{R}P^n) = \mathrm{cat}(\mathbb{C}P^n) = n+1$

Corollary

$$\mathrm{ct}(\mathbb{R}P^n) \geq \frac{(n+1)(n+2)}{2}$$

$$\mathrm{ct}(\mathbb{C}P^n) \geq \frac{(n+1)(n+2)}{2}$$

We show that the above can be improved.

More fine version of previous theorem

Theorem

$$\mathrm{ct}(X) \geq 1 + \mathrm{hdim}(X) + \frac{1}{2}\,\mathsf{cat}(X)(\mathsf{cat}(X) - 1)\,,$$

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where $\mathrm{hdim}(X)$ *is the homotopy dimension.*

A triangulation of a manifold is *combinatorial* if the links of all vertices are triangulated spheres.

Corollary

Let K be a combinatorial triangulation of a d-dimensional and c-connected closed manifold M. Then K has at least
$1 + d + c \cdot (\mathrm{cat}(M) - 2) + \tfrac{1}{2}\,\mathrm{cat}(M)(\mathrm{cat}(M) - 1)$ *vertices.*

We used the known inequality (see [6])

$$\mathrm{cat}(V) \leq \frac{\mathrm{hdim}(V)}{c+1} + 1 \qquad (1)$$

For given n-tuple of positive integers $i_1, \ldots, i_n \in \mathbb{N}$ we say that X *admits an essential* $(i_1, \ldots, i_n)$*-product* if there are coh. classes $x_k \in H^{i_k}(X)$, such that $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is non-trivial.

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We define the *covering type of the n-tuple of positive integers* $(i_1, \ldots, i_n)$ as
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Proposition

$$\mathrm{ct}(X) \geq \max\{\mathrm{ct}(|x_1|, \dots, |x_n|) \mid \text{ for all } 0 \neq x_1 \cdots x_n \in H^*(X)\}$$

Lemma

If X has non-trivial reduced homology groups in different dimensions, then $\mathrm{ct}(X) \geq \mathrm{hdim}(X) + 3$.

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We are ready to prove the main result of this section, an 'arithmetic' estimate for the covering type of a n-tuple:

Theorem

$$\mathrm{ct}(i_1, \ldots i_n) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+1)$$

If $i_1, \ldots i_n$ *are not all equal, then*

$$\mathrm{ct}(i_1, \ldots i_n) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+2)$$

Corollary

The covering type of projective spaces is bounded by:
$\mathrm{ct}(\mathbb{R}P^n) \geq \frac{1}{2}(n+1)(n+2)$, $\mathrm{ct}(\mathbb{C}P^n) \geq (n+1)^2$,
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For $\mathbb{R}P^n$ and $\mathbb{C}P^n$ these numbers are equal to the best know estimate obtained by use of the combinatorial methods, so that numerically it reproves the result of [2]. For $\mathbb{H}P^n$ there is not known an estimate of the cardinality of vertices of a "minimal" triangulation.

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Corollary

For a product $X = S^{i_1} \times \cdots \times S^{i_n}$, *where* $i_1 \leq \ldots \leq i_n$ *are not all equal, Thm. 13 yields* $\mathrm{ct}(X) \geq i_1 + 2\, i_2 + \cdots + n i_n + (n+2)$, *while for a product of spheres of the same dimension we get*

$$\mathrm{ct}((S^i)^n) \geq \frac{(n+1)(ni+2)}{2}.$$

The last estimate can be sometimes improved by ad-hoc methods

Corollary

The covering type of unitary groups is estimated as

$$\mathrm{ct}(U(n)) \geq \frac{1}{6}(4n^3 + 3n^2 + 5n + 12) \quad \textit{and} \quad \mathrm{ct}(SU(n)) \geq \frac{1}{6}(4n^3 - 3n^2 + 5n + 6).$$

These ↑ estimates are new.

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We showed that the number of vertices and top-dimensional simplices grow (at least) as a cubical function of n and that the number of **all simplices grows exponentially in** n**.**

Our computation has three main ingredients.

1. R. Stong's [19] determination of the height of the first Stiefel-Whitney class w_1 in $H^*(G_k(\mathbb{R}^n))$, and of non-trivial products in the top dimension of $H^*(G_k(\mathbb{R}^n))$ for $k = 2, 3, 4$.

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2. Lower bounds for the number of vertices in a triangul. of space whose coh. admits certain non-trivial products [10].
3. The Lower Bound Theorem (LBT) of Gromov [11], Kalai [13], or Klee and I. Novik [15] that estimates the number of faces in a triangulation of a (pseudo)manifold with a given number of vertices.

