Geodesic complexity

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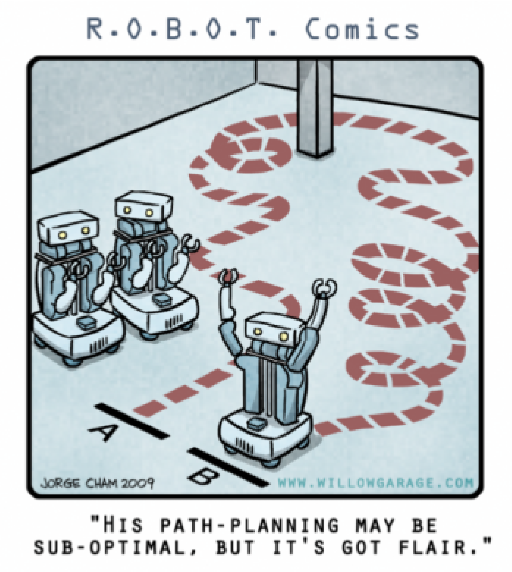
Motion planning problem



Topological complexity

Preliminary definition

A continuous motion planner assigns to each pair of points on a space X a path between them in a continuous way.
In other words, it is a section of the **free path fibration**

$$PX \to X \times X \quad , \quad \gamma \mapsto (\gamma(0), \gamma(1)).$$

Definition (Farber '03) (ENR version)

The **topological complexity** $\mathrm{TC}(X)$ of a space X is the smallest k for which there exists a decomposition

$$X \times X = E_0 \cup \ldots \cup E_k, \qquad E_i \cap E_j = \emptyset \text{ if } i \neq j,$$

such that there exists a local section of the free path fibration over each E_i.

Geodesic complexity

Definition

Let (X, d) be a metric space. We say a path γ is a *minimal geodesic* if $\ell(\gamma) = \mathrm{d}(\gamma(0), \gamma(1))$. Let $GX \subset PX$ consist of the minimal geodesics. Restricting the free path fibration to GX results in a map

$$GX \to X \times X.$$

Definition

The **geodesic complexity** $\mathrm{GC}(X)$ of a space X is the smallest k for which there exists a decomposition

$$X \times X = E_0 \cup \ldots \cup E_k, \quad E_i \cap E_j = \emptyset \text{ if } i \neq j,$$

such that there exists a local section of $GX \to X \times X$ over each E_i.

Comparing TC and GC

Question

Clearly $\mathrm{TC}(X) \leq \mathrm{GC}(X)$, but when is $\mathrm{TC}(X) = \mathrm{GC}(X)$?

TC of spheres

Theorem (Farber '03)

$$\mathsf{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \geq 2 \text{ is even} \end{cases}$$

Corollary

Because the optimal motion planners given by Farber are geodesic:

$$\mathsf{GC}(S^n) = \mathsf{TC}(S^n)$$

TC of projective spaces

Theorem (Farber–Tabachnikov–Yuzvinsky '03)

$$\mathsf{TC}(\mathbb{R}P^n) = \begin{cases} n & \text{if } n = 1, 3, 7 \\ \mathsf{Immdim}(\mathbb{R}P^n) & \text{otherwise} \end{cases}$$

Corollary

Because the motions planners given by Farber–Tabachnikov–Yuzvinsky can be modified to be geodesic:

$$\mathsf{GC}(\mathbb{R}P^n) = \mathsf{TC}(\mathbb{R}P^n)$$

Comparing TC and GC

Question

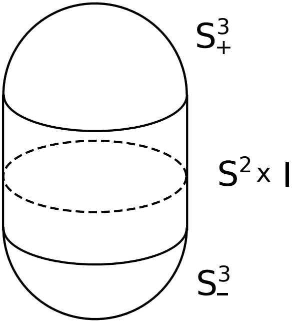
We just saw that in some cases $\mathrm{TC}(X) = \mathrm{GC}(X)$. Can we find a metric space X such that $\mathrm{TC}(X) < \mathrm{GC}(X)$?

Elongated 3-sphere

Example

Let $\tilde{S}^3$ be the result of glueing two caps on the cylinder $S^2 \times I$.
Clearly every geodesic motion planner on $\tilde{S}^3$ restricts to a motion planner on S^2. Therefore:

$$\mathrm{GC}(\tilde{S}^3) \geq \mathrm{TC}(S^2) = 2 > 1 = \mathrm{TC}(S^3) = \mathrm{TC}(\tilde{S}^3).$$



Totally convex submanifolds

Definition

A subspace Y of a metric space X is said to be **convex** if for any pair of points $x, y \in Y$, every minimal geodesic in X between x and y lies entirely in Y.

Theorem (R.-M.)

If Y is a convex subspace of X, then $\mathrm{TC}(Y) \leq \mathrm{GC}(Y) \leq \mathrm{GC}(X)$.

