The Relative Topological Complexity of Pairs of Right-Angled Artin Groups

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Intuition – Motion Planning Algorithm

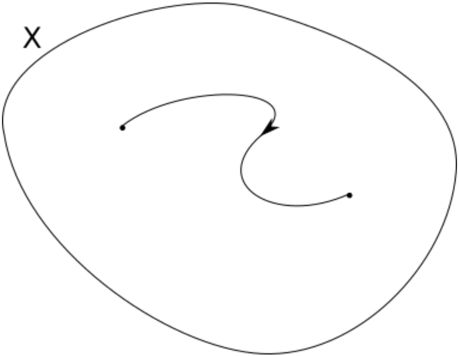

Figure 1: For motion planning algorithms, the input is a pair of points in $X \times X$, and the output is a continuous choice of path between them.

Intuition – Relative Motion Planning

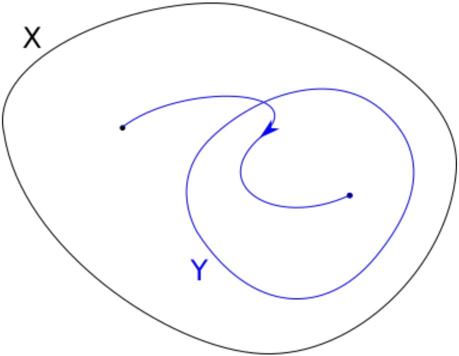

Figure 2: For this variant on motion planning algorithms, the input is a pair of points in $X \times Y$ and the output is a path between them.

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Definition

A *continuous motion planning algorithm* of size n on $X \times Y$ is an partition $\{E_i\}_{i=1}^n$ of $X \times Y$ where each E_i is a Euclidean Neighborhood Retract, and there exists continuous sections of π', $s_i : E_i \to P_{X \times Y}$, for each E_i.

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Definition

The *relative topological complexity* of the pair (X, Y), denoted $\mathrm{TC}(X, Y)$, is the smallest n for which there is a continuous motion planning algorithm of size n on $X \times Y$.

Lower Bound for $TC(X, Y)$

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Proposition (S., 2018)

If there are elements $v_1, \ldots, v_p \in \ker[H^*X \otimes H^*Y \to H^*Y]$
such that $v_1 \smile \cdots \smile v_p \neq 0$,
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Proposition (S., 2018)

Let $\iota : Y \to X$ be the inclusion map. Then, $\ker[H^*X \otimes H^*Y \to H^*Y]$ is generated as an ideal by elements of the form $\overline{v} = v \otimes 1 - 1 \otimes \iota^*(v)$ where $v \in H^*X$.

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In particular, if $n > m$,

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A group G is a *right-angled Artin group*, sometimes called a *graph group*, if it has a presentation with generators $s_1, \ldots s_n$ and where all relations are of the form $s_i s_j = s_j s_i$ for select index pairs $i < j$.

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- The free abelian group on n generators, $\mathbb{Z}(n)$, is a RAAG.
- $A = \langle a, b, c, d \mid ab = ba, ad = da, bd = db, bc = cb \rangle$ is a RAAG.

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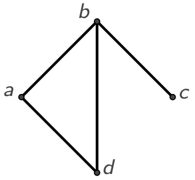

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(Forming a T^k structure with the lower-dimensional cells.)

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Dimension k	Set of k-cells
3	$\{abd\}$
2	$\{ab, ad, bd, bc\}$
1	$\{a, b, c, d\}$
0	$\{x_0\}$

The Full Picture for RAAGs

Group G		Graph Γ_G		Salvetti Complex $\mathcal{S}_G$
{generators}	$\longleftrightarrow$	{vertices}	$\longleftrightarrow$	{1-cells}
{relations}	$\longleftrightarrow$	{edges}	$\longleftrightarrow$	{2-cells}
		{k-cliques}	$\longleftrightarrow$	{k-cells}

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Some Notation

For any RAAG G with subgroup H a RAAG,

- $H^*(G) := H^*(\mathcal{S}_G)$
- $\mathrm{TC}(G) := \mathrm{TC}(\mathcal{S}_G)$
- $\mathrm{TC}(G,H) := \mathrm{TC}(\mathcal{S}_G,\mathcal{S}_H)$

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Theorem (Cohen-Pruidze, 2008)

If G is a RAAG, then $\mathrm{TC}(G) = 1 + \max\{|\mathcal{V}(K_1)| + |\mathcal{V}(K_2)|\}$ where the max is taken over dijoint cliques $K_1, K_2 \subset \Gamma_G$.

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- $\mathrm{TC}(A) = 5$.

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Let G be a RAAG. We call H a *special subgroup of* G if $\mathcal{V}(\Gamma_H) \subset \mathcal{V}(\Gamma_G)$ and $\mathcal{E}(\Gamma_H) = \mathcal{E}(\Gamma_G)|_{\mathcal{V}(\Gamma_H)}$.