Theorem (LBT)

Let K be a triangulation of a d-dimensional closed manifold, and denote by f_i, $i = 0, \ldots, d$ the number of i-dimensional simplices in K. Then

$$f_i \geq f_0 \cdot \binom{d+1}{i} - i \cdot \binom{d+2}{i+1} \quad \textit{for } i = 0, \ldots, d-1$$

and

$$f_d \geq f_0 \cdot d - (d+2)(d-1).$$

Moreover, by adding up all inequalities we obtain an estimate for the total number of simplices in K:

$$f_0 + \ldots + f_d \geq 2[(f_0 - d)(2^{d+1} - 1) + 1].$$

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The standard decomposition of $G_k(\mathbb{R}^n)$ has $\binom{n}{k}$ cells (of which only one 0-dimensional and one top-dimensional cell). A contrary

Example (The number of simplices in any triangulation is huge:)

$G_3(\mathbb{R}^9)$ is 18-dimensional and every triangulation requires at least 185 vertices. As a consequence, every triangulation of $G_3(\mathbb{R}^9)$ must have at least

$$185 \cdot 18 - (18+2) \cdot (18-1) = 2990$$

facets and at least

$$2((185-18) \cdot (2^{19}-1)+1) > 175 \cdot 10^6$$

simplices!

$G_4(\mathbb{R}^9)$ is 20-dimensional and $\Delta(G_4(\mathbb{R}^9)) \geq 242$. Therefore, every triangulation of $G_4(\mathbb{R}^9)$ requires more than 4422 facets and more than $930 \cdot 10^6$ simplices. The number of 4-dimensional simplices, whose links should be examined to compute the first rational Pontrjagin class by means of Gaifulin's formula exceeds 1.3 million.

Thank you !

Dziękuję !

References

- K. Appel and W. Haken, *Every planar map is four colorable*, Contemp. Math 98 (1989).
- P. Arnoux, A. Marin, *The Kühnel triangulation of the complex projective plane from the view-point of complex cristallography* (part II), Memoirs of Fac. Sc., Kyushu Univ. Ser. A 45 (1991), 167–244.
- E. Borghini E. G. Minian, *The covering type of closed surfaces and minimal triangulations*, arXiv:math/171202833.
- U. Brehm, W. Kühnel, *Combinatorial manifolds with few vertices*, Topology 26, 465–473 (1987).
- B. Bagchi, B. Datta, *Minimal triangulations of sphere bundles over the circle*, J. Combin. Theory Ser. A 115 (2008), no. 5, 737– 752.

O. Cornea, G. Lupton, J. Oprea, D. Tanre, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monograp hs, vol. 103, (American Mathematical Society, 2008).

B. Datta, *Minimal triangulations of manifolds*, arXiv:math/0701735.

A.A. Gaifullin, *Configuration spaces, bistellar moves, and combinatorial formulas for the first Pontryagin class.*, Proc. Steklov Inst. Math. **268** (2010), 70–86.

I.M. Gel'fand, R.D. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 304–309.

D. Govc, W. Marzantowicz and P. Pavešić, *Estimates of covering type and the number of vertices of minimal triangulations*, Discr. Comp. Geom. (2019), https://doi.org/10.1007/s00454-019-00092-z

M. Gromov, *Partial Differential Relations*, (Springer, Berlin, Heidelberg, New York, 1986).

A. Hatcher, *Algebraic Topology*, (Cambridge University Press, 2002).

G. Kalai, *Rigidity and the lower bound theorem 1*, Invent. Math. **88** (1987), 125–151.

M. Karoubi, C. Weibel, *On the covering type of a space*, arXiv/math:161200532, L'Enseignement Math., **62** (2016), 457–474.

S. Klee, I. Novik *Face enumeration on simplicial complexes*, arXiv:math/150506380.

M. Jungerman, G. Ringel, *Minimal triangulations on orientable surfaces*, Acta Math. Volume 145 (1980), 121–154.

F. Lutz, *Triangulated Manifolds with Few Vertices: Combinatorial Manifolds*, arXiv:math/0506372.

F. Lutz, *Triangulated manifolds with few vertices and vertex-transitive group actions*. (German summary) Dissertation, Technischen Universität Berlin, Berlin, 1999. Berichte aus der Mathematik. [Reports from Mathematics] Verlag Shaker, Aachen, 1999. vi+137 pp.

R. Stong, *Cup products in Grassmannians*, Top. Appl. **13** (1982), 103–113.

M. Tancer, *Intersection patterns of convex sets via simplicial complexes, a survey*, arXiv 1102.0417v2

M. Tancer, D. Tonkonog, *Nerves of good covers are algorithmically unrecognizable*. (English summary) SIAM J. Comput. 42 (2013), no. 4, 1697–1719.

G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, vol. 61, (Springer, Berlin, 1978).