Theorem (R.-M.)

There exists a metric d on S^{2k+1} such that $\mathrm{GC}(S^{2k+1}, \mathrm{d}) = 2k$ but $\mathrm{TC}(S^{2k+1}) = 1$.

Totally convex submanifolds

Theorem (R.-M.)

There exists a metric d on S^{2k+1} such that $\mathrm{GC}(S^{2k+1}, \mathrm{d}) = 2k$ but $\mathrm{TC}(S^{2k+1}) = 1$.

Remark

This shows that the difference between $\mathrm{GC}(S^{2k+1}, \mathrm{d})$ and $\mathrm{TC}(S^{2k+1})$ can be arbitrarily large. This shows that $\mathrm{GC}(X)$ is very different from the efficient topological complexity $\ell TC(X)$ of Błaszczyk–Carrasquel, for which they show that $TC(X) \leq \ell TC(X) \leq TC(X) + 1$ if X is a closed Riemannian manifold ($\ell TC(X)$ is only defined for Riemannian manifolds).

Klein bottle

Theorem (Cohen–Vandembroucq '18)

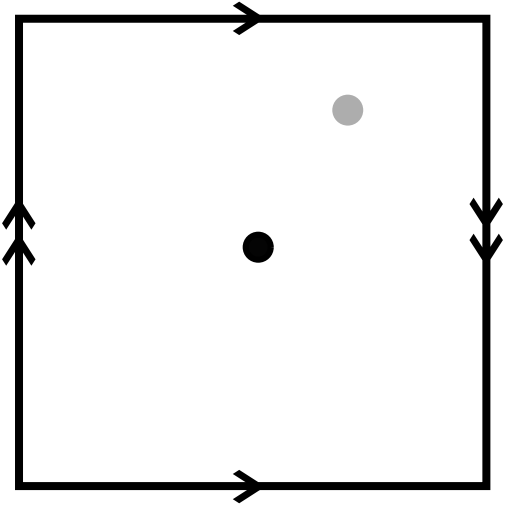
If K denotes the Klein bottle then $\mathrm{TC}(K) = 4$.

Theorem (R.-M.)

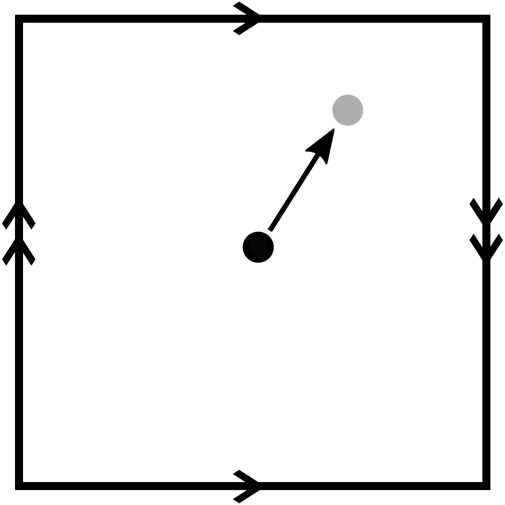
If K denotes the Klein bottle (with the flat metric) then $\mathrm{GC}(K) = 4$.

We show the **lower bound** directly. The lower bound $\mathrm{TC}(K) \geq 4$ automatically extends to $\mathrm{GC}(K) \geq \mathrm{TC}(K) \geq 4$, but it is very hard to prove.

Motion planning on K



Motion planning on K



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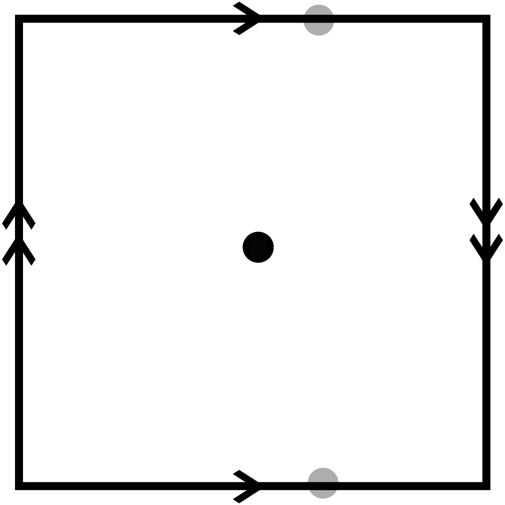
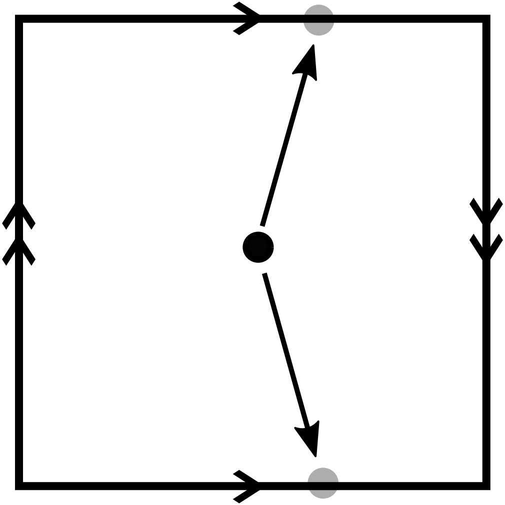