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Theorem (S., 2018)

If H is a special subgroup of a RAAG G, then $\mathrm{TC}(G, H) = 1 + \max\{|\mathcal{V}(K_G) \cup \mathcal{V}(K_H)|\}$ where the max runs over pairs of cliques $\{K_G, K_H\}$ with $K_G \subset \Gamma_G$ and $K_H \subset \Gamma_H$.

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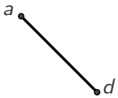

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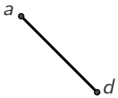

The generators of $H^*(A)$ and $H^*(B)$ are listed below.

Dimension	$H^*(A)$ Gens	$H^*(B)$ Gens
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2	ab, ad, bd, bc	ad
1	a, b, c, d	a, d
0	1	1

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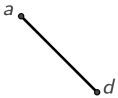

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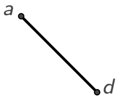

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$$\begin{aligned}\overline{a} \smile \overline{b} \smile \overline{c} \smile \overline{d} &= (a \otimes 1 - 1 \otimes \iota^*(a)) \smile (b \otimes 1 - 1 \otimes \iota^*(b)) \\ &\smile (c \otimes 1 - 1 \otimes \iota^*(c)) \smile (d \otimes 1 - 1 \otimes \iota^*(d)) \\ &= (a \otimes 1 - 1 \otimes a)(b \otimes 1)(c \otimes 1)(d \otimes 1 - 1 \otimes d) \\ &= bc \otimes ad \neq 0\end{aligned}$$

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$$\overline{a} \smile \overline{b} \smile \overline{c} \smile \overline{d} = (a \otimes 1 - 1 \otimes \iota^*(a)) \smile (b \otimes 1 - 1 \otimes \iota^*(b)) \smile (c \otimes 1 - 1 \otimes \iota^*(c)) \smile (d \otimes 1 - 1 \otimes \iota^*(d))$$
$$= (a \otimes 1 - 1 \otimes a)(b \otimes 1)(c \otimes 1)(d \otimes 1 - 1 \otimes d)$$
$$= bc \otimes ad \neq 0$$

Thus, $\mathrm{TC}(A, B) > 4 \implies \mathrm{TC}(A, B) = 5.$

□

Current/Future Work: Subcomplexes of Tori

Let X be a cellular subcomplex of a torus. Cohen and Pruidze utilize the following correspondence to compute $\mathrm{TC}(X)$:

Simplicial Complex Γ_X		Cell Complex X
{0-simplices}	$\longleftrightarrow$	{1-cells}
{1-simplicess}	$\longleftrightarrow$	{2-cells}
$\{(k-1)$-simplices$\}$	$\longleftrightarrow$	$\{k$-cells$\}$

Theorem (S., 2019)

If Y is a cellular subcomplex of X, a cellular subcomplex of T^n, then $\mathrm{TC}(X, Y) = 1 + \max\{|K_X \cup K_Y|\}$ where the max runs over pairs of $(k-1)$-simplices $\{K_X, K_Y\}$ with $K_X \subset \Gamma_X$ and $K_Y \subset \Gamma_Y$.

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Let X be a cellular subcomplex of T^n. Then, for each $\mathrm{cat}(X) \leq k \leq \mathrm{TC}(X)$, there is some cellular subcomplex $Y \subset X$ such that $\mathrm{TC}(X, Y) = k$.

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Is each $\text{cat}(X) \leq k \leq \text{TC}(X)$ realized as $\text{TC}(X, Y) = k$ by some $Y \subset X$?

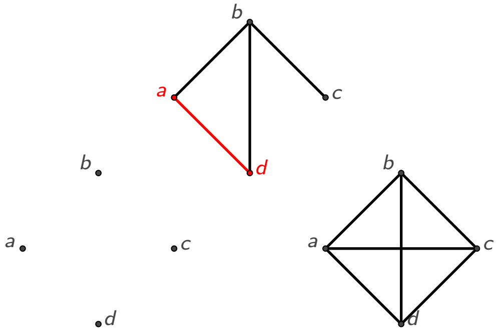
Theorem (S., 2019)

Let X be a cellular subcomplex of T^n. Then, for each $\text{cat}(X) \leq k \leq \text{TC}(X)$, there is some cellular subcomplex $Y \subset X$ such that $\text{TC}(X, Y) = k$.

Lemma (S., 2019)

Let $Y \subset X$ be a cellular subcomplex, and suppose $Y' = Y \cup e$ is another cellular subcomplex of X. Then, $\text{TC}(X, Y) \leq \text{TC}(X, Y') \leq \text{TC}(X, Y) + 1$.

Thank you for your attention!



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