Motion planning on K



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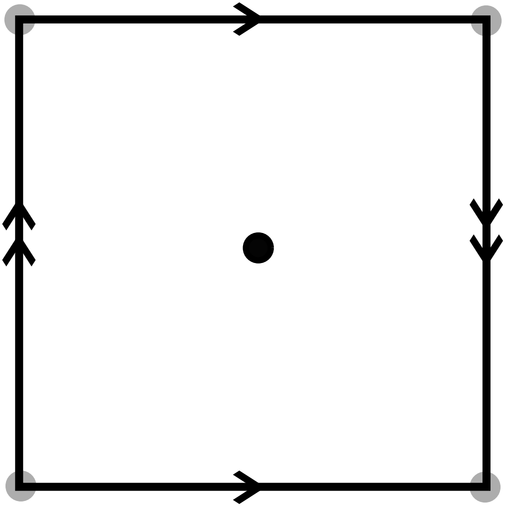
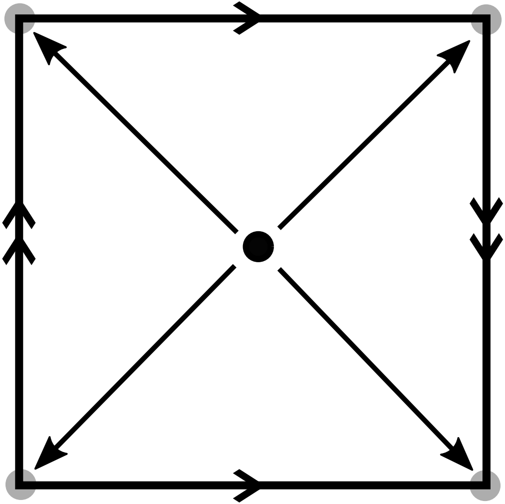
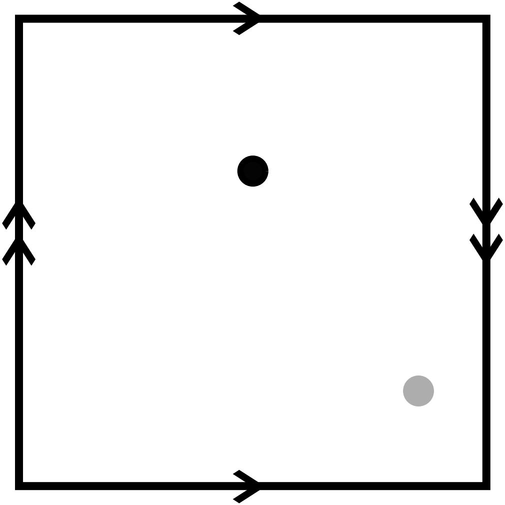


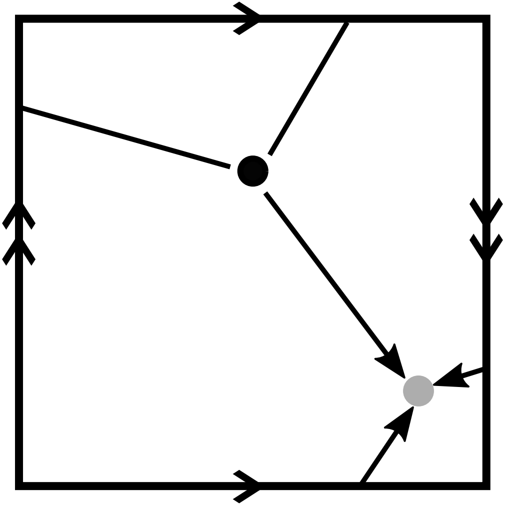
Motion planning on K



Motion planning on K



Motion planning on K



Proof of $\mathrm{GC}(K) = 4$

Definition

The **cut locus** of X is the subset $C \subset X \times X$ consisting of the pairs (x, y) for which there is more than one minimal geodesic γ from x to y.

Definition

The **cut locus slice** of a point x in X is the subset X consisting of all y such that (x, y) is in the cut locus C.

Proof of $GC(K) = 4$

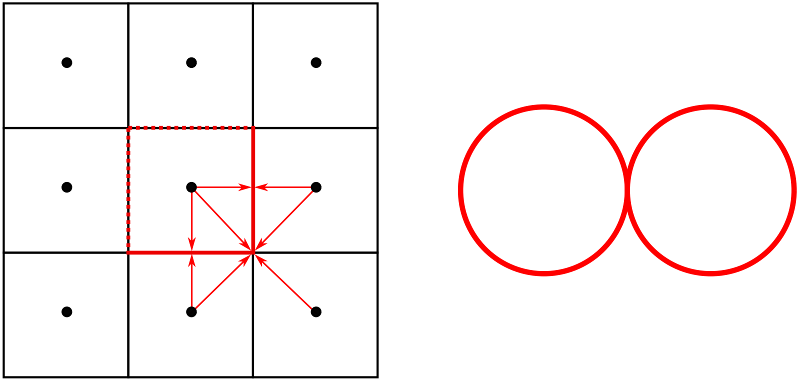
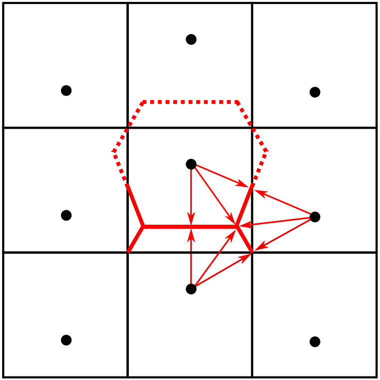


Figure: Cut locus slice for $x = (1/2, 1/2)$ in the Klein bottle.

Proof of GC(K) = 4



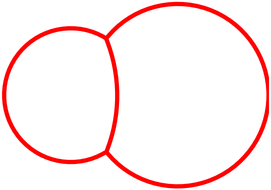


Figure: Cut locus slice for x going "up" from $(1/2, 1/2)$ to $(1/2, 1)$ in the Klein bottle. When x moves away from $(1/2, 1/2)$ a new edge appears at the vertex and then it keeps growing, while another edge gets shorter.

Proof of $\mathrm{GC}(K) = 4$

Definition

Let $S_k \subset K \times K$ consist of all pairs (x, y) such that there are precisely k minimal geodesics from x to y. Note that $GK \to K \times K$ is a branched covering. Over each S_k the map $GK \to K \times K$ restricts to a k-sheeted covering.

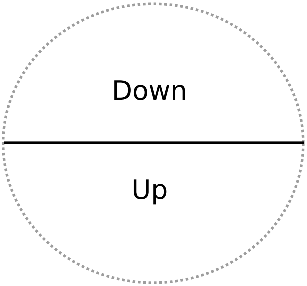


Figure: Neighborhood of y for (x, y) in S_2. Two sheets coming together.

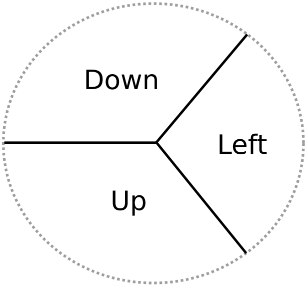


Figure: Neighborhood of y for (x, y) in S_3. Three sheets coming together.

Flat sphere

Definition

Let W^2 be the boundary of a 3-cube with the flat metric. We may call it a **flat sphere**. This example was suggested by Jarek Kędra.

Theorem (R.-M.)

$\mathrm{GC}(W^2) \geq 3 > \mathrm{TC}(W^2) = \mathrm{TC}(S^2) = 2$

Flat sphere

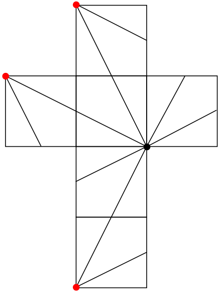


Figure: Pair (x, y) in S_6.

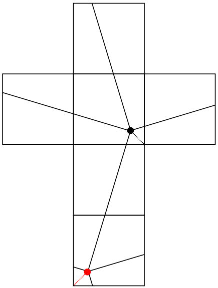


Figure: Pair (x, y) in S_4.

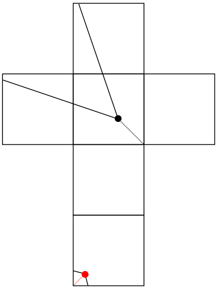


Figure: Pair (x, y) in S_2.

Further work

1. Can we use $\mathrm{GC}(K)$ to compute $\mathrm{TC}(K)$? There are currently two proofs of $\mathrm{TC}(K) = 4$ and both are very technical.
2. Compute the geodesic complexity of configuration spaces.
3. There are upper bounds $\mathrm{TC}(X) \leq \mathrm{cat}(X \times X) \leq \dim(X \times X)$. We know that the bound involving the LS-category $\mathrm{cat}(X \times X)$ does not hold for $\mathrm{GC}(X)$.
   $*$ Does $\mathrm{GC}(X) \leq \dim(X \times X)$ still hold?
   $*$ Is there a bound $\mathrm{GC}(X) \leq \mathrm{Gcat}(X \times X)$?

Thank you!